Séminaire Lotharingien de Combinatoire **80B** (2018)
Article #19, 12 pp.

Proceedings of the 30th Conference on Formal Power Series and Algebraic Combinatorics (Hanover)

On a variant of Lie_n

Sheila Sundaram*[1]

[1]*Pierrepont School, One Sylvan Road North, Westport, CT 06880*

Abstract. We introduce a new S_n-module $Lie_n^{(2)}$ which interpolates between the representation Lie_n of the symmetric group S_n afforded by the free Lie algebra, and the module $Conj_n$ of the conjugacy action of S_n on n-cycles. Using plethystic identities from our previous work, we establish a decomposition of the regular representation as a sum of *exterior* powers of the modules $Lie_n^{(2)}$. By contrast, the classical result of Thrall decomposes the regular representation into a sum of symmetric powers of the representation Lie_n. We show that nearly every known property of Lie_n in the literature appears to have a counterpart for $Lie_n^{(2)}$, suggesting connections to the cohomology of configuration spaces and other areas. The construction of $Lie_n^{(2)}$ can be generalised to a module Lie_n^S indexed by subsets S of distinct primes. This in turn yields new Schur-positivity results for multiplicity-free sums of power sums, extending our previous results.

Keywords: plethysm, Schur positivity, symmetric powers, exterior powers, power-sum symmetric functions, configuration space, braid arrangement

1 Introduction

The original motivation for this paper was the investigation begun in [11] on the positivity of the row sums in the character table of S_n. In the language of symmetric functions, one asks for what subsets T of partitions of n the sum of power sums $\sum_{\mu \in T} p_\mu$ is the Frobenius characteristic of a true representation of S_n, i.e. a symmetric function with nonnegative integer coefficients in the basis of Schur functions. A method for generating such classes of subsets T was presented in [11]. Section 2 reviews the key theorems of [11]. We describe some of our new Schur-positivity results in Section 3.

Our main focus here, however, is the unexpected discovery of a curious variant of Lie_n, the representation of S_n on the multilinear component of the free Lie algebra with n generators. The theorem of Poincaré–Birkhoff–Witt states that the universal enveloping algebra of the free Lie algebra is the full tensor algebra. By Schur–Weyl duality, this is equivalent to Thrall's decomposition of the regular representation into a sum of symmetric powers of the representations Lie_n. By contrast, here we obtain a decomposition of the regular representation as a sum of exterior powers of modules (see Theorem 4.1).

*shsund@comcast.net

The key ingredient is our variant of Lie_n, an S_n-module that we denote by $Lie_n^{(2)}$, which turns out to possess remarkable properties similar to those of Lie_n. We show that $Lie_n^{(2)}$ admits a filtration analogous to the one arising from the derived series of the free Lie algebra. Together with the plethystic identities derived in Section 2, they indicate the possibility of an underlying algebra structure for $Lie_n^{(2)}$ involving an acyclic complex. The striking similarity of our results with the Whitney homology of the partition lattice also suggests a cohomological context similar to the configuration space of the braid arrangement. There is an interesting action on derangements arising from $Lie_n^{(2)}$ as well; we show that $Lie_n^{(2)}$ gives rise to a new decomposition of the homology of the complex of injective words studied by Reiner and Webb [6], one that is different from the Hodge decomposition of Hanlon and Hersh [2]. These results are collected in Section 4.

2 Preliminaries

We follow [5] and [10] for notation regarding symmetric functions. In particular, h_n, e_n and p_n denote respectively the complete homogeneous, elementary and power-sum symmetric functions. If ch is the Frobenius characteristic map from the representation ring of the symmetric group S_n to the ring of symmetric functions with real coefficients, then $h_n = \mathrm{ch}(1_{S_n})$ is the characteristic of the trivial representation, and $e_n = \mathrm{ch}(\mathrm{sgn}_{S_n})$ is the characteristic of the sign representation of S_n. If μ is a partition of n then define $p_\mu = \prod_i p_{\mu_i}$; h_μ and e_μ are defined multiplicatively in analogous fashion. As in [5], the Schur function s_μ indexed by the partition μ is the Frobenius characteristic of the S_n-irreducible indexed by μ. Finally, ω is the involution on the ring of symmetric functions which takes h_n to e_n, corresponding to tensoring with the sign representation.

If q and r are characteristics of representations of S_m and S_n respectively, they yield a representation of the wreath product $S_m[S_n]$ in a natural way, with the property that when this representation is induced up to S_{mn}, its Frobenius characteristic is the plethysm $q[r]$. For more background about this operation, see [5].

