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Differential posets, Cayley graphs, and critical groups

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Abstract. In recent work, Benkart, Klivans, and Reiner defined the critical group of a faithful representation of a finite group, which is analogous to the critical group of a graph. In this paper we study maps between critical groups induced by injective group homomorphisms and in particular the map induced by restriction of the representation to a subgroup. We prove that in the abelian group case the critical groups are isomorphic to the critical groups of a certain Cayley graph and that the restriction map corresponds to a graph covering map. We also prove that when the group is an element in a differential tower of groups, critical groups of certain representations are closely related to words of up-down maps in the associated differential poset. We use this to generalize an explicit formula for the critical group of the permutation representation of the symmetric group given by the second author, and to enumerate the factors in such critical groups.

Keywords: Differential posets, Cayley graphs, critical groups, chip firing, Young's lattice.

1 Introduction

The critical group $K(\Gamma)$ is a well-studied abelian group invariant of a finite graph Γ which encodes information about the dynamics of a process called *chip firing* on the graph (see [6] where critical groups are called *sandpile groups*). Recent work of Benkart, Klivans, and Reiner defined analogous abelian group invariants $K(V)$, also called *critical groups*, associated to a faithful representation V of a finite group G [2]. It is known (see, for example, [16]) that graph covering maps induce surjective maps between graph critical groups. This paper investigates maps on critical groups of group representations which are induced by group homomorphisms.

Differential posets, introduced by Stanley [13], generalize many of the combinatorial and enumerative properties of Young's lattice. In [9], Miller and Reiner introduced a very strong conjecture about the Smith normal form of $UD + tI$ where U, D are the up

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and down maps in a differential poset, and t is a variable. We investigate how this conjecture, which was proven for powers of Young's lattice by Shah [12], can be used to determine the structure of critical groups in certain *differential towers of groups.*

Section 2 defines critical groups for group representations and gives background results. It also discusses background on differential posets and differential towers of groups which will be used throughout the later sections.

In Section 3 we study maps between critical groups which are induced by group homomorphisms. In particular, restriction of representations to a subgroup $H \subset G$ induces a map $\overline{\mathrm{Res}} : K(V) \to K(\mathrm{Res}_H^G V)$. When G is abelian, Theorem 3.5 shows that $K(V)$ can be identified with the critical group of a certain Cayley graph $\mathrm{Cay}(\widehat{G}, \mathcal{S}_V)$, and that the restriction map $\overline{\mathrm{Res}}$ agrees with a map on graph critical groups induced by a natural graph covering.

In [3], the second author determined the exact structure of the critical group for the permutation representation of the symmetric group $\mathfrak{S}_n$. This result depended on a relationship between tensor products with the permutation representation and the up and down maps in Young's lattice of integer partitions. Section 4 formalizes this connection and generalizes it to the context of differential towers of groups, allowing us to explicitly compute the critical group for a generalized permutation representation of the wreath product $A \wr \mathfrak{S}_n$ in Theorem 4.3. It also investigates properties of the critical groups associated to representations $V(w)$ which occur by repeatedly applying restriction and induction to the trivial representation in a differential tower of groups. The pattern of restriction and induction is specified by a word $w \in \{U, D\}^*$, where U, D are the up and down operators in the corresponding differential poset. In Theorem 4.7 we show that the structure of the critical group $K(V(w))$ is closely related to combinatorial properties of the up and down operators, as studied in [13].

Finally, in Section 5, Theorem 5.1 gives an enumeration of the factors in the elementary divisor form of $K(V(w))$ as a rank size of the corresponding differential tower of groups. Please see [1] for the proofs of the results outlined in this extended abstract.

2 Background and definitions

2.1 Critical groups of group representations

See the survey [6] for the definition and basic properties of critical groups of graphs. Except in Theorem 3.5, we will be interested in critical groups of graphs primarily by analogy.

We now briefly review critical groups of group representations, as defined in [2]. Let G be a finite group and V a faithful complex (not-necessarily-irreducible) representation of G; let $\mathbb{1}_G = V_0, V_1, ..., V_\ell$ denote the irreducible complex representations and χ_i, $i = 0, ..., \ell$ denote their characters. Let $R(G)$ denote the *representation ring* of G. This is the commutative $\mathbb{Z}$-algebra of formal integer combinations of representations of G modulo the relations $[W \oplus W'] = [W] + [W']$; the product structure is defined as $[W] \cdot [W'] = [W \otimes_{\mathbb{C}} W']$. As a $\mathbb{Z}$-module, $R(G)$ is isomorphic to $\mathbb{Z}^{\ell+1}$, since the classes of irreducible representations $[\mathbb{1}_G], [V_1], ..., [V_\ell]$ form a basis. For $g \in G$, we define elements

$$\delta^{(g)} = \sum_{i=0}^{\ell} \chi_i(g) \cdot [V_i]$$

of $R(G)$ corresponding to the columns in the character table of G. The representation ring $R(G)$ is endowed with a $\mathbb{Z}$-algebra homomorphism $\dim : R(G) \to \mathbb{Z}$ sending representations $[W]$ to their dimensions as vector spaces (which we also denote by $\dim(W)$), and extending by linearity to virtual representations. The kernel of this map, which we denote by $R_0(G)$, is the ideal of elements in $R(G)$ with virtual dimension 0. Multiplication by the element $\dim(V)[\mathbb{1}_G] - [V]$ defines a linear map $\widetilde{C}_V : R(G) \to R(G)$. Since $\dim(V)[\mathbb{1}_G] - [V] \in R_0(G)$, this descends to a linear map

$$C_V : R_0(G) \to R_0(G)$$

Definition-Proposition 2.1 ([2, Proposition 5.20])**.** If V is a faithful finite dimensional representation of G, then the linear map C_V is nonsingular, and so $\operatorname{coker}(C_V)$ is a finite abelian group. We define the *critical group* $K(V)$ to be this cokernel. We also have that $\operatorname{coker}(\widetilde{C}_V) = \mathbb{Z} \cdot \delta^{(e)} \oplus K(V)$.

Theorem 2.2 ([3, Theorem 3])**.**

a. Let $e = c_0, ..., c_\ell$ be a set of conjugacy class representatives for G. Then

$$|K(V)| = \frac{1}{|G|} \prod_{i=1}^{\ell} (\dim(V) - \chi_V(c_i)) \tag{2.1}$$

b. Suppose a is an integer value of χ_V achieved on m different conjugacy classes, then $K(V)$ contains a subgroup isomorphic to $(\mathbb{Z}/(\dim(V) - a)\mathbb{Z})^{m-1}$.

Example 2.3. Let $G = \mathfrak{S}_4$ and let $V = \mathbb{C}^4$ be the 4-dimensional representation where G acts by permuting coordinates. Working in the basis of $R(G)$ given by the classes of irreducible representations V_λ, we decompose each tensor product $V \otimes V_\lambda$ into irreducibles, giving the rows of the matrix $\widetilde{C}_V$.

$$\widetilde{C}_V = \begin{pmatrix} 3 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 3 \end{pmatrix}$$

To calculate the cokernel of $\widetilde{C}_V : R(G) \to R(G)$, we compute the Smith normal form (see Section 2.3 below) of $\widetilde{C}_V$ to get $\mathrm{diag}(0,1,1,1,4)$. This shows that $\mathrm{coker}(\widetilde{C}_V) \cong \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, and so $K(V) \cong \mathbb{Z}/4\mathbb{Z}$.

Alternatively, we could note that the non-trivial values of the character χ_V are $2,1,0,0$ and apply Theorem 2.2(a) to see that $|K(V)| = \frac{1}{4!}(4-2)(4-1)(4-0)(4-0) = 4$ and apply part (b) to see that $K(V)$ has a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z}$. This forces $K(V) \cong \mathbb{Z}/4\mathbb{Z}$. The critical groups for the permutation representation of $\mathfrak{S}_n$ were computed by the second author in [3]. In Section 4 we generalize this result further.

