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Semistable subcategories for tiling algebras

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Abstract. Semistable subcategories were introduced in the context of Mumford's GIT and interpreted by King in terms of representation theory of finite dimensional algebras. