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Shuffle-compatible descent statistics and quotients of quasisymmetric functions

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Abstract. We study shuffle-compatible permutation statistics: permutation statistics st with the property that the distribution of st over all shuffles of two permutations π and σ is completely determined by $\mathrm{st}(\pi)$, $\mathrm{st}(\sigma)$, and the lengths of π and σ. We develop a theory of shuffle-compatibility for descent statistics—permutation statistics that depend only on the descent set and length—and its connections to P-partitions, quasisymmetric functions, and noncommutative symmetric functions.

Keywords: permutation statistics, shuffles, descents, P-partitions, quasisymmetric functions, noncommutative symmetric functions

1 Introduction

We say that $\pi = \pi_1\pi_2\cdots\pi_n$ is a *permutation* of *length* n (or an *n-permutation*) if it is a sequence of n distinct letters—not necessarily from 1 to n—in $\mathbb{P}$, the set of positive integers. For example, $\pi = 47381$ is a permutation of length 5. Let $\mathfrak{P}_n$ denote the set of all permutations of length n, and let $|\pi|$ denote the length of a permutation π.

A *permutation statistic* (or *statistic*) st is a function defined on permutations such that $\mathrm{st}(\pi) = \mathrm{st}(\sigma)$ whenever π and σ are permutations with the same relative order. Three classical examples of permutation statistics are the descent set Des, the descent number des, and the major index maj. We say that $i \in [n-1]$ is a *descent* of $\pi \in \mathfrak{P}_n$ if $\pi_i > \pi_{i+1}$. Then the *descent set*

$$\mathrm{Des}(\pi) := \{\, i \in [n-1] \mid \pi_i > \pi_{i+1} \,\}$$

of π is the set of its descents, the *descent number*

$$\mathrm{des}(\pi) := |\mathrm{Des}(\pi)|$$

is its number of descents, and the *major index*

$$\mathrm{maj}(\pi) := \sum_{k \in \mathrm{Des}(\pi)} k$$

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is the sum of its descents.

Let $\pi \in \mathfrak{P}_m$ and $\sigma \in \mathfrak{P}_n$ be *disjoint* permutations, that is, permutations with no letters in common. We say that $\tau \in \mathfrak{P}_{m+n}$ is a *shuffle* of π and σ if both π and σ are subsequences of τ. The set of shuffles of π and σ is denoted $S(\pi,\sigma)$. For example, $S(53,16) = \{5316, 5136, 5163, 1653, 1536, 1563\}$. It is easy to see that the number of permutations in $S(\pi,\sigma)$ is $\binom{m+n}{m}$.

Richard Stanley's theory of P-partitions [17] implies that the descent set statistic has a remarkable property related to shuffles: for any disjoint permutations π and σ, the multiset $\{\, \mathrm{Des}(\tau) \mid \tau \in S(\pi,\sigma) \,\}$ encoding the distribution of Des over all shuffles of π and σ depends only on $\mathrm{Des}(\pi)$, $\mathrm{Des}(\sigma)$, and the lengths of π and σ [15, Exercise 3.161]. That is, if π and π' are permutations of the same length with the same descent set, and similarly with σ and σ', then the number of permutations in $S(\pi,\sigma)$ with any given descent set is the same as the number of permutations in $S(\pi',\sigma')$ with that descent set.

In fact, this property is not unique to the descent set. A result called "Stanley's shuffling theorem", which is a special case of Proposition 12.6 in Stanley's memoir on P-partitions [17], implies that the distribution of each of the statistics des, maj, and (des, maj) over the set of shuffles of two disjoint permutations depends only on the value of the statistic over the two permutations being shuffled and the lengths of these two permutations. Furthermore, it is a direct consequence of John Stembridge's "enriched P-partitions" [18] that the peak set Pk has this property, and a direct consequence of Kyle Petersen's [12] "left enriched P-partitions" that the left peak set Lpk has this property.

We call this property "shuffle-compatibility". More precisely, we say that a permutation statistic st is *shuffle-compatible* if for any disjoint permutations π and σ, the multiset $\{\, \mathrm{st}(\tau) \mid \tau \in S(\pi,\sigma) \,\}$ depends only on $\mathrm{st}(\pi)$, $\mathrm{st}(\sigma)$, $|\pi|$, and $|\sigma|$. Hence, Des, des, maj, (des, maj), Pk, and Lpk are examples of shuffle-compatible permutation statistics.

