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Schur-positivity of Equitable Ribbons

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Abstract. We study the Schur-positivity poset and its conjectured maximal connected elements, which are certain equitable ribbon Schur functions. In particular, we establish sufficient conditions for the difference of two ribbon Schur functions to be Schur-positive, and we deduce necessary conditions for the difference of two equitable ribbon Schur functions to be Schur-positive. We use this to confirm conjectures on maximal and minimal equitable ribbon Schur functions for many cases, including all chains.

Résumé. Nous étudions l'ensemble ordonné de Schur-positivité et ses éléments connexes maximaux, qui sont conjecturés comme étant certaines fonctions de Schur de ruban équitables. En particulier, nous établissons des conditions suffisantes pour que la différence de deux fonctions de Schur de ruban soit Schur-positive et nous déduisons aussi des conditions nécessaires pour que la différence de deux fonctions de Schur de ruban équitables soit Schur-positive. Nous utilisons ceci pour confirmer des conjectures portant sur les fonctions de Schur de ruban équitables maximales et minimales, dans de nombreux cas et en particulier pour les chaines.

Keywords: maximal element, minimal element, ribbon Schur function, symmetric function, Schur-positive, skew diagram

1 Introduction

Schur functions were introduced in 1901 and since then have been a central topic of research in many areas. Schur functions are characters of irreducible representations of the general linear group and they form the most interesting and important basis for the algebra of symmetric functions. In this algebra, the structure constants for the product of two Schur functions are the Littlewood–Richardson coefficients, which count particular combinatorial objects and thus are nonnegative integers. Moreover, they appear in the cup product expansion of Schubert classes in the cohomology ring of the Grassmannian and they are intimately related to eigenvalues of Hermitian matrices.

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Littlewood–Richardson coefficients also arise in the expansion of skew Schur functions, which are a generalization of Schur functions that are indexed by diagrams known as skew diagrams. Because this is a nonnegative linear combination, we say that skew Schur functions are Schur-positive and, returning to representation theory, consequently arise as the Frobenius image of the corresponding direct sum of irreducible representations of the symmetric group.

In this extended abstract, we consider the question of when the difference of two skew Schur functions is Schur-positive. It transpires that the subclass of connected skew diagrams known as equitable ribbons is central to this study.

In Section 3 we establish two ribbon Schur function inequalities in Theorem 3.2 and in Theorem 3.5, and we also explicitly determine cases where the Schur-positivity partially ordered set of equitable ribbon Schur functions is a chain in Theorem 3.8. In Section 4 we determine powerful necessary conditions for order relations in Theorem 4.2 and Theorem 4.10. We also identify families of incomparable elements in Corollary 4.14, thereby verifying that the chains identified in Theorem 3.8 are in fact the *only* chains, giving a characterization in this case. Finally, we identify a minimal element of the Schur-positivity poset of equitable ribbon Schur functions in Theorem 4.15.

2 Background

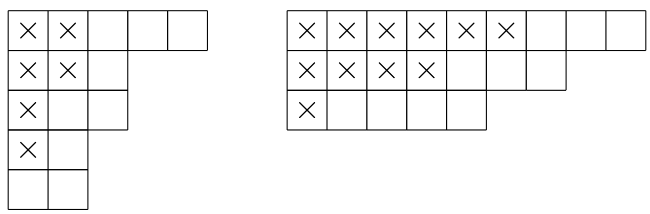
Let N be a nonnegative integer. A tuple of positive integers $\alpha = \alpha_1 \cdots \alpha_R$ is a *composition* of N if $\alpha_1 + \cdots + \alpha_R = N$. The *reverse* of α is the composition $\alpha^* = \alpha_R \cdots \alpha_1$ formed by reading the entries in reverse order. A composition α is a *partition* if $\alpha_1 \geq \cdots \geq \alpha_R$. Every composition α determines a unique partition $\lambda(\alpha)$ given by reordering its entries in weakly decreasing order. We denote by $\emptyset$ the unique composition of 0.

The *Young diagram* associated to a partition λ is the left-justified array of cells with λ_i cells in the i-th row. We use the English notation, in which rows are counted from the top. Given two partitions $\lambda = \lambda_1 \cdots \lambda_R$ and $\mu = \mu_1 \cdots \mu_S$ that satisfy $\lambda_i \geq \mu_i$ for every $1 \leq i \leq S \leq R$, their associated *skew diagram* λ/μ is the array of cells in the Young diagram of λ but not in that of μ. We call the number of cells in a row (respectively column) its *row length* (respectively *column length*). The skew diagram λ/μ is called a *ribbon* if it is connected with no 2×2 block; in other words, the adjacent rows overlap in exactly one column. In this case, we associate the ribbon with a composition by reading the row lengths from the top. We will often denote a ribbon by its associated composition.

A ribbon is *row-equitable* if its row lengths take on at most two different values and those values are consecutive. A ribbon is *column-equitable* if its column lengths take on at most two different values and those values are consecutive. A ribbon is *equitable* if it is both row- and column-equitable.

Example 2.1. The tuple $\alpha = 22335$ is a composition of 15 and determines the partition $\lambda = \lambda(\alpha) = 53322 = \alpha^*$, whose Young diagram is to the left below. The marked cells correspond to $\mu = 2211$ and the remaining cells form the skew diagram λ/μ, which is the ribbon 31212. This ribbon is not row-equitable because the row lengths take on the three values 1, 2, and 3. It is not column-equitable because the column lengths take on the values 1 and 3, which are not consecutive.

The Young diagram of the partition $\sigma = 975$ is to the right below. The marked cells form the Young diagram of $\tau = 641$ and the remaining cells form the skew diagram σ/τ, which is the ribbon 334. This ribbon is equitable because the row lengths take on the two consecutive values 3 and 4 and the column lengths take on the two consecutive values 1 and 2.



A *semistandard Young tableau* (SSYT) of *shape* λ/μ is a filling

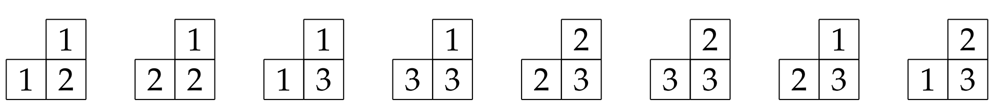
$$T : \lambda/\mu \to \mathbb{N} = \{1, 2, \ldots\}$$

of the cells of the skew diagram λ/μ so that the rows are *weakly increasing* (nondecreasing) when read from left to right and the columns are *strictly increasing* when read from top to bottom. To an SSYT T we associate the monomial x^T in countably infinitely many commuting variables $x_1, x_2, \ldots$ where the exponent of x_i is the number of cells containing i. The *skew Schur function* $s_{\lambda/\mu}$ is then

$$s_{\lambda/\mu} = \sum_{T \text{ an SSYT of shape } \lambda/\mu} x^T.$$

When $\mu = \emptyset$, so that $\lambda/\emptyset = \lambda$, we call $s_{\lambda/\mu}$ a *Schur function* and denote it by s_λ. When λ/μ is a ribbon α, we call $s_{\lambda/\mu}$ a *ribbon Schur function* and denote it by r_α. Two skew diagrams are said to be *equivalent* if their corresponding skew Schur functions are equal; in what follows, we identify a skew diagram with its equivalence class.

Example 2.2. Below are the semistandard Young tableaux of shape 22/1 that are filled with the integers at most 3 and the corresponding monomials in the skew Schur function $s_{22/1}$.



$$s_{22/1} = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + 2x_1x_2x_3 + \cdots$$

The Schur functions form a basis for the algebra of symmetric functions, and since skew Schur functions also belong to this algebra, a skew Schur function $s_{\lambda/\mu}$ expands uniquely as

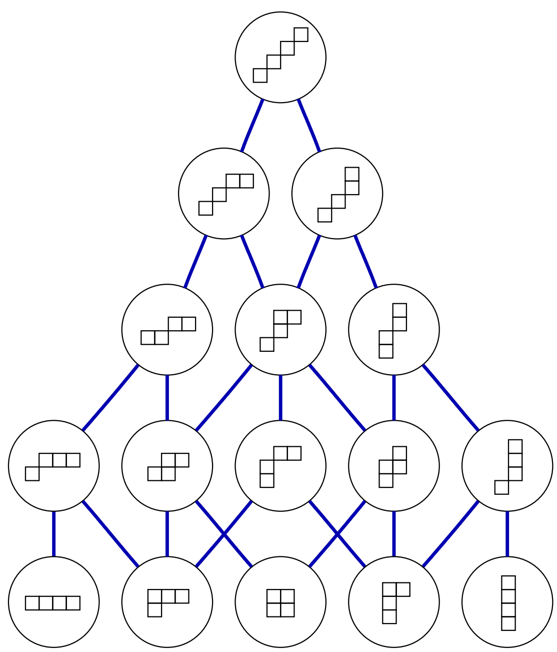
$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu,\nu} s_{\nu}.$$

The *Littlewood–Richardson coefficients* $c^{\lambda}_{\mu,\nu}$ arising in this expansion are nonnegative integers, and so this expression is called *Schur-positive*: a nonnegative linear combination of Schur functions. Given two skew diagrams λ/μ and σ/τ, we write that $\lambda/\mu \leq_s \sigma/\tau$ if the difference

$$s_{\sigma/\tau} - s_{\lambda/\mu}$$

is Schur-positive. The relation $\leq_s$ is a partial order on equivalence classes of skew diagrams, and we denote by $\mathcal{P}_N$ the *Schur-positivity poset* of those equivalence classes of skew diagrams with N cells under the order relation $\leq_s$.

Example 2.3. *Below is $\mathcal{P}_4$.*



It transpires that for every N the skew diagram consisting of N disconnected cells is the unique maximal element [2]. Therefore, we turn our attention to addressing the question of which elements are maximal among *connected* skew diagrams and focus our attention further due to the following result.

Proposition 2.4 ([2, Propositions 3.3 and 3.5])**.** For any connected skew diagram λ/μ, there is an equitable ribbon α for which

$$\lambda/\mu \leq_s \alpha.$$

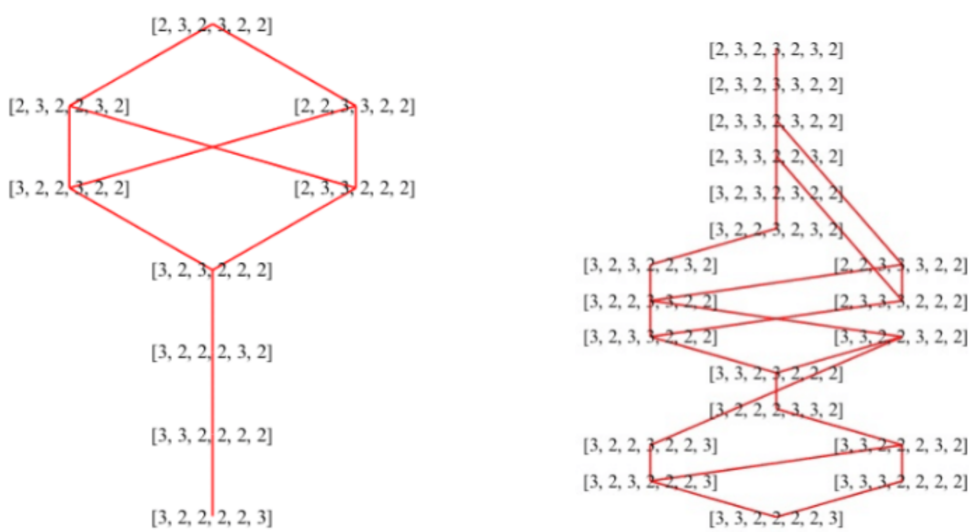
Therefore, a maximal element among connected skew diagrams must be an equitable ribbon.

Two skew diagrams are incomparable if they have a different number of cells. Moreover, two ribbons are incomparable if they have a different number of rows ([2, Lemma 3.8]). So we further narrow our search to equitable ribbons α with a fixed number N of cells and a fixed number of rows; because the row lengths of α are among $\{a, a+1\}$ for some a, this data determines the partition $\lambda(\alpha)$ of row lengths of any such ribbon. So the objects of study are the following.

Definition 2.5. The subposet of $\mathcal{P}_N$ consisting of ribbons with n rows of length $a+1$ and m rows of length a is denoted by $\mathcal{R}((a+1)^n a^m)$.

From here on we will fix N, n, m, and a. Without loss of generality, $m \neq 0$.

Example 2.6. *Below are $\mathcal{R}(3^2 2^4)$ and $\mathcal{R}(3^3 2^4)$.*



Note that if a ribbon has all row lengths at least 2, then the column lengths will all be 1 or 2 so it will automatically be column-equitable. Therefore, $\mathcal{R}((a+1)^n a^m)$ is indeed the subposet of equitable ribbons when $a \geq 2$. However, quite surprisingly, many of our results are independent of a and so we allow $a = 1$ despite such ribbons not being equitable in general.

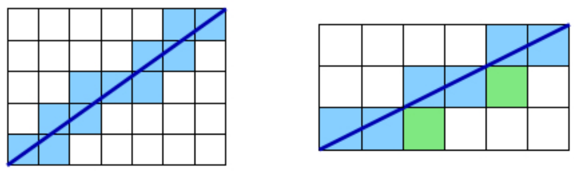
Recall that we identify skew diagrams corresponding to the same skew Schur function. While it remains conjectured when two skew Schur functions are equal, the case for ribbons has been characterized ([1, Theorem 4.1]) and for equitable ribbons we have the following result.

Proposition 2.7. The equivalence class of an equitable ribbon contains only itself and its reverse.

We now present the central conjectures that have guided our work.

Conjecture 2.8 ([2, Conjecture 1.3])**.** The poset $\mathcal{R}((a+1)^n a^m)$ has the unique maximal element given by taking the cells whose interior or upper left corner intersect the diagonal of the $(n+m) \times ((a-1)(n+m)+n+1)$ grid.

Example 2.9. The blue and green cells form the maximal elements of $\mathcal{R}(3^1 2^4)$ and $\mathcal{R}(3^2 2^1)$. The green cells are included because their top left corners intersect the diagonal.



Conjecture 2.10 ([2, Conjecture 5.2])**.** The poset $\mathcal{R}((a+1)^n a^m)$ has the unique minimal element

$$(a+1)^{\lceil \frac{n}{2} \rceil} a^m (a+1)^{\lfloor \frac{n}{2} \rfloor}.$$

One can confirm these conjectures in the partially ordered sets of Example 2.6.

3 Sufficient Conditions for Schur-positivity

We give two ribbon Schur function inequalities, which will help us to identify when $\mathcal{R}((a+1)^n a^m)$ is a chain. These inequalities can be proved by calculating Littlewood–Richardson coefficients.

Definition 3.1. Let $\alpha = \alpha_1 \cdots \alpha_R$ be a composition with $\alpha_1 \geq 2$ and $2 \leq i \leq R$. Define

$$M_i(\alpha) = (\alpha_1 - 1)\alpha_2 \cdots \alpha_{i-1}(\alpha_i + 1)\alpha_{i+1} \cdots \alpha_R;$$

that is, $M_i(\alpha)$ is given by decrementing the first part and incrementing the i-th part of α.

Theorem 3.2. Let $\alpha = \alpha_1 \cdots \alpha_R$ be a composition with $\alpha_1 \geq 2$, $2 \leq i \leq R$, and $\beta = M_i(\alpha)$. If $\beta_1 \geq \beta_2 + \cdots + \beta_i - i + 1$, then

$$\alpha \leq_s \beta.$$

Example 3.3. Let $\alpha = 7433$ and $i = 3$. Then $\beta = 6443$ and since $6 \geq 4 + 4 - 3 + 1$, we have that $7433 \leq_s 6443$.

Example 3.4. Fix $a \geq 1$ and let $\alpha = (a+1)a(a+1)a$ and $i = 2$. Then $\beta = a(a+1)(a+1)a$ and since $a \geq (a+1) - 2 + 1$, we have that $\alpha \leq_s \beta$.

Theorem 3.5. Let $\alpha = \alpha_1 \cdots \alpha_R$ be a composition with $\alpha_1 \geq 3$, $2 \leq i < j \leq R$, and $\beta = M_i(M_j(\alpha))$. If $\beta_1 \geq \beta_2 + \cdots + \beta_j - j + 1$, then

$$r_\beta - r_{M_i(\alpha)} - r_{M_j(\alpha)} + r_\alpha \text{ is Schur-positive.}$$

Example 3.6. Let $\alpha = 8333$, $i = 2$, and $j = 3$. Then $\beta = 6443$ and since $6 \geq 4 + 4 - 3 + 1$, we have that $r_{6443} - r_{7343} - r_{7433} + r_{8333}$ is Schur-positive.

Remark 3.7. Theorem 3.5 further generalizes to a Schur-positive inclusion-exclusion-type sum of ribbon Schur functions in which multiple rows have been affected. However, the formulation in Theorem 3.5 was sufficient for the calculations we performed to determine the following order relations.

Theorem 3.8. The partially ordered set $\mathcal{R}((a+1)^n a^m)$ is a chain in the following cases.

$$\begin{aligned}
\mathcal{R}((a+1)^1a^m): \quad & aa\cdots aa(a+1) \leq_s aa\cdots a(a+1)a \leq_s aa\cdots(a+1)aa \leq_s \cdots \\
& \leq_s a^{\lceil \frac{m}{2} \rceil}(a+1)a^{\lfloor \frac{m}{2} \rfloor} \\
\mathcal{R}((a+1)^na^1): \quad & (a+1)^{\lceil \frac{n}{2} \rceil}a(a+1)^{\lfloor \frac{n}{2} \rfloor} \leq_s \cdots \leq_s (a+1)(a+1)\cdots a(a+1)(a+1) \\
& \leq_s (a+1)(a+1)\cdots(a+1)a(a+1) \\
& \leq_s (a+1)(a+1)\cdots(a+1)(a+1)a \\
\mathcal{R}((a+1)^2a^2): \quad & (a+1)aa(a+1) \leq_s (a+1)(a+1)aa \leq_s (a+1)a(a+1)a \\
& \leq_s a(a+1)(a+1)a \\
\mathcal{R}((a+1)^2a^3): \quad & (a+1)aaa(a+1) \leq_s (a+1)(a+1)aaa \leq_s (a+1)aa(a+1)a \\
& \leq_s (a+1)a(a+1)aa \leq_s a(a+1)(a+1)aa \leq_s a(a+1)a(a+1)a
\end{aligned}$$

We will see in Section 4 that these are the *only* cases where $\mathcal{R}((a+1)^n a^m)$ is a chain. One can verify that Conjectures 2.8 and 2.10 hold for the partially ordered sets above and therefore for all cases where $\mathcal{R}((a+1)^n a^m)$ is a chain.

4 Necessary Conditions for Schur-positivity

4.1 Short Ends and Larger Ribbons

Theorem 3.2 suggests that a larger ribbon in $\mathcal{P}_N$ tends to have a shorter top row. Recalling that ribbons are equivalent under reversal, we suspect a larger ribbon also tends to have a shorter bottom row. This behaviour is exhibited in Example 2.6 and Conjecture 2.10. We explicitly formalize this notion in $\mathcal{R}((a+1)^n a^m)$.

Definition 4.1. A ribbon $\alpha = \alpha_1 \cdots \alpha_R \in \mathcal{R}((a+1)^n a^m)$ has *two short ends* if $\alpha_1 = \alpha_R = a$, *one short end* if either $\alpha_1 = a$ and $\alpha_R = a+1$, or $\alpha_1 = a+1$ and $\alpha_R = a$; and *zero short ends* if $\alpha_1 = \alpha_R = a+1$.

We can now state an easy-to-apply necessary condition, whose proof involves analyzing particular Littlewood–Richardson coefficients.

Theorem 4.2. Let $\alpha, \beta \in \mathcal{R}((a+1)^n a^m)$. If $\alpha \geq_s \beta$, then

$$\alpha \text{ has at least as many short ends as } \beta.$$

Example 4.3. Let $\alpha = 4334334343$ and $\beta = 3444333343$. Since α has one short end and β has two short ends, we immediately conclude that $\alpha \not\geq_s \beta$.

Example 4.4. In the following chain, note that the number of short ends is weakly increasing as the ribbons increase.

$$43334 \leq_s 44333 \leq_s 43343 \leq_s 43433 \leq_s 34433 \leq_s 34343$$

Corollary 4.5. The ribbon $a(a+1)\cdots(a+1)a$ is a maximal element of $\mathcal{R}((a+1)^n a^2)$. The ribbon $(a+1)a\cdots a(a+1)$ is a minimal element of $\mathcal{R}((a+1)^2 a^m)$.

This follows because the indicated ribbons are respectively the unique ones with the maximum and minimum numbers of short ends.

As a converse, we may ask whether a ribbon with strictly more short ends must be larger; we will see in Corollary 4.14 that this statement fails in general. However, we do have the following partial converse that is reminiscent of Proposition 2.4.

Proposition 4.6. Suppose that $m \geq 2$. For any $\alpha \in \mathcal{R}((a+1)^n a^m)$, there is a $\beta \in \mathcal{R}((a+1)^n a^m)$ with two short ends for which

$$\alpha \leq_s \beta.$$

Therefore, a maximal element of $\mathcal{R}((a+1)^n a^m)$ must have two short ends.

Note that $\mathcal{R}((a+1)^n a^1)$ is a chain and there the full converse to Theorem 4.2 holds.

4.2 Small Spacing and Larger Ribbons

We now come to our most powerful necessary condition. The conjectured maximal element of $\mathcal{R}((a+1)^n a^{n+1})$ is

$$a(a+1)a(a+1)a\cdots(a+1)a(a+1)a;$$

this suggests that we want to avoid having adjacent rows of length $(a+1)$. Similar considerations suggest minimizing the number of such rows separated by only one a, then those separated by two, and so on. Indeed, we prove all of these thresholds simultaneously. We first prepare some notation that repackages this data.

Definition 4.7. Write a composition $\alpha \in \mathcal{R}((a+1)^n a^m)$ as

$$\alpha = a^{p_1(\alpha)}(a+1)a^{p_2(\alpha)}(a+1)\cdots(a+1)a^{p_{n+1}(\alpha)},$$

where $p_i(\alpha) \geq 0$. Then we define the *profile* of α as $p(\alpha) = p_1(\alpha)\cdots p_{n+1}(\alpha)$ and the *quasi-profile* of α as $q(\alpha) = q_0(\alpha)q_1(\alpha)\cdots$ where $q_j(\alpha) = |\{i : \; p_i(\alpha) = j\}|$; that is, $q_j(\alpha)$ is the number of j's occurring in $p(\alpha)$.

Remark 4.8. As it is defined, the quasi-profile $q(\alpha)$ has an infinite tail of zeroes. We omit them for brevity.

Example 4.9. Let $a = 3$ and

$$\alpha = \; 4\;33\;4\;4\;3\;4\;333\;4\;333\;4\;333.$$

Then $p(\alpha) = 0201333$ and $q(\alpha) = 2113$.

For tuples $\delta = \delta_0\delta_1\cdots$ and $\epsilon = \epsilon_0\epsilon_1\cdots$, we denote by $\delta \leq_{lex} \epsilon$ the *lexicographic order*; that is, either $\delta = \epsilon$ or $\delta_i < \epsilon_i$ at the smallest index i at which they differ.

Theorem 4.10. Let $\alpha, \beta \in \mathcal{R}((a+1)^n a^m))$. If $\alpha \geq_s \beta$, then

$$q(\alpha) \leq_{lex} q(\beta).$$

Informally, a larger ribbon has its long rows more evenly spread out.

The proof involves a number of tools, including coarsenings of integer compositions, the complete homogeneous symmetric function basis, the Jacobi–Trudi determinantal identities, and an extensive enumeration.

Example 4.11. Let

$$\alpha = 33\;4\;3\;4\;3333333\;4\;33 \quad \text{and} \quad \beta = 333\;4\;333\;4\;333\;4\;333.$$

Then $p(\alpha) = 2172$, $p(\beta) = 3333$, $q(\alpha) = 01200001$, and $q(\beta) = 0004$. Since $q(\beta) <_{lex} q(\alpha)$, we immediately conclude that $\alpha \not\geq_s \beta$.