2.2 Differential posets and differential towers of groups

Differential posets are a class of partially ordered sets defined by Stanley in [13]. Differential posets retain many of the striking enumerative and combinatorial properties of Young's lattice Y, the lattice of integer partitions ordered by containment of Young diagrams. We refer the reader to [14] for basic definitions related to posets in what follows.

Definition 2.4 ([13, Definition 1.1 and Theorem 2.2])**.** For $r \in \mathbb{Z}_{>0}$, a poset P is called an *r-differential poset* if the following properties hold:

(DP1) P is a graded locally-finite poset with $\widehat{0}$.

(DP2) Let $\mathbb{Z}^{P_n}$ be the free abelian group spanned by elements of the n-th rank of P. Define the *up and down maps* $U_n : \mathbb{Z}^{P_n} \to \mathbb{Z}^{P_{n+1}}$ and $D_n : \mathbb{Z}^{P_n} \to \mathbb{Z}^{P_{n-1}}$ by

$$U_n x := \sum_{x \lessdot y} y, \qquad D_n y := \sum_{x \lessdot y} x$$

Where $x \lessdot y$ means that y covers x. Then we require that for all n we have

$$D_{n+1}U_n - U_{n-1}D_n = rI$$

When the context is clear we omit the subscripts from the up and down maps.

When P is a differential poset, we let $p_n = |P_n|$ denote the size of the n-th rank, and we let $\Delta p_n = p_n - p_{n-1}$ denote the difference in the sizes of consecutive ranks. We make the convention that $p_i = 0$ for $i < 0$. In the case of Young's lattice, $p_n = p(n)$ where $p(n)$ denotes the number of integer partitions of n. The following results of Stanley characterize the eigenspaces of UD in terms of the rank sizes.

Theorem 2.5 ([13, Theorem 4.1])**.** Let P be an r-differential poset and let $n \in \mathbb{N}$. Then UD_n is semisimple and has characteristic polynomial:

$$\mathrm{ch}(UD_n) = \prod_{i=0}^{n} (x - ri)^{\Delta p_{n-i}}.$$

Theorem 2.6 ([13, Proposition 4.6])**.** Let $E_n(ri)$ denote the eigenspace of UD_n belonging to the eigenvalue ri, then $E_n(0) = \ker(D_n) = (UP_{n-1})^{\perp}$ and $E_n(ri) = U^i E_{n-i}(0)$ for $1 \leq i \leq n$.

We will be interested in differential posets which come from the branching rules of a tower of finite groups.

Definition 2.7 ([9, Definition 6.1])**.** For $r \in \mathbb{Z}_{>0}$ define an *r-differential tower of groups* $\mathfrak{G}$ to be an infinite tower of finite groups:

$$\mathfrak{G} : \{e\} = G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots$$

such that for all n:

(DTG1) The branching rules for restricting irreducibles from G_n to G_{n-1} are multiplicity-free, and

(DTG2) $\mathrm{Res}^{G_{n+1}}_{G_n} \mathrm{Ind}^{G_{n+1}}_{G_n} - \mathrm{Ind}^{G_n}_{G_{n-1}} \mathrm{Res}^{G_n}_{G_{n-1}} = r \cdot \mathrm{id}$ where both sides are regarded as linear operators on $R(G_n)$.

An r-differential tower of groups $\mathfrak{G}$ corresponds to an r-differential poset $P = P(\mathfrak{G})$ whose n-th rank P_n is in bijection with the set $\mathrm{Irr}(G_n)$ of irreducible representations of G_n. We will use Greek letters like λ to denote elements of $P(\mathfrak{G})$ and V_λ to denote the corresponding irreducible representation. We write $|\lambda| = n$ if $\lambda \in P_n$, or equivalently if V_λ is a representation of G_n. For $\lambda \in P_n$ and $\mu \in P_{n+1}$, $\lambda \lessdot \mu$ in P if and only if $\mathrm{Res}^{G_{n+1}}_{G_n} V_\mu$ contains V_λ in its irreducible decomposition, thus condition (DTG2) becomes condition (DP2).

Example 2.8. Let Y denote Young's lattice of integer partitions. It is well known that irreducible representations of the symmetric group $\mathfrak{S}_n$ are indexed by partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ with $|\lambda| = \sum_i \lambda_i = n$; we refer the reader to [7] for background on the representation theory of the symmetric group. Young's rule says that $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} V_\lambda$ decomposes as a direct sum of V_ν where ν ranges over all possible ways to remove a single box from the Young diagram for λ. It is well known [13] that Y is a 1-differential poset, so (DP2) holds, and by the above identification (DTG2) also holds. Thus

$$\mathfrak{S} : \{e\} \subset \mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \cdots$$

is a 1-differential tower of groups, with $P(\mathfrak{S}) = Y$.

More generally, if A is an abelian group of size r, then Okada [10] showed that the tower of wreath products $A \wr \mathfrak{S} : \{e\} \subset A \subset A \wr \mathfrak{S}_2 \subset A \wr \mathfrak{S}_3 \subset \cdots$ is an r-differential tower of groups with $P(A \wr \mathfrak{S}) = Y^r$. Recent work of the second author [5] shows that these are the *only* differential towers of groups when r is one or prime.

The following result shows that the groups in any differential tower of groups have the same order as those in $A \wr \mathfrak{S}$.

Proposition 2.9. Let $\mathfrak{G} : \{e\} = G_0 \subset G_1 \subset \cdots$ be an r-differential tower of groups, then $|G_n| = r^n \cdot n!$ for all $n \geq 0$.

2.3 Smith normal form and cokernels of linear maps

The cokernel of a linear map over a PID is described by the Smith normal form of the corresponding matrix; see [15] for a review of the basic properties of Smith forms and their applications in combinatorics.

We will primarily be interested in determining Smith normal forms over $\mathbb{Z}$, but we will use some results about Smith normal forms over $\mathbb{Z}[t]$ as a computational tool. When $R = \mathbb{Z}$ we will always assume that the s_i are nonnegative (this can be achieved since ± 1 are the units in $\mathbb{Z}$). When referring to an abelian group $A = \mathrm{coker}(M)$, we say that A has k *factors* if exactly k of the s_i are different from 1; dually, we write $\mathrm{ones}(A) = k$ if exactly k of the s_i are equal to 1.

In [9] Miller and Reiner make the following remarkable conjecture; note that $\mathbb{Z}[t]$ is not a PID, so Smith forms are not guaranteed to exist:

Conjecture 2.10 ([9, Conjecture 1.1])**.** For all differential posets P, and for all n, the map $U_{n-1}D_n + tI : \mathbb{Z}[t]^{p_n} \to \mathbb{Z}[t]^{p_n}$ has a Smith normal form over $\mathbb{Z}[t]$.

Shah showed that Conjecture 2.10 is true for Y^r, giving us the following corollaries.

Theorem 2.11. ([12]) For any $r \geq 1$:

a. The Smith form $\mathrm{diag}(s_1, ..., s_{p_n})$ of UD_n in Y^r is given by

$$s_{n+1-i} = \prod_{\substack{k \\ m(k) \geq i}} k$$

where $m(k)$ denotes the multiplicity of the eigenvalue k of UD_n.

b. For all n the down maps $D_n : \mathbb{Z}^{p_n} \to \mathbb{Z}^{p_{n-1}}$ in Y^r are surjective.

3 Maps induced between critical groups

For $\sigma : H \to G$ a group homomorphism and W a representation of G, we let W^σ denote the representation of H given by $h \cdot w := \sigma(h)w$ for all $h \in H, w \in W$. If σ is the inclusion of a subgroup, then $W^\sigma = \mathrm{Res}^G_H W$. If σ is an automorphism of G, then W^σ corresponds to the usual notion of *twisting* by σ. We extend by linearity to define W^σ for W a virtual representation.

Theorem 3.1. Let $\sigma : H \hookrightarrow G$ be an injective group homomorphism and V a faithful representation of G, then $\overline{\sigma} : [W] \mapsto [W^\sigma]$ is a well-defined group homomorphism $K(V) \to K(V^\sigma)$. If σ is an isomorphism, then so is $\overline{\sigma}$.

Example 3.2. Let σ denote the unique outer automorphism of $\mathfrak{S}_6$ (the map $\overline{\sigma}$ is uninteresting for inner automorphisms since $W \cong W^\sigma$). Indexing the irreducible representations of $\mathfrak{S}_6$ by partitions in the usual way, it can easily be calculated that the action of $\overline{\sigma}$ sends $V_{(5,1)} \leftrightarrow V_{(2,2,2)}$. One can calculate that

$$K(V_{(5,1)}) \cong K(V_{(2,2,2)}) \cong (\mathbb{Z}/6\mathbb{Z})^2 \oplus \mathbb{Z}/120\mathbb{Z}.$$

In the case $\sigma : H \hookrightarrow G$, one might have hoped that, in analogy with the known surjective homomorphism between critical groups of graphs induced by a graph covering map, the map $\overline{\sigma} : [W] \mapsto [\mathrm{Res}^G_H W]$ would be surjective on critical groups; the following example shows that this is not the case for general groups $H \subset G$.

Example 3.3. Let $G = D_5$ be the dihedral group of order 10, and let V be the direct sum of a two-dimensional irreducible and the non-trivial one-dimensional irreducible. This is the complexification of the action of G in $\mathbb{R}^3$ by rotation of a fixed plane and reflection across that plane. One can calculate (see [4, Appendix C]) that $K(V) \cong \mathbb{Z}/2\mathbb{Z}$. Letting $H = C_5$ be the cyclic subgroup, however, one can show that $K(\mathrm{Res}^G_H V) \cong \mathbb{Z}/5\mathbb{Z}$. Thus $\overline{\mathrm{Res}} : K(V) \to K(\mathrm{Res}^G_H V)$ cannot be surjective. This is a natural counterexample to pick, since C_5 has more conjugacy classes than D_5, and so $\mathrm{Res} : R(D_5) \to R(C_5)$ cannot be surjective.

There are two classes of groups for which $\overline{\mathrm{Res}}$ can be seen to be surjective for all V, both of which will be investigated further throughout the paper.

Proposition 3.4. The map $\overline{\mathrm{Res}} : K(V) \to K(\mathrm{Res}^G_H V)$ is surjective if:

(i) G is abelian,

(ii) $G = A \wr \mathfrak{S}_n$ and $H = A \wr \mathfrak{S}_m$ for A an abelian group and $m \leq n$.

3.1 Cayley graph covering maps

In this section we investigate the relationship between critical groups of group representations and critical groups of graphs when G is abelian.

For any finite group G, we let $\widehat{G} = \mathrm{Hom}(G, \mathbb{C})$ denote the Pontryagin dual group. When G is abelian, all irreducible representations are 1-dimensional, and so $\widehat{G}$ is equal to the group of irreducible characters of G under point-wise multiplication. If V is a faithful representation of an abelian group G, then the multiset $\mathcal{S}_V$ of characters of irreducible components appearing in V generates $\widehat{G}$ as a group. This follows from the standard fact that all irreducible representations of a finite group appear as factors in a sufficiently large tensor power of a fixed faithful representation.

If G is a group with generating multiset $\mathcal{S}$, the *Cayley graph* $\mathrm{Cay}(G, \mathcal{S})$ is the directed multigraph with vertex set G and directed edges $g \to gx$ whenever $x \in \mathcal{S}$.

Theorem 3.5. For V a faithful representation of an abelian group G the critical groups $K(V)$ and $K(\mathrm{Cay}(\widehat{G}, \mathcal{S}_V))$ can be naturally identified, and the diagram

$$\begin{array}{ccc} K(V) & = & K(\mathrm{Cay}(\widehat{G}, \mathcal{S}_V)) \\ \downarrow \overline{\mathrm{Res}} & & \downarrow \overline{\varphi} \\ K(\mathrm{Res}^G_H V) & = & K(\mathrm{Cay}(\widehat{H}, \mathcal{S}_{\mathrm{Res} V})) \end{array}$$

commutes, where $\overline{\varphi}$ is the surjection on critical groups induced by the natural graph covering map $\varphi : \mathrm{Cay}(\widehat{G}, \mathcal{S}_V) \to \mathrm{Cay}(\widehat{H}, \mathcal{S}_{\mathrm{Res} V})$.

Remark 3.6. For graph covering maps $\varphi : \Gamma \to \Gamma'$, Reiner and Tseng [11] give an interpretation of the kernel of $\overline{\varphi} : K(\Gamma) \twoheadrightarrow K(\Gamma')$ as a certain "voltage graph critical group". Thus the identification in Theorem 3.5 allows one to describe the kernel of $\overline{\mathrm{Res}}$ in these same terms in the abelian group case.

4 Critical groups and differential posets

By a *word* of length $2k$, we mean a sequence $w = w_1 \ldots w_{2k}$ of U's and D's. A word w is *balanced* if the number of U's is equal to the number of D's. When a tower of groups $G_0 \subset G_1 \subset \cdots$ is clear from context, we let $w(\mathrm{Ind}, \mathrm{Res})$ denote the linear operator $\bigoplus_i R(G_i) \to \bigoplus_i R(G_i)$ defined by replacing the U's in w with Ind and the D's with Res and viewing the resulting sequence as a composition of linear operators. We always assume that induction and restriction are between consecutive groups in the sequence and that $\mathrm{Res}[V] = 0$ for $[V] \in R(G_0)$. Similarly, if P is a differential poset, then we let the linear map $w(U, D) : \bigoplus_i \mathbb{Z}^{P_i} \to \bigoplus_i \mathbb{Z}^{P_i}$ be defined as the natural composition of linear operators. When w is balanced, then for each i, $w(\mathrm{Ind}, \mathrm{Res})$ (resp. $w(U, D)$) restricts to a map $R(G_i) \to R(G_i)$ (resp. $\mathbb{Z}^{P_i} \to \mathbb{Z}^{P_i}$) which we denote by $w(\mathrm{Ind}, \mathrm{Res})_i$ (resp. $w(U, D)_i$).

Example 4.1. Let $\mathfrak{S}$ be the tower of symmetric groups, and $Y = P(\mathfrak{S})$ denote Young's lattice. Fix $i \geq 1$, then $w(U, D) = UD$ is a linear map $\mathbb{Z}^{Y_i} \to \mathbb{Z}^{Y_i}$ and $w(\mathrm{Ind}, \mathrm{Res})$ is a linear map $R(\mathfrak{S}_i) \to R(\mathfrak{S}_i)$ sending $[W] \mapsto [\mathrm{Ind}^{\mathfrak{S}_i}_{\mathfrak{S}_{i-1}} \mathrm{Res}^{\mathfrak{S}_i}_{\mathfrak{S}_{i-1}} W]$. It is easy to see that $w(\mathrm{Ind}, \mathrm{Res})[\mathbb{1}_{\mathfrak{S}_i}] = [\mathrm{Ind}^{\mathfrak{S}_i}_{\mathfrak{S}_{i-1}} \mathbb{1}_{\mathfrak{S}_{i-1}}]$ is the class of the permutation representation of $\mathfrak{S}_i$. If we identify $\mathbb{Z}^{Y_i}$ with $R(\mathfrak{S}_i)$ via the differential tower of group structure, then one can check that $w(\mathrm{Ind}, \mathrm{Res})[\mathbb{1}_{\mathfrak{S}_i}] \cdot (-)$ and $w(U, D)$ in fact agree as linear maps. This fact is generalized in Proposition 4.2 below.

When $f = \sum_i c_i w^{(i)}$ is a finite nonnegative sum of balanced words, and a differential tower of groups $\mathfrak{G}$ is understood, write $V(f)_n$ for the representation of G_n given by:

$$f(\mathrm{Ind}, \mathrm{Res})[\mathbb{1}_{G_n}] = \sum_i c_i w^{(i)}(\mathrm{Ind}, \mathrm{Res})[\mathbb{1}_{G_n}]$$

For example, if f consists of the single word $U^k D^k$, and we are working in the tower $\mathfrak{S}$ of symmetric groups, then $V(f)_n = \mathrm{Ind}_{\mathfrak{S}_{n-k}}^{\mathfrak{S}_n} \mathbb{1}$ is a basic object of study in the representation theory of the symmetric group. Under the standard characteristic map $\mathrm{ch} : R(\mathfrak{S}_n) \xrightarrow{\sim} \Lambda_n$ between the representation ring of $\mathfrak{S}_n$ and the ring of degree-n symmetric functions, this representation is sent to the complete homogeneous symmetric function $h_{(n-k,1^k)}$ indexed by a "hook shape".

Proposition 4.2. Let $\mathfrak{G} : G_0 \subset G_1 \subset \cdots$ be an r-differential tower of groups with corresponding differential poset $P = P(\mathfrak{G})$, let f be a finite nonnegative sum of balanced words. Then, identifying $\mathbb{Z}^{P_n}$ and $R(G_n)$, the maps $f(U, D)_n$ and $[V(f)_n] \cdot (-)$ are equal. Furthermore the character values of $V(f)_n$ are equal to the eigenvalues of $f(U, D)_n$.

4.1 The generalized permutation representation

The representation $\mathrm{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \mathbb{1}$ of the symmetric group $\mathfrak{S}_n$ is easily seen to be isomorphic to the n-dimensional permutation representation, where $\mathfrak{S}_n$ acts by permuting coordinates in $\mathbb{C}^n$. In [3], the second author was able to explicitly compute the critical group for this representation, generalizing Example 2.3 to arbitrary n. Here we extend that result to a broader class of differential towers of groups:

Theorem 4.3. Let $\mathfrak{G} = G_0 \subset G_1 \subset \cdots$ be an r-differential tower of groups such that the associated differential poset $P = P(\mathfrak{G})$ satisfies Conjecture 2.10 (such as $\mathfrak{G} = A \wr \mathfrak{S}$ with A abelian of order r). Let $V = V(UD)_n = \mathrm{Ind}_{G_{n-1}}^{G_n} \mathbb{1}_{G_{n-1}}$. Then

$$K(V) = \bigoplus_{i=2}^{p_n} \mathbb{Z}/q_i\mathbb{Z}$$

where

$$q_i = \prod_{\substack{1 \le j \le n \\ \Delta p_j \ge i}} rj.$$

Remark 4.4. In Corollary 2.8 of [8], Miller showed that for any differential poset the largest Smith factor of UD agrees with the form predicted by Conjecture 2.10. We can use this to prove that the largest factor in the critical group is given by q_2, without assuming that $P(\mathfrak{G})$ satisfies this conjecture.

4.2 The structure of $K(V(f))$

In this section we investigate the order and subgroup structure of $K(V(f)_n)$ for general finite sums of balanced words f. Although exact formulas for the critical group, like that given in Theorem 4.3 for the case $f(U,D) = UD$ remain elusive in general, the results below considerably restrict the structure of $K(V(f)_n)$.