This extended abstract is a summary of the recent paper [6], which presents the first in-depth investigation of shuffle-compatibility and focuses in particular on the shuffle-compatibility of "descent statistics": permutation statistics that depend only on the descent set and length of a permutation. In Section 2, we introduce some preliminary notions from the theory of descent statistics and define the "shuffle algebra" of a shuffle-compatible permutation statistic st, whose multiplication encodes the distribution of st over shuffles of permutations (or more precisely, equivalence classes of permutations induced by the statistic st). In Section 3, we give a shuffle-compatibility criterion which implies that the shuffle algebra of any shuffle-compatible descent statistic is isomorphic to a quotient of the algebra QSym of quasisymmetric functions, and also a "dual" shuffle-compatibility criterion exploiting the duality between QSym and the coalgebra **Sym** of noncommutative symmetric functions. These are used to establish the shuffle-compatibility of a number of well-known descent statistics and to give explicit descriptions of their shuffle algebras; we present three of these algebras in Section 4. Finally, we state several open problems surrounding the notion of shuffle-compatibility in Section 5.

Before proceeding, we note that there is another class of algebras that are related to permutations and their descent sets. If st is a function defined on the nth symmetric group $\mathfrak{S}_n$, we may consider the elements

$$K_\alpha = \sum_{\substack{\pi \in \mathfrak{S}_n \\ \mathrm{st}(\pi)=\alpha}} \pi$$

in the group algebra of $\mathfrak{S}_n$, where α ranges over the image of st. Louis Solomon [14] proved that if st is the descent set, then the K_α span a subalgebra of the group algebra of $\mathfrak{S}_n$, called the *descent algebra* of $\mathfrak{S}_n$. Several other descent statistics are known to give subalgebras of the descent algebra [1, 3, 9, 10, 11, 12, 13]; these statistics have the property that given values α and β of st and a permutation $\tau \in \mathfrak{S}_n$, the number of pairs (π, σ) of permutations in $\mathfrak{S}_n$ with $\mathrm{st}(\pi) = \alpha$, $\mathrm{st}(\alpha) = \beta$, and $\pi\sigma = \tau$ depends only on $\mathrm{st}(\tau)$. Hence, our theory of shuffle-compatible descent statistics can be interpreted as an analogue of Solomon's descent theory for statistics compatible under the shuffle product (as opposed to the ordinary product).

2 Preliminary definitions and results

2.1 Descent compositions and descent statistics

Given a subset $S \subseteq [n-1]$ with elements $s_1 < s_2 < \cdots < s_j$, let $\mathrm{Comp}(S)$ be the composition $(s_1, s_2 - s_1, \ldots, s_j - s_{j-1}, n - s_j)$ of n, and given a composition $L = (L_1, L_2, \ldots, L_k)$, let $\mathrm{Des}(L) := \{L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{k-1}\}$ be the corresponding subset of $[n-1]$. Then, Comp and Des are inverse bijections. If $\pi \in \mathfrak{P}_n$ has descent set $S \subseteq [n-1]$, then we call $\mathrm{Comp}(S)$ the *descent composition* of π, which we also denote by $\mathrm{Comp}(\pi)$. Conversely, if π has descent composition L, then its descent set $\mathrm{Des}(\pi)$ is $\mathrm{Des}(L)$.

A permutation statistic st is called a *descent statistic* if it depends only on the descent composition, that is, if $\mathrm{Comp}(\pi) = \mathrm{Comp}(\sigma)$ implies $\mathrm{st}(\pi) = \mathrm{st}(\sigma)$ for any two permutations π and σ. Equivalently, st is a descent statistic if it depends only on the descent set and length of a permutation. We saw four examples of descent statistics in the introduction: the descent set Des, descent number des, major index maj, and the joint statistic $(\mathrm{des}, \mathrm{maj})$. The following are some additional descent statistics that we consider in our investigation of shuffle-compatibility:

- The peak set Pk and peak number pk. We say that i (where $2 \leq i \leq n-1$) is a *peak* of $\pi \in \mathfrak{P}_n$ if $\pi_{i-1} < \pi_i > \pi_{i+1}$. The *peak set* $\mathrm{Pk}(\pi)$ of π is defined to be the set of peaks of π, and the *peak number* $\mathrm{pk}(\pi)$ of π is defined to be $\mathrm{pk}(\pi) := |\mathrm{Pk}(\pi)|$.

