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# Schur polynomials, entrywise positivity preservers, and weak majorization

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**Abstract.** We prove a monotonicity phenomenon for ratios of Schur polynomials. In this we are motivated by – and apply our result to – understanding polynomials and power series that preserve positive semidefiniteness (psd) when applied entrywise to psd matrices. We then extend these results to classify polynomial preservers of total positivity. As a further application, we extend a conjecture of Cuttler, Greene, and Skandera (2011) to obtain a novel characterization of weak majorization using Schur polynomials. Our proofs proceed through a Schur positivity result of Lam, Postnikov, and Pylyavskyy (2007), and computing the leading terms of Schur polynomials.

**Keywords:** Schur polynomial, Cauchy–Binet formula, Schur positivity, positive semidefinite matrix, totally positive matrix, weak majorization

*Notation:* Given integers $1 \leqslant k \leqslant N$ and a domain $I \subset \mathbb{R}$, let $\mathbb{P}_N^k(I)$ denote the positive semidefinite Hermitian $N \times N$ matrices $A = (a_{jk})$, with all entries $a_{jk}$ in $I$ and rank at most $k$; and let $\mathbb{P}_N(I) := \mathbb{P}_N^N(I)$. We will mostly be concerned with the case $I \subset [0, \infty)$.

A function $f : I \to \mathbb{R}$ acts *entrywise* on matrices in $\mathbb{P}_N(I)$ via: $f[A] := (f(a_{jk}))_{j,k=1}^N$. We say $f$ is *entrywise positivity preserving*[1] if $f[A]$ is positive semidefinite whenever $A$ is. The present work is motivated by the classical question of classifying the entrywise positivity preserving functions; this question has been studied for over a century.

By the Schur product theorem [16], and the fact that $\mathbb{P}_N(\mathbb{R})$ is a convex closed cone, it follows that if $f : [0, \rho) \to \mathbb{R}$ is of the form $f(x) = \sum_{k \geqslant 0} c_k x^k$, and $f$ is *absolutely monotonic* – i.e., $c_k \geqslant 0 \; \forall k$ – then $f$ is entrywise positivity preserving on $\mathbb{P}_N$ for all $N$. The converse was proved for continuous functions by Schur's student, Schoenberg:

**Theorem 0.1** (Schoenberg, [15]). *Suppose $f : (-1, 1) \to \mathbb{R}$ is continuous and $f[-]$ preserves positivity on $\mathbb{P}_N((-1, 1)$. Then $f$ is absolutely monotonic.*

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[1]We work with the notions of positivity (i.e., positive semidefiniteness), total positivity, and Schur positivity in this note.

Schoenberg's theorem has subsequently been generalized to other settings, including $(-\rho, \rho), [0, \rho)$, and $(0, \rho)$ for $0 < \rho \leqslant \infty$, and the robustness of the absolute monotonicity condition is valid in each of these settings. We mention the noteworthy sequel by Rudin [14], who strengthened the result by showing that (i) the continuity hypothesis can be removed, and (ii) one only needs to preserve positivity on Toeplitz matrices of all dimensions. Rudin was motivated by his work with Kahane (and Helson and Katznelson) on preservers of Fourier–Stieltjes sequences for positive measures on the torus.

**Remark 0.2.** *In a parallel vein to Rudin's work, the recent work [3] studied preservers of moment sequences for positive measures on the line. As it shows, Theorem 0.1 holds upon (i) preserving positivity on Hankel matrices of all dimensions, and (ii) without the continuity hypothesis.*

A natural and challenging mathematical refinement of the above problem is to classify entrywise positivity preservers in *fixed* dimension. This problem has also attained modern relevance owing to its connections to high-dimensional covariance estimation; for more details see the discussion and references in [2]. While it has been the subject of significant research, and (relatively straightforward) characterizations are known for $N = 2$, the problem remains open for all $N \geqslant 3$.

# 1 Polynomials preserving positive semidefiniteness

In this note we focus on the case of polynomials and power series $f(x) = \sum_{k \geqslant 0} c_k x^k$ that preserve positivity on $\mathbb{P}_N(I)$ for fixed $N$. As we reveal, the study of positivity and its preservation by such functions has remarkable connections to type $A$ representation theory: Schur polynomials, Schur positivity, Gelfand–Tsetlin patterns, and the Harish-Chandra–Itzykson–Zuber formula. For full proofs of the results below, we refer the reader to the paper [10], of which this note is an extended abstract.

As mentioned above, a full classification of the entrywise endomorphisms of $\mathbb{P}_N$ for fixed $N$ remains elusive to date. Essentially the only known necessary condition in fixed dimension is due to Horn, who in his thesis [8] ascribes the result to his advisor, Loewner. The result states that if $I = (0, \infty)$ and $f$ is entrywise positivity preserving on $\mathbb{P}_N(I)$ for $N \geqslant 3$, then $f \in C^{N-3}(I)$ and $f$ has non-negative derivatives on $I$ of orders $0, \dots, N-3$. In a similar vein, for polynomials and power series, a straightforward Taylor series argument shows similar conclusions under weaker hypotheses on the test sets:

**Lemma 1.1** (Horn-type necessary conditions)**.** *Let $N \geqslant 2$ and $0 < \rho \leqslant \infty$, and $f(x) = \sum_{k \geqslant 0} c_k x^k$ is a convergent power series on $(0, \rho)$.*

*(i) (See [2, Lemma 2.4].) If $f$ is entrywise positivity preserving on $\mathbb{P}_N^1((0, \rho))$, and $c_{n_0} < 0$ for some $n_0$, then $c_n > 0$ for at least $N$ values of $n < n_0$. (Thus the first $N$ non-zero Maclaurin coefficients of $f$, if they exist, must be positive.)*

*(ii) Suppose $\rho = \infty$, and $f$ is convergent on $(0,\infty)$ and entrywise positivity preserving on $\mathbb{P}_N^1((0,\infty))$. If $c_{n_0} < 0$ for some $n_0$, then $c_n > 0$ for at least $N$ values of $n < n_0$ and at least $N$ values of $n > n_0$. (Hence if $f$ is a polynomial, then the first $N$ non-zero coefficients and the last $N$ non-zero coefficients of $f$, if they exist, must be positive.)*

The question now arises, if any other coefficient of a preserver $f$ can be negative. (Certainly if no coefficient is negative then $f[-] : \mathbb{P}_n \to \mathbb{P}_n \ \forall n \geqslant 1$ by the Schur product theorem, as explained above.) Until very recently, not a single example was known of a power series which preserved $\mathbb{P}_N((0,\rho))$ entrywise for any $N \geqslant 3$. The only such 'atomic' examples were discovered in recent work [2, 4], for polynomials

$$f(x) = x^r(c_0 + c_1 x + \cdots + c_{N-1}x^{N-1} + c_M x^M)$$

with real coefficients (where $M \geqslant N \geqslant 3$, $r \in \mathbb{Z}^{\geqslant 0}$), which preserve positivity on $\mathbb{P}_N((0,\rho))$ for bounded domains, i.e., $\rho < \infty$.

However, the methods used in [2, 4] provably do not extend to any other set of (non-consecutive) powers; nor to the case of unbounded domains. Consequently, there were no other examples known to date, nor whether such examples can even exist. In particular, apart from Lemma 1.1(i), no other constraint on the coefficients was known.

In this note, we not only address this gap and resolve it completely, but more strongly, settle the question: *What are all possible sign patterns of power series that entrywise preserve positivity on $\mathbb{P}_N((0,\rho))$, for finite $\rho$?*[2] Explicitly, we show:

**Theorem 1.2.** *Fix integers $N > 0$ and $0 \leqslant n_0 < n_1 < \cdots < n_{N-1}$, as well as a sign $\epsilon_M \in \{-1, 0, +1\}$ for each $M > n_{N-1}$. Given positive reals $0 < \rho < \infty$ and $c_{n_0}, \ldots, c_{n_{N-1}}$, there exists a convergent power series on $(0,\rho)$ that is an entrywise positivity preserver on $\mathbb{P}_N((0,\rho))$:*

$$f(x) = c_{n_0}x^{n_0} + c_{n_1}x^{n_1} + \cdots + c_{n_{N-1}}x^{n_{N-1}} + \sum_{M > n_{N-1}} c_M x^M,$$

*such that $c_M$ has the sign of $\epsilon_M$ for every $M > n_{N-1}$.*

Theorem 1.2 shows that the necessary Horn-type condition in Lemma 1.1(i) is sharp.[3] In particular, it demonstrates the existence of polynomials and power series that preserve positivity on $\mathbb{P}_N((0,\rho))$ but not on $\mathbb{P}_{N+1}((0,\rho))$.

To prove Theorem 1.2, we first present a 'fewnomial' version that is equivalent. Namely, suppose we can show the result for exactly one $c_M < 0$ and all other $c_M = 0$ for $M > n_{N-1}$, i.e.,

*There exists $c_M$ with the same sign as $\epsilon_M$ such that*

$$f_M(x) := \sum_{j=0}^{N-1} c_{n_j} x^{n_j} + c_M x^M \text{ entrywise preserves positivity on } \mathbb{P}_N((0,\rho)).$$

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[2] We also answer the question for $\rho = \infty$ in the full paper [10]; here we restrict ourselves to $\rho < \infty$.

[3] Similarly for the unbounded domain case $\rho = \infty$, we show in [10] that the necessary condition in Lemma 1.1(ii) is also sharp, i.e. the only restriction in the possible sign patterns.

Then by suitably choosing each $c_M$ and rescaling $f_M$ by $2^{n_{N-1}-M}$, and adding together the rescaled functions (together with the Schur product theorem), one obtains the desired power series in Theorem 1.2.

Thus, it suffices to prove the above polynomial version. The following result achieves this goal; and moreover, obtains the exact threshold bound for $c_M < 0$.