Example 4.12. Returning to Example 4.4,

$$43334 \leq_s 44333 \leq_s 43343 \leq_s 43433 \leq_s 34433 \leq_s 34343,$$

the profiles from left to right are 030, 003, 021, 012, 102, and 111, and so the quasi-profiles are 2001, 2001, 111, 111, 111, and 03, which are indeed weakly decreasing in lexicographic order as the ribbons increase.

Corollary 4.13. Suppose that $m = d(n+1)$ for some integer d. Then the ribbon

$$a^d(a+1)a^d(a+1)\cdots a^d(a+1)a^d$$

is a maximal element of $\mathcal{R}((a+1)^n a^m)$.

This follows from Theorem 4.10 because the indicated ribbon is the unique element of $\mathcal{R}((a+1)^n a^m)$ with lexicographically minimal quasi-profile. This corollary confirms Conjecture 2.8 whenever m is a multiple of $(n+1)$.

Corollary 4.14. For any $k \geq 0$, the ribbons

$$a(a+1)(a+1)aaaa^k \text{ and } (a+1)aa(a+1)aaa^k$$

are incomparable. For any γ consisting of a's and $(a+1)$'s, the ribbons

$$aa(a+1)(a+1)(a+1)\gamma \text{ and } (a+1)a(a+1)a(a+1)\gamma$$

are incomparable. It follows that the chains identified in Theorem 3.8 are a complete classification of all cases in which $\mathcal{R}((a+1)^n a^m)$ is a chain.

In each case, the result follows from applying Theorem 4.2 and Theorem 4.10.

Theorem 4.10, along with extending the techniques of its proof, are instrumental in proving the following direction of Conjecture 2.10.

Theorem 4.15. The ribbon

$$(a+1)^{\lceil \frac{n}{2} \rceil} a^m (a+1)^{\lfloor \frac{n}{2} \rfloor}$$

is a minimal element of $\mathcal{R}((a+1)^n a^m)$.

In further studying $\mathcal{R}((a+1)^n a^m)$, our focus turns to ribbons α and β for which $q(\alpha) = q(\beta)$; that is, $p(\alpha)$ and $p(\beta)$ are permutations of each other. The following result is an analogue of Theorem 4.2 applied to these.

Theorem 4.16. Suppose that $\alpha, \beta \in \mathcal{R}((a+1)^n a^m)$ satisfy $q(\alpha) = q(\beta)$. By equivalence under reversal, assume without loss of generality that $p_1(\alpha) \leq p_{n+1}(\alpha)$ and $p_1(\beta) \leq p_{n+1}(\beta)$. If $\alpha \geq_s \beta$, then

$$p_1(\alpha) \cdot p_{n+1}(\alpha) \geq_{lex} p_1(\beta) \cdot p_{n+1}(\beta).$$

Example 4.17. Let

$$\alpha = 33\,4\,333\,4\,333\,4\,333\,4\,33333 \quad \text{and} \quad \beta = 333\,4\,33\,4\,33333\,4\,333\,4\,333.$$

Then $p(\alpha) = 23335$ and $p(\beta) = 32533$ and indeed $q(\alpha) = q(\beta) = 001301$. Since

$$p_1(\alpha) \cdot p_5(\alpha) = 25 <_{lex} 33 = p_1(\beta) \cdot p_5(\beta),$$

we immediately conclude that $\alpha \not\geq_s \beta$.

5 Future Directions

Theorem 4.10 suggests that a larger ribbon in $\mathcal{R}((a+1)^n a^m)$ has its rows of length $(a+1)$ more evenly spaced out. We extend our notion of equitability to refine this aspect of balancedness. This perspective may be useful in tackling Conjecture 2.8.

Definition 5.1. A tuple $\epsilon = \epsilon_1 \cdots \epsilon_R$ of nonnegative integers is *equitable* if the ϵ_i take on at most two distinct values $\{b, b+1\}$ where $b \geq 0$. Writing an equitable ϵ as

$$\epsilon = b^{p_1(\epsilon)}(b+1)b^{p_2(\epsilon)}(b+1)\cdots(b+1)b^{p_{n+1}(\epsilon)},$$

where $p_i(\epsilon) \geq 0$, we define $p(\epsilon) = p_1(\epsilon)\cdots p_{n+1}(\epsilon)$. We may also write an equitable ϵ as

$$\epsilon = (b+1)^{\hat{p}_1(\epsilon)}b(b+1)^{\hat{p}_2(\epsilon)}b\cdots b(b+1)^{\hat{p}_{m+1}(\epsilon)},$$

where $\hat{p}_i(\epsilon) \geq 0$, we define $\hat{p}(\epsilon) = \hat{p}_1(\epsilon)\cdots\hat{p}_{m+1}(\epsilon)$.

Example 5.2. Let

$$\epsilon = 33443343334.$$

Then $p(\epsilon) = 20230$ and $\hat{p}(\epsilon) = 00201001$.

We now define a sequence of operations to perform on a composition α. The number of operations that can be performed represents the degree of equitability of α.

Definition 5.3. Let α be an equitable ribbon. Define $p^{(1)}(\alpha) = p(\alpha)$. Then for $k \geq 2$, if $p^{(k-1)}(\alpha)$ is equitable and has as few short ends as possible given $\lambda(p^{(k-1)}(\alpha))$, define

$$p^{(k)}(\alpha) = \hat{p}(p^{(k-1)}(\alpha)).$$

Now α is *k-equitable* if $p^{(k)}(\alpha)$ is defined. It is *infinitely equitable* if it is k-equitable for all k. The *degree of equitability* $\mathrm{eq}(\alpha)$ is the largest k for which α is k-equitable (or ∞ if α is infinitely equitable).

Example 5.4. Let

$$\alpha = 3333\ 4\ 33\ 4\ 33\ 4\ 33\ 4\ 333.$$

Then $p^{(1)}(\alpha) = p(\alpha) = 42223$. This is not equitable so we stop here and $\mathrm{eq}(\alpha) = 1$.

Let

$$\beta = 333\ 4\ 33\ 4\ 33\ 4\ 333\ 4\ 333.$$

Then $p^{(1)}(\beta) = p(\beta) = 32233$. This is equitable and has as few short ends as possible. So we define $p^{(2)}(\beta) = \hat{p}(32233) = 102$. However, this is not equitable, so we stop here and $\mathrm{eq}(\beta) = 2$.

Let

$$\gamma = 333\ 4\ 333\ 4\ 33\ 4\ 333\ 4\ 333\ 4\ 333\ 4\ 33\ 4\ 333\ 4\ 333.$$

Then $p^{(1)}(\gamma) = 332333233$, $p^{(2)}(\gamma) = 232$, but while this is equitable it does not have as few short ends as possible, so $\mathrm{eq}(\gamma) = 2$.

Let

$$\delta = 333\ 4\ 333\ 4\ 33\ 4\ 333\ 4\ 333\ 4\ 33\ 4\ 333\ 4\ 333\ 4\ 333.$$

Then $p^{(1)}(\delta) = 332332333$, $p^{(2)}(\delta) = 223$, and $p^{(3)}(\delta) = 001$, as is every subsequent $p^{(k)}(\delta)$. Therefore δ is infinitely equitable.

The following result illustrates why we make this definition.

Theorem 5.5. The conjectured maximal element of $\mathcal{R}((a+1)^n a^m)$ is the unique infinitely equitable element of $\mathcal{R}((a+1)^n a^m)$.

Therefore, we conclude with a stronger reformulation of the maximal element conjecture.

Conjecture 5.6. Let $\alpha, \beta \in \mathcal{R}((a+1)^n a^m)$ and suppose that $\alpha \geq_s \beta$. Then

$$\mathrm{eq}(\alpha) \geq \mathrm{eq}(\beta).$$

The case where $\mathrm{eq}(\beta) = 1$ is simply a restatement of Proposition 2.4. The case where $\mathrm{eq}(\beta) = 2$ follows from Theorems 4.10 and 4.16.

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