- The left peak set Lpk and left peak number lpk. We say that $i \in [n-1]$ is a *left peak* of $\pi \in \mathfrak{P}_n$ if i is a peak of π or if $i = 1$ and i is a descent of π. The *left peak set* $\mathrm{Lpk}(\pi)$ of π is defined to be the set of left peaks of π, and the *left peak number* $\mathrm{lpk}(\pi)$ of π is defined to be $\mathrm{lpk}(\pi) := |\mathrm{Lpk}(\pi)|$.

- The number udr of up-down runs. An *up-down run* of $\pi \in \mathfrak{P}_n$ is either a maximal monotone consecutive subsequence or π_1 when $\pi_1 > \pi_2$. For example, the up-down runs of $\pi = 871542$ are 8, 871, 15, and 542, so $\mathrm{udr}(\pi) = 4$.

- Ordered tuples of descent statistics, such as $(\mathrm{pk}, \mathrm{des})$, $(\mathrm{lpk}, \mathrm{des})$, and so on.

2.2 Shuffle algebras

Every permutation statistic st induces an equivalence relation on permutations; we say that permutations π and σ are st-*equivalent* if $\mathrm{st}(\pi) = \mathrm{st}(\sigma)$ and $|\pi| = |\sigma|$. We write the st-equivalence class of π as $[\pi]_{\mathrm{st}}$. For a shuffle-compatible statistic st, we can then associate to st a $\mathbb{Q}$-algebra in the following way. First, associate to st a $\mathbb{Q}$-vector space by taking as a basis the st-equivalence classes of permutations. We give this vector space a multiplication by taking

$$[\pi]_{\mathrm{st}}[\sigma]_{\mathrm{st}} = \sum_{\tau \in S(\pi,\sigma)} [\tau]_{\mathrm{st}},$$

which is well-defined (i.e., the choice of π and σ does not matter) because st is shuffle-compatible. Conversely, if such a multiplication is well-defined, then st is shuffle-compatible. We denote the resulting algebra $\mathcal{A}_{\mathrm{st}}$ and call it the *shuffle algebra* of st. Observe that $\mathcal{A}_{\mathrm{st}}$ is graded, and $[\pi]_{\mathrm{st}}$ belongs to the nth homogeneous component of $\mathcal{A}_{\mathrm{st}}$ if π has length n.

As a preliminary example, the following theorem gives a description of the major index shuffle algebra, which we prove in [6] using a formula arising from Stanley's theory of P-partitions [17]. Recall that the nth q-factorial $[n]_q!$ is defined by

$$[n]_q! := (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

Theorem 2.1. *The map given by*

$$[\pi]_{\mathrm{maj}} \mapsto \frac{q^{\mathrm{maj}(\pi)}}{[|\pi|]_q!} x^{|\pi|}$$

is a $\mathbb{Q}$-algebra isomorphism from $\mathcal{A}_{\mathrm{maj}}$ to the span of

$$\left\{ \frac{q^j}{[n]_q!} x^n \right\}_{n \geq 1,\, 0 \leq j \leq \binom{n}{2}},$$

a subalgebra of $\mathbb{Q}[[q]][x]$.

For descent statistics st, not only does st induce a equivalence relation on permutations, but it also induces a equivalence relation on compositions because permutations with the same descent composition are necessarily st-equivalent. Thus, we can regard the shuffle algebra of a descent statistic st as consisting of st-equivalence classes of compositions rather than permutations, and we will do this when presenting our theory of shuffle-compatible descent statistics in Section 3.

We say that st_1 is a *refinement* of st_2 if for all permutations π and σ of the same length, $\mathrm{st}_1(\pi) = \mathrm{st}_1(\sigma)$ implies $\mathrm{st}_2(\pi) = \mathrm{st}_2(\sigma)$. For example, the statistics of which the descent set is a refinement are exactly what we call descent statistics.