**Theorem 1.3.** *Fix integers $N > 0$ and $0 \leqslant n_0 < n_1 < \cdots < n_{N-1} < M$, as well as a real scalar $0 < \rho < \infty$. Given real scalars $c_{n_0}, \ldots, c_{n_{N-1}}, c_M$, define $f(x) := \sum_{j=0}^{N-1} c_{n_j} x^{n_j} + c_M x^M$. Then the following are equivalent:*

1. *The function $f$ entrywise preserves positivity on rank-one matrices in $\mathbb{P}_N((0,\rho))$.*
2. *Either all $c_{n_j}, c_M \geqslant 0$; or $c_{n_j} > 0\ \forall j$, and $c_M \geqslant -\mathcal{C}^{-1}$, where*

$$\mathcal{C} = \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n})^2} \frac{\rho^{M-n_j}}{c_{n_j}}. \tag{1.1}$$

*Here $\mathbf{n} := (n_0, \ldots, n_{N-1})^T$, $\mathbf{n}_j := (n_0, \ldots, n_{j-1}, n_{j+1}, \ldots, n_{N-1}, M)^T$, and given a vector $\mathbf{t} = (t_0, \ldots, t_{k-1})^T$, its 'Vandermonde determinant' is $V(\mathbf{t}) := \prod_{1 \leqslant i < j \leqslant k} (t_j - t_i)$.*

3. *The function $f$ entrywise preserves positivity on $\mathbb{P}_N([0,\rho])$.*

Theorem 1.3 provides a quantitative form of Schoenberg's theorem in fixed dimension. As mentioned above, it moreover provides the first examples (for non-consecutive powers $n_j$) of polynomials preserving positivity on $\mathbb{P}_N$ but not on $\mathbb{P}_{N+1}$. Also note that the threshold $\mathcal{C}$ in (1.1) is attained on the boundary of the cone, on rank-one matrices.

We conclude this section by recording the following strengthening of Theorem 1.2. First note that Theorem 1.3 can be reformulated as a linear matrix inequality:

$$A^{\circ M} \preceq \mathcal{C} \sum_{j=0}^{N-1} c_{n_j} A^{\circ n_j}, \qquad \forall A \in \mathbb{P}_N([0,\rho]), \tag{1.2}$$

where $\preceq$ denotes the positive semidefinite ordering: $A \preceq B \Leftrightarrow B - A \in \mathbb{P}_N(\mathbb{R})$. The sharp bound $\mathcal{C}$ is also tight enough, to enable generalizing (1.2) to arbitrary power series:

**Corollary 1.4** (Analytic functions)**.** *Notation as in Theorem 1.3. Given a power series $g(x) = \sum_{M > n_{N-1}} g_M x^M$ that is convergent at $\rho$, there exists a finite threshold $\mathcal{K}$ such that the function $x \mapsto \mathcal{K} \sum_{j=0}^{N-1} c_{n_j} x^{n_j} - g(x)$ is entrywise positivity preserving on $\mathbb{P}_N([0,\rho])$, i.e.,*

$$g[A] \preceq \mathcal{K} \sum_{j=0}^{N-1} c_{n_j} A^{\circ n_j}, \qquad \forall A \in \mathbb{P}_N([0,\rho]). \tag{1.3}$$

The proof can be found in [10, Section 3].

## 2 Schur positivity and ratios of Schur polynomials

This section is devoted to proving Theorem 1.3. The key ingredient in the proof, missing from previous work in the literature, is the use of Schur polynomials; in some sense, the classification of sign patterns starts with the Schur product theorem, and comes back full circle to Schur, via Schur polynomials and Schur positivity.

We now set notation. Fix an integer $N > 0$, and define $\mathbf{n}_{\min}$ to be the vector $(0, 1, \ldots, N-1)^T$. Given a vector $\mathbf{n} = (n_0, \ldots, n_{N-1})^T$ of strictly increasing non-negative integers, define the Schur polynomial $s_{\mathbf{n}}$ in the vector $\mathbf{u} = (u_1, \ldots, u_N)^T$ by the formula

$$s_{\mathbf{n}}(\mathbf{u}) := \sum_T \prod_{j=0}^{N-1} u_{j+1}^{a_j}, \tag{2.1}$$

where $T$ runs over all column-strict Young tableaux of shape $(n_{N-1} - (N-1), \ldots, n_0 - 0)$ and with cell entries $1, \ldots, N$, and $a_j$ is the number of occurrences of $j$ in the tableau $T$. Schur polynomials are homogeneous symmetric polynomials of total degree $\sum_j n_j - \binom{N}{2}$, and are characters of irreducible representations of $\mathfrak{sl}_N(\mathbb{C})$. We now mention two further properties of Schur polynomials that will be of use below: (i) their relation to generalized Vandermonde determinants, and (ii) the 'Weyl Dimension Formula':

$$\det(\mathbf{u}^{\circ n_0} | \cdots | \mathbf{u}^{\circ n_{N-1}}) = s_{\mathbf{n}}(\mathbf{u}) V(\mathbf{u}), \qquad s_{\mathbf{n}}((1, \ldots, 1)^T) = \frac{V(\mathbf{n})}{V(\mathbf{n}_{\min})}. \tag{2.2}$$

As we explain below, a key step in the proof of Theorem 1.3 is the following monotonicity phenomenon for Schur polynomials, which is interesting for additional reasons explained in the final section.