Define

$$H(t) = \sum_{i\geq 0} t^i h_i, \quad E(t) = \sum_{i\geq 0} t^i e_i; \qquad H = \sum_{i\geq 0} h_i, \quad E = \sum_{i\geq 0} e_i. \tag{2.1}$$

Now let $\{q_i\}_{i\geq 1}$ be a sequence of symmetric functions, each q_i homogeneous of degree i. Let $Q = \sum_{i\geq 1} q_i$, $Q(t) = \sum_{n\geq 1} t^n q_n$. For each partition λ of $n \geq 1$ with $m_i(\lambda) = m_i$ parts equal to i, let $|\lambda| = n = \sum_i i m_i$ be the size of λ, and $\ell(\lambda) = \sum_i m_i(\lambda) = \sum_i m_i$ be the length (total number of parts) of λ.

Define $H_\lambda[Q] = \prod_{i:m_i(\lambda)\geq 1} h_{m_i}[q_i]$ and $E_\lambda[Q] = \prod_{i:m_i(\lambda)\geq 1} e_{m_i}[q_i]$.

For the empty partition (of zero) we define $H_\emptyset[Q] = 1 = E_\emptyset[Q] = H_\emptyset^{\pm}[Q] = E_\emptyset^{\pm}[Q]$.

Consider the generating functions $H[Q](t)$ and $E[Q](t)$. With the convention that Par, the set of all partitions of nonnegative integers, includes the unique empty partition of zero, by the preceding observations and standard properties of plethysm [5] we have

$$H[Q](t) = \sum_{\lambda \in Par} t^{|\lambda|} H_\lambda[Q], \qquad \text{and } E[Q](t) = \sum_{\lambda \in Par} t^{|\lambda|} E_\lambda[Q];$$

Also write $Q^{alt}(t)$ for the alternating sum $\sum_{n \geq 1} (-1)^{i-1} t^i q_i = tq_1 - t^2 q_2 + t^3 q_3 - \dots$.

Let $\psi(n)$ be any real-valued function defined on the positive integers. Define symmetric functions f_n by

$$f_n = \frac{1}{n} \sum_{d|n} \psi(d) p_d^{\frac{n}{d}}, \qquad \text{so that} \qquad \omega(f_n) = \frac{1}{n} \sum_{d|n} \psi(d) (-1)^{n - \frac{n}{d}} p_d^{\frac{n}{d}}. \tag{2.2}$$

Note that, when $\psi(1)$ is a positive integer, this makes f_n the Frobenius characteristic of a possibly virtual S_n-module whose dimension is $(n-1)!\psi(1)$.

Also define the associated polynomial in one variable, t, by

$$f_n(t) = \frac{1}{n} \sum_{d|n} \psi(d) t^{\frac{n}{d}}. \tag{2.3}$$

The following theorem regarding the sequence of symmetric functions f_n is a key element in our work. For the purposes of this abstract we have taken $v = 1$ in the original statement of [11, Theorem 3.2].

Theorem 2.1 ([11, Theorem 3.2])**.** *Let $F = \sum_{n \geq 1} f_n$, $H = \sum_{n \geq 0} h_n$ and $E = \sum_{n \geq 0} e_n$. We have the following plethystic generating functions:*

$$\textit{(Symmetric powers)} \quad H[F](t) = \sum_{\lambda \in Par} t^{|\lambda|} H_\lambda[F] = \prod_{m \geq 1} (1 - t^m p_m)^{-f_m(1)} \tag{2.4}$$

$$\textit{(Exterior powers)} \quad E[F](t) = \sum_{\lambda \in Par} t^{|\lambda|} E_\lambda[F] = \prod_{m \geq 1} (1 - t^m p_m)^{f_m(-1)} \tag{2.5}$$

$$\textit{(Alternating exterior powers)} \quad \sum_{\lambda \in Par} t^{|\lambda|} (-1)^{|\lambda| - \ell(\lambda)} \omega(E_\lambda[F])$$
$$= \sum_{\lambda \in Par} t^{|\lambda|} H_\lambda[\omega(F)^{alt}] = H[\omega(F)^{alt}](t) = \prod_{m \geq 1} (1 + t^m p_m)^{f_m(1)} \tag{2.6}$$

$$\textit{(Alternating symmetric powers)} \quad \sum_{\lambda \in Par} t^{|\lambda|} (-1)^{|\lambda| - \ell(\lambda)} \omega(H_\lambda[F])$$
$$= \sum_{\lambda \in Par} t^{|\lambda|} E_\lambda[\omega(F)^{alt}] = E[\omega(F)^{alt}](t) = \prod_{m \geq 1} (1 + t^m p_m)^{-f_m(-1)} \tag{2.7}$$

We also have, with F, H, and E as defined above, the following generating functions for the symmetric and exterior powers on conjugacy classes of derangements.

Theorem 2.2 ([11, Proposition 2.3, Corollary 2.4]). *Assume that $\psi(1) = 1$ in the definition (2.3) of f_n, i.e. $f_1 = p_1$.*

$$H[\textstyle\sum_{n\geq 2} f_n](t) = \left(\sum_{n\geq 0} t^n(-1)^n e_n\right)\prod_{m\geq 1}(1-t^m p_m)^{-f_m(1)}$$

$$E[\textstyle\sum_{n\geq 2} f_n](t) = \left(\sum_{n\geq 0} t^n(-1)^n h_n\right)\prod_{m\geq 1}(1-t^m p_m)^{f_m(-1)}$$

3 A class of symmetric functions indexed by subsets of primes

Define Lie_n to be f_n with the choice $\psi(d) = \mu(d)$ where μ is the number-theoretic Möbius function, and $Conj_n$ to be f_n with the choice $\psi(d) = \phi(d)$ where ϕ is Euler's phi-function. It is well known that Lie_n is the Frobenius characteristic of the action of S_n on the multilinear component of the free Lie algebra, and also of the induced representation $\exp(\frac{2i\pi}{n})\uparrow_{C_n}^{S_n}$, where C_n is the cyclic group generated by an n-cycle in S_n (see [7] for background and history.) Likewise $Conj_n$ is the Frobenius characteristic of the induced representation $1\uparrow_{C_n}^{S_n}$, i.e. the conjugacy action of S_n on the class of n-cycles.

Definition 3.1. Let $S = \{q_1, \ldots, q_k, \ldots\}$ be a set of distinct primes. Every positive integer n factors uniquely into $n = Q_n \ell_n$ where $Q_n = \prod_{q\in S} q^{a_q(n)}$ for nonnegative integers $a_q(n)$, and $(\ell_n, q) = 1$ for all $q \in S$. We associate to the set S two representations, defined via their Frobenius characteristics, as follows. For each $n \geq 1$:

$$L_n^S = \frac{1}{n}\sum_{d|n}\psi(d)p_d^{\frac{n}{d}} \quad \text{with } \psi(d) = \phi(Q_d)\mu(\ell_d), \text{ and} \tag{3.1}$$

$$L_n^{\bar{S}} = \frac{1}{n}\sum_{d|n}\bar{\psi}(d)p_d^{\frac{n}{d}} \quad \text{with } \bar{\psi}(d) = \phi(\ell_d)\mu(Q_d). \tag{3.2}$$

Let $\mathcal{P}$ denote the set of all primes. Clearly $L_n^{\emptyset} = Lie_n = L_n^{\bar{\mathcal{P}}}$ and $L_n^{\bar{\emptyset}} = Conj_n = L_n^{\mathcal{P}}$. More generally, we have:

Theorem 3.2. *L_n^S and $L_n^{\bar{S}}$ are Schur-positive symmetric functions. If $S = \{q\}$ where q is prime, and k is the largest power of q which divides n, and C_n is the cyclic subgroup of S_n generated by an n-cycle, then $L_n^S = Lie_n^{(q)}$ is the Frobenius characteristic of the induced representation*

$$\exp(\tfrac{2i\pi}{n}\cdot q^k)\uparrow_{C_n}^{S_n}.$$

Let $P(S)$ denote the set of positive integers whose prime divisors constitute a subset of the set of primes S; note that $1 \in P(S)$. Thus $P(S) = \{n \geq 1 : q|n$ and q prime $\implies q \in$

$S\}$. Similarly let $P(\bar{S})$ be the set of positive integers whose set of prime divisors is disjoint from S; note that $P(\bar{S})$ is precisely the set of integers that are relatively prime to every prime in $S : P(\bar{S}) = \{n \geq 1 : q \text{ is a prime factor of } n \Longrightarrow q \notin S\}$. Also $P(S) \cap P(\bar{S}) = \{1\}$.

Using a technical calculation of the values $L_n^S(1)$ and $L_n^{\bar{S}}(1)$, we now obtain, from equations (2.4) and (2.6) of Theorem 2.1 applied to the sequences of symmetric functions $f_n = L_n^S$ and $f_n = L_n^{\bar{S}}$:

Theorem 3.3. *Let* $L^S = \sum_{n\geq 1} L_n^S$, $L^{\bar{S}} = \sum_{n\geq 1} L_n^{\bar{S}}$. *Let DPar denote the subset of all partitions Par consisting of partitions with distinct parts, and the empty partition. Then one has the generating functions*

$$H[L^S](t) = \prod_{n\in P(S)} (1 - t^n p_n)^{-1} = \sum_{\lambda \in Par:\lambda_i \in P(S)} t^{|\lambda|} p_\lambda; \tag{3.3}$$

$$H[\omega(L^S)^{alt}](t) = \prod_{n\in P(S)} (1 + t^n p_n) = \sum_{\lambda \in DPar:\lambda_i \in P(S)} t^{|\lambda|} p_\lambda; \tag{3.4}$$

$$H[L^{\bar{S}}](t) = \prod_{n\in P(\bar{S})} (1 - t^n p_n)^{-1} = \sum_{\lambda \in Par:\lambda_i \in P(\bar{S})} t^{|\lambda|} p_\lambda; \tag{3.5}$$

$$H[\omega(L^{\bar{S}})^{alt}](t) = \prod_{n\in P(\bar{S})} (1 + t^n p_n) = \sum_{\lambda \in DPar:\lambda_i \in P(\bar{S})} t^{|\lambda|} p_\lambda; \tag{3.6}$$

When $S = \emptyset$, equations (3.3), (3.4) and (3.5) reduce to known formulas of Thrall [14], Cadogan [1], and Solomon [9], respectively. See also [10].

$$(Thrall) \qquad H[\sum_{n\geq 1} Lie_n](t) = (1 - tp_1)^{-1} \iff \sum_{\lambda \vdash n} H_\lambda[Lie] = p_1^n, \quad n \geq 1. \tag{3.7}$$

$$(Cadogan) \qquad H[\sum_{n\geq 1} (-1)^{n-1} \omega(Lie_n)](t) = 1 + tp_1 \iff \sum_{\lambda \vdash n} H_\lambda[\omega(Lie)^{alt}] = 0, \quad n \geq 1. \tag{3.8}$$

$$(Solomon) \qquad H[\sum_{n\geq 1} Conj_n](t) = \prod_{n\geq 1} (1 - t^n p_n)^{-1} \iff \sum_{\lambda \vdash n} H_\lambda[Conj] = \sum_{\lambda \vdash n} p_\lambda, \quad n \geq 1. \tag{3.9}$$

In similar fashion, by computing the values of $L_n^S(-1)$ and $L_n^{\bar{S}}(-1)$ and invoking equations (2.5) and (2.7) of Theorem 2.1, we obtain the exterior power analogue of Theorem 3.3.

Theorem 3.4. *Let S be a set of primes, and let* $L_n^S, L_n^{\bar{S}}$ *be as defined in Theorem 3.3. Then*

$$E[L^S(t)] = \begin{cases} \prod_{n\in P(S)} (1 - t^n p_n)^{-1} \prod_{n\, even, \frac{n}{2} \in P(S)} (1 - t^n p_n), & 2 \notin S \\ \prod_{n\, odd, n\in P(S)} (1 - t^n p_n)^{-1}, & 2 \in S. \end{cases} \tag{3.10}$$

(Note n is necessarily odd in the first product of the case $2 \notin S$.)

$$\omega(E[L^S(t)]) = \prod_{n \in P(S)} (1-t^n p_n)^{-1} \prod_{n\, even, \frac{n}{2} \in P(S)} (1+t^n p_n), \text{ provided } 2 \notin S \tag{3.11}$$

$$E[\omega(L^S)^{alt}(t)] = \begin{cases} \prod_{n \in P(S)} (1+t^n p_n) \prod_{n\, even, \frac{n}{2} \in P(S)} (1+t^n p_n)^{-1}, & 2 \notin S \\ \prod_{n\, odd, n \in P(S)} (1+t^n p_n), & 2 \in S. \end{cases} \tag{3.12}$$

$$E[L^{\bar{S}}(t)] = \begin{cases} \prod_{n\ odd, n \in P(\bar{S})} (1-t^n p_n)^{-1}, & 2 \notin S \\ \prod_{n \in P(\bar{S})} (1-t^n p_n)^{-1} \prod_{n\ even,\ \frac{n}{2} \in P(\bar{S})} (1-t^n p_n), & 2 \in S. \end{cases} \tag{3.13}$$

(Note n is necessarily odd in the first product of the case $2 \in S$, as is $\frac{n}{2}$ in the second product.)

$$\omega(E[L^{\bar{S}}(t)]) = \prod_{n \in P(\bar{S})} (1-t^n p_n)^{-1} \prod_{n\ even,\ \frac{n}{2} \in P(\bar{S})} (1+t^n p_n), \text{ provided } 2 \in S. \tag{3.14}$$

$$E[\omega(L^{\bar{S}})^{alt}(t)] = \begin{cases} \prod_{n\ odd, n \in P(\bar{S})} (1+t^n p_n), & 2 \notin S \\ \prod_{n \in P(\bar{S})} (1+t^n p_n) \prod_{n\ even,\ \frac{n}{2} \in P(\bar{S})} (1+t^n p_n)^{-1}, & 2 \in S. \end{cases} \tag{3.15}$$

For any subset T_n of partitions of n, denote by P_{T_n} the sum of power-sum symmetric functions $\sum_{\lambda \in T_n} p_\lambda$. Since L_n^S and $L_n^{\bar{S}}$ are Schur-positive, so are their symmetric and exterior powers. Hence we deduce the following from the preceding two theorems.

Theorem 3.5. *For a fixed set of primes S, the sums P_{T_n} are Schur-positive for the following choices of T_n:*

1. $T_n = \{\lambda \vdash n : \lambda_i \in P(S)\}$;
2. *If* $2 \in S$, $T_n = \{\lambda \vdash n : \lambda_i\, odd,\ \lambda_i \in P(S)\}$;
3. *If $2 \notin S$, T_n consists of all partitions λ of n such that the parts are (necessarily odd and) in $P(S)$, or the parts are twice an odd number in $P(S)$, the even parts occurring at most once.*

4 The case $q = 2$: A comparison of Lie_n and $Lie_n^{(2)}$

In this section we will describe some remarkable properties of the S_n-module $Lie_n^{(2)}$. Theorem 3.2 implies that $Lie_2^{(2)} = h_2$ and $Lie_n^{(2)} = Lie_n$ if n is odd. We use Theorems 2.1 and 2.2, and plethystic computations with the identities of Theorems 3.3 and 3.4, to show that the representation $Lie_n^{(2)}$ has properties that curiously parallel those of Lie_n. We write Lie for the sum of symmetric functions $\sum_{n \geq 1} Lie_n$ and $Lie^{(2)}$ for the sum of symmetric functions $\sum_{n \geq 1} Lie_n^{(2)}$.

Theorem 4.1.

$$\sum_{\lambda\vdash n} H_\lambda[Lie] = p_1^n; \qquad \sum_{\lambda\vdash n} E_\lambda[Lie^{(2)}] = p_1^n; \tag{4.1}$$

$$H[\sum_{n\geq 1}(-1)^{n-1}\omega(Lie_n)] = 1+p_1; \qquad E[\sum_{n\geq 1}(-1)^{n-1}\omega(Lie_n^{(2)})] = 1+p_1 \tag{4.2}$$

$$\text{If } n\geq 2: \sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)}E_\lambda[Lie] = 0; \qquad \sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)}H_\lambda[Lie^{(2)}] = 0; \tag{4.3}$$

$$\text{If } n\geq 2: \sum_{\lambda\vdash n} E_\lambda[Lie] = 2e_2p_1^{n-2}; \qquad \sum_{\lambda\vdash n} H_\lambda[Lie^{(2)}] = \sum_{\lambda\vdash n,\lambda_i=2^{a_i}} p_\lambda. \tag{4.4}$$

We now discuss the implications of Theorem 4.1.

In (4.1) of the preceding theorem, the second equation gives a decomposition of the regular representation of S_n as a sum of *exterior* powers of induced modules, whereas the first equation is precisely Thrall's theorem [14], restated in (3.7), that the regular representation of S_n decomposes into a sum of *symmetrised* Lie modules.

In (4.2), the first equation contains the known result of Cadogan [1] (see (3.8)) giving the plethystic inverse of the homogeneous symmetric functions $\sum_{n\geq 1} h_n$, and the second equation is a new result, giving the plethystic inverse of the elementary symmetric functions $\sum_{n\geq 1} e_n$.

The equations in (4.3) and (4.4) are particularly significant. It is well known that the degree n term in $e_{n-r}[Lie]$ is the Frobenius characteristic of the rth Whitney homology $WH_r(\Pi_n)$ of the partition lattice Π_n, tensored with the sign (see [12, Remark 1.8.1]), and hence of the sign-twisted rth cohomology of the pure configuration space arising as the complement of the braid arrangement. The rth Whitney homology also coincides as an S_n-module with the rth cohomology of the pure braid group, see [3]. The first equation in (4.3) therefore restates the acyclicity of Whitney homology for the partition lattice [12]. Writing $WH_{odd}(\Pi_n)$ for $\oplus_{k=0}^{n/2} WH_{2k+1}(\Pi_n)$, and $WH_{even}(\Pi_n)$ for $\oplus_{k=0}^{n/2} WH_{2k}(\Pi_n)$, we have the isomorphism of S_n-modules

$$WH_{odd}(\Pi_n) \simeq WH_{even}(\Pi_n). \tag{4.5}$$

Now consider (4.4). Denote by $WH(\Pi_n)$ the sum of all the graded pieces of the Whitney homology of Π_n. The first equation in (4.4) says (recall that we have tensored with the sign representation) that $WH(\Pi_n) = 2h_2p_1^{n-2}$, a result originally due to Lehrer [4, Proposition 5.6 (i)]. We may rewrite this in our notation as

$$\mathrm{ch}\, WH(\Pi_n) = \mathrm{ch}\,(WH_{odd}(\Pi_n)\oplus WH_{even}(\Pi_n)) = 2h_2p_1^{n-2}. \tag{4.6}$$

(Note that it then follows that $\mathrm{ch}\, WH(\Pi_{n+1}) = p_1\cdot \mathrm{ch}\, WH(\Pi_n), n\geq 2.$)

By combining this with (4.5), we obtain

$$\mathrm{ch}\,(WH_{odd}(\Pi_n)) = \mathrm{ch}\,(WH_{even}(\Pi_n)) = h_2p_1^{n-2}, \tag{4.7}$$

yielding the decomposition of the regular representation which appears in the recent paper [3] of Hyde and Lagarias:

$$\text{ch } WH_{odd}(\Pi_n) + \omega(\text{ch } WH_{odd}(\Pi_n)) = \sum_{r=0}^{n-1} \omega^r(\text{ch } WH_r(\Pi_n)) = p_1^n. \tag{4.8}$$

We will now describe analogous results arising from the representations $Lie_n^{(2)}$. Define a new module $Vh_r(n)$ whose Frobenius characteristic is the degree n term in $h_{n-r}[Lie^{(2)}]$. By Theorem 3.2, this is a true S_n-module. The second equation of (4.3) can now be interpreted as an acyclicity statement:

$$Vh_n(n) - Vh_{n-1}(n) + Vh_{n-2}(n) - \ldots + (-1)^r Vh_r(n) + \ldots = 0,$$

and hence, in analogy with (4.5), letting $Vh_{odd}(n) = \oplus_{k=0}^{n/2} Vh_{2k+1}$ and $Vh_{even} = \oplus_{k=0}^{n/2} Vh_{2k}$:

$$Vh_{odd}(n) \simeq Vh_{even}(n). \tag{4.9}$$

The second equation in (4.4) gives, similarly,

$$\text{ch } (Vh_{odd}(n) \oplus Vh_{even}(n)) = \sum_{\lambda \vdash n; \lambda_i = 2^{a_i}} p_\lambda \tag{4.10}$$

Hence, combining this with (4.9) we have established the following results, showing that the modules $Vh_r(n)$ share the same features as the Whitney homology modules and hence of the cohomology of the configuration space for the braid arrangement:

Theorem 4.2. *The following symmetric functions are Schur-positive with integer coefficients:*

$$\text{ch } (Vh_{odd}(n)) = \text{ch } (Vh_{even}(n)) = \frac{1}{2} \sum_{\lambda \vdash n; \lambda_i = 2^{a_i}} p_\lambda \tag{4.11}$$

$$\text{ch } Vh(n) = \text{ch } (Vh_{odd}(n)) + \text{ch } (Vh_{even}(n)) = \sum_{\lambda \vdash n; \lambda_i = 2^{a_i}} p_\lambda \tag{4.12}$$

$$\text{ch } Vh_{odd}(n) + \omega(\text{ch } Vh_{odd}(n)) = \sum_{\lambda \vdash n; n-\ell(\lambda) even; \lambda_i = 2^{a_i}} p_\lambda \tag{4.13}$$

Also, $\text{ch } Vh(2n+1) = p_1 \cdot \text{ch } Vh(2n)$.

We now have at least four decompositions of the regular representation, namely the two in (4.1) and two from (4.8) (tensoring the latter with the sign representation gives two), into sums of modules indexed by the conjugacy classes, each module obtained by inducing a linear character from a centraliser of S_n. We write these out for S_4.

Example 4.3. The first two decompositions are from (4.1) of Theorem 4.1; the third is from (4.8). In all cases, of course, the four pieces each have the same dimension, equal to the size of the corresponding conjugacy class, namely, $1, 6, 11, 6$. That these four decompositions are all distinct is clear, since each has a distinguishing feature. Both copies of the irreducible for the partition (2^2) appear only in one piece for **[PBW]**, while the natural representation is a submodule of one piece only in the third.

PBW: $h_4[Lie] = (4)$; $h_3[Lie] = (3,1) + (2,1^2)$;
$$h_2[Lie] = (3,1) + 2(2^2) + (2,1^2) + (1^4);\ h_1[Lie] = Lie_4 = (3,1) + (2,1^2).$$

Ext. $Lie^{(2)}$**:** $e_4[Lie^{(2)}] = (1^4)$; $e_3[Lie^{(2)}] = (3,1) + (2,1^2)$;
$$e_2[Lie^{(2)}] = 2(3,1) + (2^2) + (2,1^2);\ e_1[Lie^{(2)}] = Lie_4^{(2)} = (4) + (2^2) + (2,1^2).$$

Whitney: $\omega(WH_0) = (1^4)$; $WH_1 = (4) + (3,1) + (2^2)$;
$$\omega(WH_2) = (3,1) + 2(2,1^2) + (2^2);\ WH_3 = Lie_4 = (3,1) + (2,1^2).$$

There is yet another analogy between $WH_k(\Pi_n)$ and the modules $V_k(n)$ arising from the identities of Theorem 4.1. In [12], it was shown that the Whitney homology of the partition lattice (and more generally of any Cohen–Macaulay poset) has the following important property:

Theorem 4.4 ([12, Proposition 1.9])**.** *For $0 \le k \le n-1$, the truncated alternating sum*
$$WH_k(\Pi_n) - WH_{k-1}(\Pi_n) + \ldots + (-1)^k WH_0(\Pi_n)$$
is a true S_n-module, and is isomorphic as an S_n-module to the unique nonvanishing homology of the rank-selected subposet of Π_n obtained by selecting the first k ranks. Equivalently, the degree n term in the plethysm
$$(e_{n-k} - e_{n-k+1} + \ldots + (-1)^k e_n)[Lie]$$
is Schur-positive. In particular, the kth Whitney homology decomposes into a sum of two S_n-modules as follows:
$$\mathrm{ch}\, WH_k(\Pi_n) = \omega\left(e_{n-k}[Lie]|_{deg\ n}\right) = \beta_n([1,k]) + \beta_n([1,k-1]),$$
where $\beta_n([1,k])$ denotes the Frobenius characteristic of this rank-selected homology (of the first k ranks of Π_n) as in [12, Proposition 1.9].

We conjecture that a similar decomposition exists for the S_n-modules $Vh_k(n)$. More precisely, we have

Conjecture 4.5 (Verified up to $n = 7$)**.** *Let $Vh_k(n)$ be the S_n-module whose Frobenius characteristic is the degree n term in the plethysm $h_{n-k}[Lie^{(2)}]$, for $k = 0, 1, \ldots, n-1$. Then for $0 \leq k \leq n-1$, the truncated alternating sum $Vh_k(n) - Vh_{k-1}(n) + \ldots + (-1)^k Vh_0(n) = U_k(n)$ is a true S_n-module, and hence one has the S_n-module decomposition*

$$\mathrm{ch}\, Vh_k(n) = h_{n-k}[Lie^{(2)}]|_{deg\ n} = \mathrm{ch}\, U_k(n) + \mathrm{ch}\, U_{k-1}(n).$$

(Here we define $U_{-1}(n)$ to be the zero module and $U_0(n)$ to be the trivial S_n-module.) Equivalently, the degree n term in the plethysm

$$(h_{n-k} - h_{n-k+1} + \ldots + (-1)^k h_n)[Lie^{(2)}]$$

is Schur-positive for $0 \leq k \leq n-1$.

This conjecture is easily verified for $0 \leq k \leq 2$; in that case there are simple formulas for $\mathrm{ch}\, Vh_k(n)$.

Recent work of Hyde and Lagarias [3] rediscovers the representations $\beta_n([1,k])$ in a cohomological setting. Our results suggest the existence of a similar topological context in which the modules $Vh_k(n)$ and $U_k(n)$ appear. If X is any topological space, then the ordered configuration space $PConf_n(X)$ of n distinct points in X is defined to be the set $\{(x_1, \ldots, x_n) : i \neq j \implies x_i \neq x_j\}$. The symmetric group S_n acts on $PConf_n(X)$ by permuting coordinates, and hence induces an action on the cohomology $H^k(PConf_n(X), Q), k \geq 0$. The following theorem of [13] recasts the "Lie" identities of Theorem 4.1 in the context of $X = \mathbf{R}^d$:

Theorem 4.6 ([13, Theorem 4.4])**.** *The Frobenius characteristic of $H^k(PConf_n(\mathbf{R}^2), Q)$ is given by $\omega\left(e_{n-k}[Lie]|_{deg\ n}\right)$.*

The Frobenius characteristic of $H^{2k}(PConf_n(\mathbf{R}^3), Q)$ is given by $h_{n-k}[Lie]|_{deg\ n}$.

Question 4.7. Is there a similar cohomological context for the "$Lie^{(2)}$" identities of Theorem 4.1?

We turn next to the action on fixed-point-free permutations. If $F = \sum_{n\geq 1} f_n$ is any series of symmetric functions where f_n is of homogeneous degree n, let $F_{\geq 2}$ denote the series $F - f_1 = \sum_{n\geq 2} f_n$. Reiner and Webb study the Cohen–Macaulay complex of injective words, and compute the S_n-action on its top homology [6]. Using Theorem 2.2, we show that the representations $Lie_n^{(2)}$ make an appearance here as well:

Theorem 4.8. *Let Δ_n^k denote the degree n term in $e_k[Lie_{\geq 2}^{(2)}]$, and let $\Delta_n = \sum_{k\geq 1} \Delta_n^k$. Then Δ_n coincides with the homology representation on the complex of injective words.*

Hanlon and Hersh have shown that this representation has a Hodge decomposition [2]. Writing D_n^k for the degree n term in $h_k[Lie_{\geq 2}]$, their result may be stated as follows:

$$\Delta_n = \sum_{k\geq 1} \omega(D_n^k).$$

Surprisingly, the decomposition of Δ_n given by Theorem 4.8 is different from the Hodge decomposition. The first nontrivial example appears below.

Example 4.9. For $n = 4$, we have $\Delta_4 = p_1^2 h_2 - p_1 h_3 + h_4 = (4) + (3,1) + (2^2) + (2,1^2)$. Also $\Delta_4^2 = e_2[h_2] = (3,1)$, $\Delta_4^1 = Lie_4^{(2)} = (4) + (2^2) + (2,1^2)$. The two Hodge pieces, however, each consist of two irreducibles: $\omega(h_2[Lie_2]) = (2^2) + (4)$ and $\omega(h_1[Lie_4]) = (3,1) + (2,1^2)$.

A classical result asserts that the free Lie algebra has the following filtration arising from its derived series [7, Section 8.6.12]. Let $\kappa = \sum_{n\geq 2} s_{(n-1,1)}$. Then $Lie_{\geq 2} = \kappa + \kappa[\kappa] + \kappa[\kappa[\kappa]] + \ldots$.

Theorem 4.1 allows us to deduce a similar decomposition for $Lie_n^{(2)}$:

Theorem 4.10. $Lie_{\geq 2}^{(2)} = \omega(\kappa) + \omega(\kappa)[\omega(\kappa)] + \omega(\kappa)[\omega(\kappa)[\omega(\kappa)]] + \ldots$

We conclude with yet another feature of the Lie representation that reappears in $Lie_n^{(2)}$. Recall that Lie_{n-1} admits a lifting W_n which is a true S_n-module, the Whitehouse module, appearing in many different contexts [8], [10, Exercise 7.88 (d)], whose Frobenius characteristic is given by $\operatorname{ch} W_n = p_1 Lie_{n-1} - Lie_n$.

One can ask if the same construction for $Lie_n^{(2)}$ yields a true S_n-module. Clearly one obtains a possibly virtual module which restricts to $Lie_{n-1}^{(2)}$ as an S_{n-1}-module. We have the following conjecture, verified in Maple (with Stembridge's SF package) up to $n = 32$:

Conjecture 4.11. *The symmetric function $p_1 Lie_{n-1}^{(2)} - Lie_n^{(2)}$ is Schur-positive if and only if n is* not *a power of 2.*

One direction of this conjecture is easy to verify. Let $n = 2^k \geq 2$. Then $n-1$ is odd, so $Lie_{n-1}^{(2)} = Lie_n$. Also (by Theorem 3.2) $Lie_n^{(2)} = Conj_n$ is just the permutation module afforded by the conjugacy action on the class of n-cycles of S_n. Consequently it contains the trivial representation (exactly once). But Lie_n never contains the trivial representation ($n \geq 2$). Hence, when n is a power of 2, the trivial module appears with negative multiplicity (-1) in $p_1 Lie_{n-1}^{(2)} - Lie_n^{(2)}$.

References

[1] C.C. Cadogan. "The Möbius function and connected graphs". *J. Combin. Theory Ser. B* **11** (1971), pp. 193–200. DOI: 10.1016/0095-8956(71)90029-3.

[2] P. Hanlon and P. Hersh. "A Hodge decomposition for the complex of injective words". *Pacific J. Math.* **214**.1 (2004), pp. 109–125. DOI: 10.2140/pjm.2004.214.109.

[3] T. Hyde and J.C. Lagarias. "Polynomial splitting measures and cohomology of the pure braid group". *Arnold Math. J.* **3**.2 (2017), pp. 219–249. DOI: 10.1007/s40598-017-0064-z.

[4] G.I. Lehrer. "On the Poincaré series associated with Coxeter group actions on complements of hyperplanes". *J. London Math. Soc. (2)* **36**.2 (1987), pp. 275–294. DOI: 10.1112/jlms/s2-36.2.275.

[5] I.G. Macdonald. *Symmetric functions and Hall polynomials*. Second. Oxford Mathematical Monographs. With contributions by A. Zelevinsky, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995, pp. x+475.

[6] V. Reiner and P. Webb. "The combinatorics of the bar resolution in group cohomology". *J. Pure Appl. Algebra* **190**.1-3 (2004), pp. 291–327. DOI: 10.1016/j.jpaa.2003.12.006.

[7] C. Reutenauer. *Free Lie algebras*. Vol. 7. London Mathematical Society Monographs. New Series. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993, pp. xviii+269.

[8] A. Robinson and S. Whitehouse. "The tree representation of Σ_{n+1}". *J. Pure Appl. Algebra* **111**.1-3 (1996), pp. 245–253. DOI: 10.1016/0022-4049(95)00116-6.

[9] L. Solomon. "On the sum of the elements in the character table of a finite group". *Proc. Amer. Math. Soc.* **12** (1961), pp. 962–963. DOI: 10.2307/2034401.

[10] R.P. Stanley. *Enumerative combinatorics. Vol. 2*. Vol. 62. Cambridge Studies in Advanced Mathematics. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. Cambridge University Press, Cambridge, 1999, pp. xii+581. DOI: 10.1017/CBO9780511609589.

[11] S. Sundaram. "The conjugacy action of S_n and modules induced from centralisers". *J. Algebraic Combin.* (2017). DOI: 10.1007/s10801-017-0796-9.

[12] S. Sundaram. "The homology representations of the symmetric group on Cohen-Macaulay subposets of the partition lattice". *Adv. Math.* **104**.2 (1994), pp. 225–296. URL.

[13] S. Sundaram and V. Welker. "Group actions on arrangements of linear subspaces and applications to configuration spaces". *Trans. Amer. Math. Soc.* **349**.4 (1997), pp. 1389–1420. DOI: 10.1090/S0002-9947-97-01565-1.

[14] R.M. Thrall. "On symmetrized Kronecker powers and the structure of the free Lie ring". *Amer. J. Math.* **64** (1942), pp. 371–388. DOI: 10.2307/2371691.