Theorem 2.2. *Suppose that* st_1 *is shuffle-compatible and is a refinement of* st_2*. Let A be a $\mathbb{Q}$-algebra with basis $\{u_\alpha\}$ indexed by* st_2*-equivalence classes α, and suppose that there exists a $\mathbb{Q}$-algebra homomorphism $\phi\colon \mathcal{A}_{\mathrm{st}_1} \to A$ such that for every* st_1*-equivalence class β, we have $\phi(\beta) = u_\alpha$ where α is the* st_2*-equivalence class containing β. Then* st_2 *is shuffle-compatible and the map $u_\alpha \mapsto \alpha$ extends by linearity to an isomorphism from A to $\mathcal{A}_{\mathrm{st}_2}$.*

3 Theory of shuffle-compatible descent statistics

3.1 Quasisymmetric functions

Let us review some basic definitions and results surrounding quasisymmetric functions. A formal power series $f \in \mathbb{Q}[[x_1, x_2, \ldots]]$ in countably many commuting variables $x_1, x_2, \ldots$ of bounded degree is called a *quasisymmetric function* if for any $a_1, a_2, \ldots, a_k \in \mathbb{P}$, $i_1 < i_2 < \cdots < i_k$, and $j_1 < j_2 < \cdots < j_k$, we have

$$[x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}]\, f = [x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}]\, f.$$

Let QSym_n be the set of quasisymmetric functions homogeneous of degree n. Then QSym_n is a $\mathbb{Q}$-vector space and is known to have dimension equal to 2^{n-1}, the number of compositions of n. The most important basis of QSym_n for our purposes is the basis of *fundamental quasisymmetric functions* $\{F_L\}_{L \vDash n}$ given by

$$F_L := \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n \\ i_j < i_{j+1} \text{ if } j \in \mathrm{Des}(L)}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

If $f \in \mathrm{QSym}_m$ and $g \in \mathrm{QSym}_n$, then $fg \in \mathrm{QSym}_{m+n}$. Thus $\mathrm{QSym} := \bigoplus_{n=0}^{\infty} \mathrm{QSym}_n$ is a graded $\mathbb{Q}$-algebra called the *algebra of quasisymmetric functions* with coefficients in $\mathbb{Q}$, a subalgebra of $\mathbb{Q}[[x_1, x_2, \ldots]]$. Motivated by Richard Stanley's theory of P-partitions, the first author introduced quasisymmetric functions in [5] and developed the basic algebraic properties of QSym. Further properties of QSym and connections with many topics of study in combinatorics and algebra were developed in the subsequent decades.

In particular, QSym is a Hopf algebra and is the terminal object in the category of combinatorial Hopf algebras in the sense of Aguiar–Bergeron–Sottile [2].

Using P-partitions, one can show that the multiplication rule for the fundamental basis is given by

$$F_J F_K = \sum_L c^L_{J,K} F_L. \tag{3.1}$$

where $c^L_{J,K}$ is the number of permutations with descent composition L among the shuffles of a permutation with descent composition J and one with descent composition K; see [16, Exercise 7.93]. This implies that QSym is isomorphic to the descent set shuffle algebra $\mathcal{A}_{\mathrm{Des}}$ with the fundamental basis corresponding to the basis of Des-equivalence classes.

Later, John Stembridge [18] introduced a variant of the notion of P-partitions called "enriched P-partitions", which is closely related to the combinatorics of peaks. Using enriched P-partitions, Stembridge defined the peak quasisymmetric functions $\{K_{n,\Lambda}\}$ which are indexed by peak sets Λ of n-permutations. These peak functions multiply by a rule similar to Equation (3.1) but with the role of descent compositions (equivalently, descent sets) replaced with peak sets, which shows that the peak set Pk is shuffle-compatible with shuffle algebra $\mathcal{A}_{\mathrm{Pk}}$ isomorphic to the span of the peak functions, called the *algebra of peaks*.

In a similar vein, Kyle Petersen [12] introduced "left enriched P-partitions" which play an analogous role to enriched P-partitions but for left peaks; it follows from Petersen's work that the left peak set Lpk is shuffle-compatible and that the shuffle algebra $\mathcal{A}_{\mathrm{Lpk}}$ is isomorphic to Petersen's *algebra of left peaks*. The shuffle-compatibility of the left peak set also follows from the work of Aguiar, Bergeron, and Nyman [1], which showed the existence of the coalgebra dual to $\mathcal{A}_{\mathrm{Lpk}}$.

Both Stembridge's algebra of peaks and Petersen's algebra of left peaks can be realized as quotients of QSym. In fact, this is true in general for shuffle algebras of shuffle-compatible descent statistics. The following is one of our central results.

Theorem 3.1. *A descent statistic* st *is shuffle-compatible if and only if there exists a* $\mathbb{Q}$*-algebra homomorphism* $\phi_{\mathrm{st}} : \mathrm{QSym} \to A$*, where* A *is a* $\mathbb{Q}$*-algebra with basis* $\{u_\alpha\}$ *indexed by* st*-equivalence classes* α *of compositions, such that* $\phi_{\mathrm{st}}(F_L) = u_\alpha$ *whenever* $L \in \alpha$*. In this case, the map given by*

$$[\pi]_{\mathrm{st}} \mapsto u_\alpha,$$

where $\mathrm{Comp}(\pi) \in \alpha$*, is a* $\mathbb{Q}$*-algebra isomorphism from* $\mathcal{A}_{\mathrm{st}}$ *to* A*.*

One direction of this theorem follows immediately from Theorem 2.2; we omit the proof of the other direction.

Corollary 3.2. *The shuffle algebra of any shuffle-compatible descent statistic is isomorphic to a quotient algebra of* QSym*.*

3.2 Noncommutative symmetric functions

Let $\mathbb{Q}\langle\langle X_1, X_2, \dots \rangle\rangle$ be the $\mathbb{Q}$-algebra of formal power series in countably many noncommuting variables $X_1, X_2, \dots$. Consider the elements

$$\mathbf{r}_L := \sum_{i_1,\dots,i_n} X_{i_1} X_{i_2} \cdots X_{i_n}$$

of $\mathbb{Q}\langle\langle X_1, X_2, \dots \rangle\rangle$, where the sum is over all $i_1, \dots, i_n$ satisfying

$$\underbrace{i_1 \leq \cdots \leq i_{L_1}}_{L_1} > \underbrace{i_{L_1+1} \leq \cdots \leq i_{L_1+L_2}}_{L_2} > \cdots > \underbrace{i_{L_1+\cdots+L_{k-1}+1} \leq \cdots \leq i_n}_{L_k}$$

for $L = (L_1, L_2, \dots, L_k)$. These $\mathbf{r}_L$ are called *ribbon functions*; they are linearly independent and generate a subalgebra **Sym** of $\mathbb{Q}\langle\langle X_1, X_2, \dots \rangle\rangle$ called the *algebra of noncommutative symmetric functions* with coefficients in $\mathbb{Q}$. The algebra **Sym** is graded and decomposes as $\mathbf{Sym} := \bigoplus_{n=0}^{\infty} \mathbf{Sym}_n$ where $\mathbf{Sym}_n$ is the vector space with basis $\{\mathbf{r}_L\}_{L \vDash n}$.

The study of **Sym** was initiated by Gelfand et al. [4], who showed that **Sym** is also a Hopf algebra and is in fact the graded dual of the Hopf algebra QSym of quasisymmetric functions. In particular, the ribbon basis of **Sym** is dual to the fundamental basis of QSym, which means that the structure constants for the comultiplication of the $\mathbf{r}_L$ are precisely the structure constants for the multiplication of the F_L. In other words, we have

$$\Delta \mathbf{r}_L = \sum_{J,K} c_{J,K}^{L} \mathbf{r}_J \otimes \mathbf{r}_K$$

where Δ denotes the comultiplication of **Sym** and the $c_{J,K}^{L}$ are the same as in Equation (3.1). This duality gives rise to the following shuffle-compatibility criterion, which can be seen as a dual version of Theorem 3.1.

Theorem 3.3. *Let* st *be a descent statistic. For each* st*-equivalence class α of compositions, let* $\mathbf{r}_\alpha^{\mathrm{st}} := \sum_{L \in \alpha} \mathbf{r}_L$. *Then* st *is shuffle-compatible if and only if for every equivalence class α, there exist constants $c_{\beta,\gamma}^{\alpha}$ for which*

$$\Delta \mathbf{r}_\alpha^{\mathrm{st}} = \sum_{\beta,\gamma} c_{\beta,\gamma}^{\alpha} \mathbf{r}_\beta^{\mathrm{st}} \otimes \mathbf{r}_\gamma^{\mathrm{st}},$$

that is, the $\mathbf{r}_\alpha^{\mathrm{st}}$ span a subcoalgebra of **Sym**.

When the $\mathbf{r}_\alpha^{\mathrm{st}}$ span a subcoalgebra of **Sym**, this coalgebra is the graded dual of the shuffle algebra $\mathcal{A}_{\mathrm{st}}$, so the $c_{\beta,\gamma}^{\alpha}$ are the structure constants for $\mathcal{A}_{\mathrm{st}}$. While Theorem 3.1 tells us that we can prove the shuffle-compatibility of a descent statistic by constructing suitable quotients of QSym, Theorem 3.3 tells us that we could, alternatively, construct suitable subcoalgebras of **Sym**, and this is often easier. Moreover, because it is straightforward to compute coproducts of noncommutative symmetric functions, Theorem 3.3 is useful for showing that a descent statistic is not shuffle-compatible and for conjecturing that a statistic is shuffle-compatible, which is not the case for Theorem 3.1.

Theorem 3.3 does not give us a way to explicitly describe the dual algebra $\mathcal{A}_{\mathrm{st}}$, but this can be done with the help of a theorem involving the notion of grouplike noncommutative symmetric functions which is proven using Theorem 3.3. We omit the theorem here, but it appears as Theorem 5.8 of [6].

4 Explicit descriptions of shuffle algebras

4.1 The $(\mathrm{pk}, \mathrm{des})$ shuffle algebra

So far, we know that Des, Pk, and Lpk are shuffle-compatible and we have identified their shuffle algebras. We also know that the statistics des, maj, and $(\mathrm{des}, \mathrm{maj})$ are shuffle-compatible; Theorem 2.1 characterizes the shuffle algebra of maj, and characterizations of the shuffle algebras of des and $(\mathrm{des}, \mathrm{maj})$ can be found in our paper [6].

In [6], we also characterize using quasisymmetric functions and noncommutative symmetric functions the shuffle algebras of the statistics pk, $(\mathrm{pk}, \mathrm{des})$, lpk, $(\mathrm{lpk}, \mathrm{des})$, udr, and $(\mathrm{udr}, \mathrm{des})$, thus showing that all of these statistics are shuffle-compatible. In this extended abstract, we focus our attention on the $(\mathrm{pk}, \mathrm{des})$, pk, and des shuffle algebras.

We begin with our result for $\mathcal{A}_{(\mathrm{pk},\mathrm{des})}$, which we prove in [6] using noncommutative symmetric functions. The operation of *Hadamard product* $*$ on formal power series in t is defined by

$$\Big(\sum_{n=0}^{\infty} a_n t^n\Big) * \Big(\sum_{n=0}^{\infty} b_n t^n\Big) := \sum_{n=0}^{\infty} a_n b_n t^n.$$

Theorem 4.1 (Shuffle-compatibility of $(\mathrm{pk}, \mathrm{des})$)**.**

(a) *The map given by*

$$[\pi]_{(\mathrm{pk},\mathrm{des})} \mapsto \begin{cases} \dfrac{t^{\mathrm{pk}(\pi)+1}(y+t)^{\mathrm{des}(\pi)-\mathrm{pk}(\pi)}(1+yt)^{|\pi|-\mathrm{pk}(\pi)-\mathrm{des}(\pi)-1}(1+y)^{2\,\mathrm{pk}(\pi)+1}}{(1-t)^{|\pi|+1}} x^{|\pi|}, & \text{if } |\pi| \geq 1, \\ 1/(1-t), & \text{if } |\pi| = 0, \end{cases}$$

is a $\mathbb{Q}$*-algebra isomorphism from* $\mathcal{A}_{(\mathrm{pk},\mathrm{des})}$ *to the span of*

$$\left\{\frac{1}{1-t}\right\} \bigcup \left\{\frac{t^{j+1}(y+t)^{k-j}(1+yt)^{n-j-k-1}(1+y)^{2j+1}}{(1-t)^{n+1}} x^n\right\}_{\substack{n\geq 1,\\ 0\leq j\leq \lfloor (n-1)/2\rfloor,\\ j\leq k\leq n-j-1}},$$

a subalgebra of $\mathbb{Q}[[t]][x,y]$ *where multiplication is the Hadamard product in* t*.*

(b) *The* $(\mathrm{pk}, \mathrm{des})$ *shuffle algebra* $\mathcal{A}_{(\mathrm{pk},\mathrm{des})}$ *is isomorphic to the span of*

$$\{1\} \cup \{p^{n-j}(1+y)^n(1-y)^{n-2k}x^n\}_{n\geq 1, 0\leq j\leq n-1, 0\leq k\leq \lfloor j/2\rfloor},$$

a subalgebra of $\mathbb{Q}[p, x, y]$.