**Proposition 2.1.** *Fix integers $0 \leqslant n_0 < \cdots < n_{N-1}$ and $0 \leqslant m_0 < \cdots < m_{N-1}$, such that $n_j \leqslant m_j\ \forall j$. Define the function $f : (0, \infty)^N \to \mathbb{R}$ via: $f(\mathbf{u}) := \dfrac{s_{\mathbf{m}}(\mathbf{u})}{s_{\mathbf{n}}(\mathbf{u})}$. Then $f$ is non-decreasing in each coordinate.*

We show that this result is in fact the analytical shadow of a deeper, algebraic Schur positivity phenomenon. To show the result, by the quotient rule and symmetry it suffices to show that the polynomial $P_{\mathbf{m},\mathbf{n}}(\mathbf{u}) := s_{\mathbf{n}} \cdot \partial_{u_1}(s_{\mathbf{m}}) - s_{\mathbf{m}} \cdot \partial_{u_1}(s_{\mathbf{n}})$ sends $(0, \infty)^N$ to $[0, \infty)$. This is assured if $P_{\mathbf{m},\mathbf{n}}$ is a $\mathbb{Z}^{\geqslant 0}$-linear combination of monomials, i.e. monomial-positive. Even stronger: note that expanding $s_{\mathbf{n}}$ as a polynomial in $u_1$, the coefficient of $u_1^k$ is a skew-Schur polynomial $s_{\mathbf{n}/(k)}((u_2, \ldots, u_N)^T)$; and similarly for $s_{\mathbf{m}}(\mathbf{u})$. (See [12, Chapter I.5] for details and further properties.) Now we claim:

**Proposition 2.2.** *Writing $P_{\mathbf{m},\mathbf{n}}(\mathbf{u}) := s_{\mathbf{n}} \cdot \partial_{u_1}(s_{\mathbf{m}}) - s_{\mathbf{m}} \cdot \partial_{u_1}(s_{\mathbf{n}})$ as a polynomial in $u_1$, the coefficient of every power of $u_1$ is* Schur positive*, i.e., a non-negative integer linear combination of Schur polynomials in $u_2, \ldots, u_N$.*

Clearly, this result implies Proposition 2.1 by the above discussion.

*Sketch of proof.* Write $\mathbf{u}' := (u_2, \dots, u_N)^T \in \mathbb{R}^{N-1}$ and the vector $(0, \dots, 0, 1)^T \in \mathbb{Z}^N$ as $\mathbf{e}_N$. Then $s_{\mathbf{n}}(\mathbf{u}) = \sum_{j \geqslant 0} s_{(\mathbf{n}-\mathbf{n}_{\min})/j\mathbf{e}_N}(\mathbf{u}')$, where the right-hand summand is understood to vanish whenever $n_{N-1} - (N-1) < j$. Similarly one writes out $s_{\mathbf{m}}(\mathbf{u})$ in terms of skew-Schur polynomials. Now a symmetrization procedure shows that

$$P_{\mathbf{m},\mathbf{n}}(\mathbf{u}) = \sum_{k > j \geqslant 0} \Big( s_{(\mathbf{n}-\mathbf{n}_{\min})/j\mathbf{e}_N}(\mathbf{u}') s_{(\mathbf{m}-\mathbf{n}_{\min})/k\mathbf{e}_N}(\mathbf{u}') - s_{(\mathbf{n}-\mathbf{n}_{\min})/k\mathbf{e}_N}(\mathbf{u}') s_{(\mathbf{m}-\mathbf{n}_{\min})/j\mathbf{e}_N}(\mathbf{u}') \Big) \times (k-j) u_1^{j+k-1}.$$

Thus it suffices to show that each summand is Schur positive when $k > j$ and $\mathbf{m} - \mathbf{n}_{\min}$ dominates $\mathbf{n} - \mathbf{n}_{\min}$ coordinatewise, where $\mathbf{n}_{\min} = (0, \dots, N-1)$ as above. But this is a special case of a Schur positivity result by Lam, Postnikov, and Pylyavskyy [11, Theorem 4]. Notice that if $n_{N-1} - (N-1) < k$ then the 'negative coefficient' summand above vanishes, and then the Schur positivity of $s_{(\mathbf{n}-\mathbf{n}_{\min})/j\mathbf{e}_N}(\mathbf{u}') s_{(\mathbf{m}-\mathbf{n}_{\min})/k\mathbf{e}_N}(\mathbf{u}')$ already follows from the Littlewood–Richardson rule [12, Chapter I, Equations (5.2), (5.3)], since it implies that skew-Schur polynomials are Schur positive, whence so are their products. □

With Proposition 2.1 in hand, we now complete the proof of Theorem 1.3.

**Definition 2.3.** *For $S \subset \mathbb{R}$ a subset, $S^N_<$ comprises all vectors in $S^N$ with pairwise distinct coordinates that are sorted in increasing order.*

*Sketch of proof of Theorem 1.3.* Clearly (3) $\implies$ (1). Next, suppose (1) holds. If $c_{n_j}, c_M$ are not all non-negative, then $c_{n_j} > 0 \ \forall j$ by Lemma 1.1(i). We claim:

**Lemma 2.4.** *Suppose $\mathbb{F}$ is a field, and $h(x) = \sum_{n \in S} c_n x^n \in \mathbb{F}[x]$ is a polynomial, with $S \subset \mathbb{Z}^{\geqslant 0}$ having at least $n$ elements. Then for any $\mathbf{u}, \mathbf{v} \in (\mathbb{F}^N)^T$, we have:*

$$\det h[\mathbf{u}\mathbf{v}^T] = \sum_{\mathbf{n} \in S^N_<} s_{\mathbf{n}}(\mathbf{u}) s_{\mathbf{n}}(\mathbf{v}) V(\mathbf{u}) V(\mathbf{v}) \prod_{n \in \mathbf{n}} c_n. \tag{2.3}$$

*Proof.* Write $S = \{n_1, \dots, n_K\}$ where the $n_j$ are in increasing order. Then,

$$h[\mathbf{u}\mathbf{v}^T] = \sum_{j=1}^K c_{n_j} \mathbf{u}^{\circ \mathbf{n}_j} (\mathbf{v}^{\circ \mathbf{n}_j})^T = (\mathbf{u}^{\circ n_1} | \dots | \mathbf{u}^{\circ n_K}) \operatorname{diag}(c_{n_1}, \dots, c_{n_K}) (\mathbf{v}^{\circ n_1} | \dots | \mathbf{v}^{\circ n_K}).$$

Now (2.3) follows from the Cauchy–Binet formula. □

The next step is to reformulate hypothesis (1), by assuming $c_M < 0$. Set $t := |c_M|^{-1}$ and define

$$p_t(x) := t\, h(x) - x^M, \qquad \text{where} \qquad h(x) := \sum_{j=0}^{N-1} c_{n_j} x^{n_j}. \tag{2.4}$$

Then hypothesis (1) in the theorem is equivalent to assuming that $p_t[\mathbf{u}\mathbf{u}^T]$ is positive semidefinite for $\mathbf{u} \in (0, \sqrt{\rho})^N$, and hypothesis (2) seeks to find the smallest threshold $t$ that works for all such vectors $\mathbf{u}$.

Our approach is to (a) first produce the optimal threshold for a single vector $\mathbf{u}$ (i.e., matrix $\mathbf{u}\mathbf{u}^T$), and then to (b) maximize over a suitable set of vectors $\mathbf{u}$ to obtain the constant in (1.1). The first of these steps will follow from the following basic result.

**Lemma 2.5.** *Fix a vector $\mathbf{w} \in \mathbb{R}^N$ and a positive definite (real symmetric) matrix $H$. Define $P_t := tH - \mathbf{w}\mathbf{w}^T$ for $t \in \mathbb{R}$. Then the following are equivalent:*

1. *$P_t$ is positive semidefinite.*
2. $\det P_t \geqslant 0$.
3. $t \geqslant \mathbf{w}^T H^{-1} \mathbf{w} = 1 - \dfrac{\det(H - \mathbf{w}\mathbf{w}^T)}{\det H}$.

Returning to the proof of $(1) \implies (2)$ in the theorem, define the vectors

$$\mathbf{u}(\epsilon) := (1, \epsilon, \ldots, \epsilon^{N-1})^T, \quad \mathbf{u}_\epsilon := \sqrt{\rho\epsilon}\, \mathbf{u}(\epsilon), \qquad \epsilon \in (0,1). \tag{2.5}$$

Now apply Lemma 2.5 with $H := \sum_{j=0}^{N-1} c_{n_j} (\mathbf{u}_\epsilon \mathbf{u}_\epsilon^T)^{\circ n_j}$ and $\mathbf{w} := \mathbf{u}_\epsilon^{\circ M}$, noting via Vandermonde determinants that $H$ is nonsingular for all $\epsilon \in (0,1)$. Expanding the formula in Lemma 2.5(3) and using (2.2), it follows that $t$ must exceed the quantity

$$1 - \prod_{j=0}^{N-1} c_{n_j}^{-1} V(\mathbf{u}_\epsilon)^{-2} s_{\mathbf{n}}(\mathbf{u}_\epsilon)^{-2} \left( \prod_{j=0}^{N-1} c_{n_j} V(\mathbf{u}_\epsilon)^2 s_{\mathbf{n}}(\mathbf{u}_\epsilon)^2 - \sum_{j=0}^{N-1} \prod_{k \neq j} c_{n_k} V(\mathbf{u}_\epsilon)^2 s_{\mathbf{n}_j}(\mathbf{u}_\epsilon)^2 \right)$$

for every $\epsilon \in (0,1)$. This expression simplifies to yield:

$$t \geqslant \sup_{\epsilon \in (0,1)} \sum_{j=0}^{N-1} \frac{s_{\mathbf{n}_j}(\mathbf{u}_\epsilon)^2}{c_{n_j} s_{\mathbf{n}}(\mathbf{u}_\epsilon)^2}.$$

Applying Proposition 2.1, the supremum is attained as $\epsilon \to 1^-$, and by (2.2) yields precisely the constant $\mathcal{C}$ in (1.1). This proves (2).

Conversely, assuming (2), the proof of (1) proceeds similarly using Lemma 2.5. By continuity and symmetry, it suffices to prove $p_t[\mathbf{u}\mathbf{u}^T] \in \mathbb{P}_N$ for all $\mathbf{u} \in (0, \sqrt{\rho})^N$ with strictly increasing coordinates, if $t \geqslant \mathcal{C}$ as in (1.1). But again by Proposition 2.1, for such $t$ it follows that

$$t \geqslant \sup_{\mathbf{u} \in (0,\sqrt{\rho})^N_<} \sum_{j=0}^{N-1} \frac{s_{\mathbf{n}_j}(\mathbf{u})^2}{c_{n_j} s_{\mathbf{n}}(\mathbf{u})^2}.$$

Repeating (in reverse order) the arguments following (2.5), the assertion (1) follows.

Finally, the proof of $(1) \implies (3)$ uses the following 'extension principle':

**Theorem 2.6.** *Fix $0 < \rho \leqslant \infty$ and a continuously differentiable function $h : (0, \rho) \to \mathbb{R}$. If $h$ and $h'$ are entrywise positivity preserving on $\mathbb{P}_N^1((0, \rho))$ and $\mathbb{P}_{N-1}((0, \rho))$ respectively, then $h$ does the same on all of $\mathbb{P}_N((0, \rho))$.*

We refer the reader to [10, Section 3] for more on this result and how it completes the proof of Theorem 1.3. □

We conclude this section by remarking that the recent works [2, 4] prove Theorem 1.3 in the special case when $\mathbf{n} = \mathbf{n}_{\min}$, i.e., $n_j = j \ \forall 0 \leqslant j < N$. In that case, the hypotheses in the theorem are in fact equivalent to:

(3′) *The entrywise map $f$ preserves positivity on $\mathbb{P}_N(D(0, \rho))$, where $D(0, \rho)$ is the complex disc centered at the origin and of radius $\rho \in (0, \infty)$; and $\mathbb{P}_N$ here denotes complex Hermitian positive semidefinite matrices.*

However, as we discuss in [10, Section 7], such a general statement necessarily does not hold if $\mathbf{n}$ is not an integer translate of $\mathbf{n}_{\min}$. In fact for *every* $\mathbf{n} \neq n_0 + \mathbf{n}_{\min}$, there are infinitely many $M > n_{N-1}$ for which $f[-]$ fails to preserve positivity on $\mathbb{P}_N(D(0, \rho))$. As a simple example, even with negative real entries one cannot have a 'structured' classification of sign patterns as in Lemma 1.1(i). Consider the polynomials

$$p_{k,t}(x) := t(1 + x^2 + \cdots + x^{2k}) - x^{2k+1}, \qquad k \geqslant 0, \ t > 0,$$

acting on $\mathbb{P}_2((-\rho, \rho))$. Setting $\mathbf{u} := (1, -1)^T$ and $A = (\rho/2)\mathbf{u}\mathbf{u}^T \in \mathbb{P}_2((-\rho, \rho))$, one computes: $\mathbf{u}^T p_{k/2}[A]\mathbf{u} = -4(\rho/2)^{2k+1} < 0$. Consequently, $p_{k,t}$ is not entrywise positivity preserving for any $t > 0$ and integer $k \geqslant 1$. Thus, the general problem is provably harder than the one studied in [2, 4], and new methods were required to solve it.

**Remark 2.7.** *If one is merely interested in classifying the sign patterns of positivity preserving power series on $\mathbb{P}_N((0, \rho))$, then the bound in (1.1) is stronger than what is required. As explained in [10, Section 3], a more 'qualitative' approach suffices to show Theorem 1.2. The key result required is a (novel) 'first-order approximation' of every Schur polynomial – see Proposition 4.2, which is used below to prove an extension of the Cuttler–Greene–Skandera conjecture [5].*

**Remark 2.8.** *In fact Theorem 1.3 holds for all **real** powers that lie in $\mathbb{Z}^{\geqslant 0} \cup [N-2, \infty)$ (for the reasons behind this set of powers, see [7]). One proof involves first extending Proposition 2.1 to real powers, i.e., using generalized Vandermonde determinants. A second, 'qualitative' proof involves obtaining 'first-order approximations' for such determinants (generalizing the ones mentioned in the previous remark) using Gelfand–Tsetlin polytopes and the Harish-Chandra–Itzykson–Zuber integral. For details, see [10, Sections 5, 8].*

# 3 Application 1: Polynomials preserving total positivity

A rectangular real matrix is *totally positive* [9] if every minor is non-negative. Such matrices appear in representation theory, discrete mathematics, stochastic processes, and other areas; for instance, generalized Vandermonde matrices are (strictly) totally positive.

It was recently shown in [3] that, in the spirit of Schoenberg and Rudin's theorems, an entrywise map $f : [0, \infty) \to \mathbb{R}$ preserves total positivity on *Hankel* matrices of all sizes, if and only if $f|_{(0,\infty)}$ is absolutely monotonic and $0 \leqslant f(0) \leqslant \lim_{\epsilon \to 0^+} f(\epsilon)$. Thus, totally positive Hankel matrices serve as a 'well-behaved' test set. (In fact, the Schur product theorem also holds for this class of matrices.) In contrast, if we work with the larger set of all symmetric (equivalently, positive semidefinite) totally positive matrices, then preservers $f$ of total positivity on this set are necessarily constant or linear; and this holds even if we restrict to just the $5 \times 5$ symmetric matrices in this set and $f$ analytic.

In light of these remarks, we work only with the Hankel totally positive matrices with entries in an interval $I = (0, \rho)$ or $[0, \rho]$. Denoting such sets by $HTN_N(I)$, we have:

**Theorem 3.1.** *Notation as in Theorem 1.3. The following are equivalent.*

1. *The entrywise map $f[-]$ preserves total positivity on rank-one matrices in $HTN_N((0, \rho))$.*
2. *The entrywise map $f[-]$ preserves positivity on rank-one matrices in $HTN_N((0, \rho))$.*
3. *Either all $c_{n_j}, c' \geqslant 0$; or $c_{n_j} > 0 \ \forall j$, and $c' \geqslant -\mathcal{C}^{-1}$, where $\mathcal{C} = \displaystyle\sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n})^2} \frac{\rho^{M-n_j}}{c_{n_j}}$.*
4. *The entrywise map $f[-]$ preserves total positivity on $HTN_N([0, \rho])$.*

Thus, preserving total positivity on rank-one Hankel matrices in $\mathbb{P}_N((0, \rho))$ is the same as preserving positivity on this test set, and also equivalent to the hypotheses in Theorem 1.3.

To prove the result, we make use of the following connection between positive and totally positive Hankel matrices.

**Lemma 3.2** (see [6, Corollary 3.5])**.** *Let $A_{N \times N}$ be a Hankel matrix. Then $A$ is totally positive if and only if $A$ and its truncation $A^{(1)}$ have non-negative principal minors. Here, $A^{(1)}$ denotes the submatrix of $A$ with the first column and last row removed.*

*Sketch of proof of Theorem 3.1.* Clearly (4) $\implies$ (1) $\implies$ (2). Now observe from the computations following Lemma 2.5 and equation (2.5) that one only needs to work with the matrices $\mathbf{u}_\epsilon \mathbf{u}_\epsilon^T$, and these are all rank-one matrices in $HTN_N((0, \rho))$ by Lemma 3.2. Similarly, Lemma 1.1(i) can be shown working only with $\mathbf{u}_\epsilon \mathbf{u}_\epsilon^T$. Thus, (2) $\implies$ (3).

Finally, if (3) holds, the map $f[-]$ preserves positivity on $\mathbb{P}_N([0, \rho))$ by Theorem 1.3. Now we are done by the following 'extension principle', which can be shown using

Lemma 3.2: *If $0 < \rho \leqslant \infty$ and $f : [0,\rho) \to \mathbb{R}$ entrywise preserves positivity on $\mathbb{P}_N([0,\rho))$, then $f[-]$ preserves total positivity on $HTN_N([0,\rho)) \cap \mathbb{P}_N([0,\rho))$.* □

**Remark 3.3.** *The extension principle in the proof just above, allows one to also classify the sign patterns of all power series that preserve total positivity in fixed dimension, for both bounded and unbounded domains. These results follow from their counterparts for entrywise positivity preservers; see [10, Section 9] for details.*

# 4 Application 2: The Cuttler-Greene-Skandera conjecture, and weak majorization via Schur polynomials

Given a real vector $\mathbf{u} = (u_1, \dots, u_N)^T$, denote its decreasing rearrangement by $u_{[1]} \geqslant \cdots \geqslant u_{[N]}$. We say $\mathbf{u}$ *weakly majorizes* $\mathbf{v}$ for vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$ – and write $\mathbf{u} \succ_w \mathbf{v}$ – if

$$\sum_{j=1}^{k} u_{[j]} \geqslant \sum_{j=1}^{k} v_{[j]}, \ \forall 0 < k < N, \qquad \sum_{j=1}^{N} u_{[j]} \geqslant \sum_{j=1}^{N} v_{[j]}. \tag{4.1}$$

If moreover the final inequality is an equality, we say $\mathbf{u}$ *majorizes* $\mathbf{v}$.

We begin by recalling a conjecture by Cuttler, Greene, and Skandera [5, Conjecture 7.4], which says that given $\mathbf{m}, \mathbf{n} \in (\mathbb{Z}^{\geqslant 0})^N_<$ (see Definition 2.3),

$$\frac{s_{\mathbf{m}}(\mathbf{u})}{s_{\mathbf{n}}(\mathbf{u})} \geqslant \frac{s_{\mathbf{m}}((1,\dots,1)^T)}{s_{\mathbf{n}}((1,\dots,1)^T)}, \qquad \forall \mathbf{u} \in (0,\infty)^N, \tag{4.2}$$

if $\mathbf{m}$ majorizes $\mathbf{n}$. The conjecture was very recently proved in [17] and also in [1].

In a parallel direction, observe that if $\mathbf{m}$ dominates $\mathbf{n}$ coordinatewise, and $\mathbf{m} \neq \mathbf{n}$, then (4.2) cannot hold at points $\epsilon(1,\dots,1)^T$ for $\epsilon \in (0,1)$, by homogeneity considerations. However, (4.2) holds on $[1,\infty)^N$, as an immediate corollary of Proposition 2.1:

$$m_j \geqslant n_j \ \forall j \quad \Longrightarrow \quad \frac{s_{\mathbf{m}}(\mathbf{u})}{s_{\mathbf{n}}(\mathbf{u})} \geqslant \frac{s_{\mathbf{m}}((1,\dots,1)^T)}{s_{\mathbf{n}}((1,\dots,1)^T)} = \frac{V(\mathbf{m})}{V(\mathbf{n})}, \qquad \forall \mathbf{u} \in [1,\infty)^N. \tag{4.3}$$

A common unification of both of these settings is thus a natural question – restricting to $\mathbf{u} \in [1,\infty)^N$ as above. The aforementioned works [1, 5, 17] all assume $\sum_j m_j = \sum_j n_j$; replacing this by an inequality allows us to achieve the desired common generalization, and to show the converse. In fact, this *characterizes* weak majorization, for real tuples:

**Theorem 4.1.** *Given vectors $\mathbf{m}, \mathbf{n} \in (\mathbb{R}^{\geqslant 0})^N_<$, we have*

$$\frac{\det(\mathbf{u}^{\circ m_0} | \dots | \mathbf{u}^{\circ m_{N-1}})}{V(\mathbf{m})} \geqslant \frac{\det(\mathbf{u}^{\circ n_0} | \dots | \mathbf{u}^{\circ n_{N-1}})}{V(\mathbf{n})}, \qquad \forall \mathbf{u} \in [1,\infty)^N \tag{4.4}$$

*if and only if $\mathbf{m}$ weakly majorizes $\mathbf{n}$.*

While the integer tuple case was our main motivation, it is more convenient to work with the more general real tuples in (one half of) the proof. The (other half of the) proof proceeds through a 'first-order approximation' of generalized Vandermonde determinants; here we write down its special case for Schur polynomials.

**Proposition 4.2.** *Fix integers $N > 0$ and $0 \leqslant n_0 < \cdots < n_{N-1}$, and scalars $0 \leqslant u_1 \leqslant \cdots \leqslant u_N$. With $\mathbf{n}, \mathbf{u}$ as in (2.1), we have the following two sharp inequalities:*

$$1 \times \mathbf{u}^{\mathbf{n}-\mathbf{n}_{\min}} \leqslant s_{\mathbf{n}}(\mathbf{u}) \leqslant \frac{V(\mathbf{n})}{V(\mathbf{n}_{\min})} \times \mathbf{u}^{\mathbf{n}-\mathbf{n}_{\min}}. \tag{4.5}$$

*Proof.* By (2.2), $s_{\mathbf{n}}(\mathbf{u})$ is the sum of $\frac{V(\mathbf{n})}{V(\mathbf{n}_{\min})}$ monomials, one of which equals $\mathbf{u}^{\mathbf{n}-\mathbf{n}_{\min}}$, and this dominates all other monomials. The sharpness can be shown by using the principal specialization of the Weyl Character Formula [12, Chapter I.3]. ☐

*Sketch of proof of Theorem 4.1.* Suppose (4.4) holds. Define the partial sums of $\mathbf{m}, \mathbf{n}$:

$$\widetilde{n}_j := n_{N-j} + \cdots + n_{N-1}, \qquad \widetilde{m}_j := m_{N-j} + \cdots + m_{N-1}, \qquad 0 \leqslant j \leqslant N-1.$$

Now fix $j$ and set $\mathbf{u} = \mathbf{u}(t) := (1, \dots, N-j, (N-j+1)t, \dots, Nt)$ for $t \in [1, \infty)$. Using (4.4) and Proposition 4.2, we compute for all $t \geqslant 1$:

$$t^{\widetilde{n}_j} \prod_{k=1}^{N} k^{n_{k-1}} = \mathbf{u}^{\mathbf{n}} \leqslant \mathbf{u}^{\mathbf{n}_{\min}} s_{\mathbf{n}}(\mathbf{u}) \leqslant \mathbf{u}^{\mathbf{n}_{\min}} \frac{V(\mathbf{n})}{V(\mathbf{m})} s_{\mathbf{m}}(\mathbf{u}) \leqslant \frac{V(\mathbf{n})\mathbf{u}^{\mathbf{m}}}{V(\mathbf{n}_{\min})} = t^{\widetilde{m}_j} \frac{V(\mathbf{n})}{V(\mathbf{n}_{\min})} \prod_{k=1}^{N} k^{m_{k-1}}.$$

We infer from taking $t \to \infty$ that the growth rate $\widetilde{m}_j$ of the right-hand side must dominate that on the left, which is $\widetilde{n}_j$. Therefore $\mathbf{m}$ weakly majorizes $\mathbf{n}$.[4]

Conversely, suppose $\mathbf{m}, \mathbf{n}$ are non-negative real vectors with $\mathbf{m} \succ_w \mathbf{n}$. Given $\mathbf{u} \in [1, \infty)^N$, define $F_{\mathbf{u}} : [0, \infty)^N \to \mathbb{R}$ to be the Harish-Chandra–Itzykson–Zuber integral:

$$F_{\mathbf{u}}(\mathbf{m}) := \int_{U(N)} \exp \operatorname{tr} \left( \operatorname{diag}(m_0, \dots, m_{N-1}) U \operatorname{diag}(\log(u_1), \dots, \log(u_N)) U^* \right) \, dU. \tag{4.6}$$

By continuity, it suffices to show (4.4) for $\mathbf{u} \in (1, \infty)^N_<$. If we show $F_{\mathbf{u}}(\mathbf{m}) \geqslant F_{\mathbf{u}}(\mathbf{n})$, then we would be done by the Harish-Chandra–Itzykson–Zuber formula. Now this property of $F_{\mathbf{u}}$ follows from [13, Chapter 3, C.2.d]. See [10, Section 10] for details. ☐

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