The following proposition of Stanley characterizes eigenspaces for sums of balanced words in a differential poset:

Proposition 4.5 ([13, Proposition 4.12])**.** Let P be an r-differential poset and let $f(U,D)$ be a finite sum of balanced words. Uniquely write $f(U,D) = \sum_{j\geq 0} \beta_j (UD)^j$ and define $\alpha_i = \sum_{j\geq 0} \beta_j (ri)^j$. Then the characteristic polynomial of $f(U,D)_n : \mathbb{Z}^{P_n} \to \mathbb{Z}^{P_n}$ is given by $\mathrm{ch} f(U,D)_n = \prod_{j=0}^{n} (x - \alpha_i)^{\Delta p_{n-i}}$.

Proposition 4.5 allows us to characterize the order and subgroup structure of critical groups $K(V(f)_n)$. Since it is clear from the definition that $K(V \oplus \mathbb{1}) = K(V)$ for all representations V, we are free to assume in Proposition 4.5 that $\beta_0 = 0$, and we use this convention in what follows.

Proposition 4.6. Let f be a nonnegative finite sum of balanced words, and maintain the notation of Proposition 4.5. Assume further that $P = P(\mathfrak{G})$ for $\mathfrak{G}$ a differential tower of groups. Then $\dim(V(f)_n) = \alpha_n$, and $V(f)_n$ is a faithful representation.

Theorem 4.7. Let $\mathfrak{G}$ be an r-differential tower of groups and let $f(U,D)$ be a nonnegative finite sum of balanced words. Then, using the notation of Proposition 4.5, we have:

a. The size of the critical group $K(V(f)_n)$ is given by:

$$|K(V(f)_n)| = \frac{1}{r^n \cdot n!} \prod_{i=0}^{n-1} (\alpha_n - \alpha_i)^{\Delta p_{n-i}}.$$

b. For each $i = 1, ..., n-1$, the critical group $K(V(f)_n)$ has a subgroup isomorphic to $(\mathbb{Z}/(\alpha_n - \alpha_i)\mathbb{Z})^{\Delta p_{n-i} - 1}$.

5 Enumeration of factors in critical groups

In what follows, when a differential tower of groups and a rank n are understood, we let $\mathrm{ones}(w)$ denote the number of ones in the Smith normal form of $\widetilde{C}_{V(w)_n}$, where w is a balanced word. Then the number of nontrivial factors in the critical group $K(V(w)_n)$ is $p_n - 1 - \mathrm{ones}(w)$, since $\widetilde{C}$ is a $p_n \times p_n$-matrix and there is always a unique zero in the Smith form, by Definition-Proposition 2.1.

Theorem 5.1. Let w be any balanced word of length $2k \leq 2n$. Consider the r-differential tower of groups $A \wr \mathfrak{S}$, where A is abelian group of order $r \geq 2$, and the corresponding differential poset Y^r. Then

$$\mathrm{ones}(w) = |(Y^r)_{n-k}| = \sum_{\substack{i_1+\cdots+i_r=n-k \\ i_1,\ldots,i_r \geq 0}} \prod_{j=1}^{r} p(i_j).$$

In particular, $\mathrm{ones}(w)$ depends only on r and $n-k$, and not on the particular w chosen.

Example 5.2. This example shows that the hypothesis $r \geq 2$ in Theorem 5.1 is necessary. Let $w = (UD)^2$ and work in the tower $\mathfrak{S}$ of symmetric groups. Then for $n = 7$ one can calculate that $\mathrm{ones}(w) = 9 \neq p(7-2) = 7$.

We can still give some upper and lower bounds in the $r = 1$ case. For a balanced word w of length $2k$, write

$$w(U, D) = \sum_{i=0}^{k} c_i U^i D^i \tag{5.1}$$

Then define $\ell(w) = \min\{i | c_i \neq 0\}$; clearly $0 \leq \ell(w) \leq k$, with equality on the right if and only if $w = U^k D^k$.

Proposition 5.3. Let w be a balanced word of length $2k \leq 2n$. Then, working in the tower $\mathfrak{S}$ of symmetric groups, we have $p(n-k) \leq \mathrm{ones}(w) \leq p(n - \ell(w))$.

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