As a simple application of Theorem 4.1, let us consider shuffling a permutation $\pi \in \mathfrak{P}_n$ with j peaks and k descents with a permutation $\sigma \in \mathfrak{P}_1$ (i.e., a single letter). Let

$$u_{n,j,k} := \frac{t^{j+1}(y+t)^{k-j}(1+yt)^{n-j-k-1}(1+y)^{2j+1}}{(1-t)^{n+1}}x^n,$$

the basis element of $\mathcal{A}_{(\mathrm{pk},\mathrm{des})}$ corresponding to the $(\mathrm{pk}, \mathrm{des})$-equivalence class of permutations with j peaks and k descents. To count permutations in $S(\pi, \sigma)$ by peaks and descents, we need to expand the Hadamard product $u_{n,j,k} * u_{1,1,1}$ as a linear combination of $u_{n+1,j',k'}$ for $0 \leq j' \leq \lfloor n/2 \rfloor$ and $j' \leq k' \leq n - j'$. We have

$$u_{1,0,0} = \frac{(1+y)t}{(1-t)^2}x = (1+y)x\sum_{m=0}^{\infty} mt^m.$$

Thus for any power series $f(t)$, we have

$$f(t) * u_{1,0,0} = (1+y)xtf'(t),$$

which can be used to derive the formula

$$\begin{aligned} u_{n,j,k} * u_{1,0,0} = (j+1)u_{n+1,j,k} + (j+1)u_{n+1,j,k+1} \\ + (k-j)u_{n+1,j+1,k} + (n-j-k-1)u_{n+1,j+1,k+1}. \end{aligned}$$

Thus, among the $n+1$ shuffles of π and σ, $j+1$ have j peaks and k descents, $j+1$ have j peaks and $k+1$ descents, $k-j$ have $j+1$ peaks and k descents, and $n-j-k-1$ have $j+1$ peaks and $k+1$ descents. This can be proven using a simple combinatorial argument if the single letter of σ is greater than or smaller than all of the letters of π, but it is not so easy in general.

4.2 The pk and des shuffle algebras

Next, we state our results for the shuffle algebras $\mathcal{A}_{\mathrm{pk}}$ and $\mathcal{A}_{\mathrm{des}}$. Both $\mathcal{A}_{\mathrm{pk}}$ and $\mathcal{A}_{\mathrm{des}}$ are homomorphic images of $\mathcal{A}_{(\mathrm{pk},\mathrm{des})}$ obtained by setting $y = 1$ and $y = 0$, respectively. In fact, Theorems 4.2 and 4.3 presented below can be derived from Theorem 4.1 using Theorem 2.2 and these homomorphisms. An alternative approach is to use Theorem 3.1 along with a specialization of Stembridge's peak quasisymmetric functions obtained via enriched P-partitions (for Theorem 4.2) or an analogous specialization of the fundamental quasisymmetric functions (for Theorem 4.3).

Theorem 4.2 (Shuffle-compatibility of the peak number)**.**

(a) *The map given by*

$$[\pi]_{\mathrm{pk}} \mapsto \begin{cases} \dfrac{2^{2\,\mathrm{pk}(\pi)+1} t^{\mathrm{pk}(\pi)+1}(1+t)^{|\pi|-2\,\mathrm{pk}(\pi)-1}}{(1-t)^{|\pi|+1}} x^{|\pi|}, & \text{if } |\pi| \geq 1, \\ 1/(1-t), & \text{if } |\pi| = 0, \end{cases}$$

is a $\mathbb{Q}$*-algebra isomorphism from* $\mathcal{A}_{\mathrm{pk}}$ *to the span of*

$$\left\{ \frac{1}{1-t} \right\} \bigcup \left\{ \frac{2^{2j+1} t^{j+1}(1+t)^{n-2j-1}}{(1-t)^{n+1}} x^n \right\}_{n \geq 1,\, 0 \leq j \leq \lfloor \frac{n-1}{2} \rfloor},$$

a subalgebra of $\mathbb{Q}[[t]][x]$ *where multiplication is the Hadamard product in* t*.*

(b) *The* pk *shuffle algebra* $\mathcal{A}_{\mathrm{pk}}$ *is isomorphic to the span of*

$$\{1\} \cup \{p^j x^n\}_{n \geq 1,\, 1 \leq j \leq n,\, j \equiv n \,(\mathrm{mod}\, 2)},$$

a subalgebra of $\mathbb{Q}[p, x]$*.*

Theorem 4.3 (Shuffle-compatibility of the descent number)**.**

(a) *The map given by*

$$[\pi]_{\mathrm{des}} \mapsto \begin{cases} \dfrac{t^{\mathrm{des}(\pi)+1}}{(1-t)^{|\pi|+1}} x^{|\pi|}, & \text{if } |\pi| \geq 1, \\ 1/(1-t), & \text{if } |\pi| = 0, \end{cases}$$

is a $\mathbb{Q}$*-algebra isomorphism from* $\mathcal{A}_{\mathrm{des}}$ *to the span of*

$$\left\{ \frac{1}{1-t} \right\} \bigcup \left\{ \frac{t^{j+1}}{(1-t)^{n+1}} x^n \right\}_{n \geq 1,\, 0 \leq j \leq n-1},$$

a subalgebra of $\mathbb{Q}[[t]][x]$ *where multiplication is the Hadamard product in* t*.*

(b) *The map given by*

$$[\pi]_{\mathrm{des}} \mapsto \begin{cases} \dbinom{p - \mathrm{des}(\pi) + |\pi| - 1}{|\pi|} x^{|\pi|}, & \text{if } |\pi| \geq 1, \\ 1, & \text{if } |\pi| = 0, \end{cases}$$

is a $\mathbb{Q}$*-algebra isomorphism from* $\mathcal{A}_{\mathrm{des}}$ *to the span of*

$$\{1\} \bigcup \left\{ \binom{p - j + n - 1}{n} x^n \right\}_{n \geq 1,\, 0 \leq j \leq n-1},$$

a subalgebra of $\mathbb{Q}[p, x]$*.*

(c) *The* des *shuffle algebra* $\mathcal{A}_{\mathrm{des}}$ *is isomorphic to the span of*

$$\{1\} \cup \{p^j x^n\}_{n \geq 1,\, 1 \leq j \leq n},$$

a subalgebra of $\mathbb{Q}[p, x]$*.*

5 Open problems

We now state a couple permutation statistics that we conjecture to be shuffle-compatible.

Conjecture 5.1. *The descent statistics* $(\mathrm{udr}, \mathrm{pk})$, *and* $(\mathrm{udr}, \mathrm{pk}, \mathrm{des})$ *are shuffle-compatible.*

Define the *exterior peak set* Epk by

$$\mathrm{Epk}(\pi) := \begin{cases} \mathrm{Lpk}(\pi), & \text{if } n-1 \in \mathrm{Des}(\pi) \\ \mathrm{Lpk}(\pi) \cup \{n\}, & \text{otherwise,} \end{cases}$$

for $\pi \in \mathfrak{P}_n$. In a previous version of [6], we conjectured that Epk is shuffle-compatible; this has been verified by Darij Grinberg [8] using a P-partition argument. Prior to this, Grinberg had shown that QSym is a "dendriform algebra" [7], an algebra whose multiplication can be split into a "left multiplication" and a "right multiplication" satisfying certain axioms. With the shuffle-compatibility of Epk, Grinberg proved that $\mathcal{A}_{\mathrm{Epk}}$ is a dendriform quotient of QSym. Other statistics that Grinberg has shown to define dendriform quotients of QSym include des, $(\mathrm{des}, \mathrm{maj})$, and Lpk. On the other hand, maj and Pk do not have this property. In addition to investigating whether the statistics stated in Conjecture 5.1 are in fact shuffle-compatible, it is also an open problem to determine which other shuffle-compatible descent statistics define dendriform quotients of QSym.

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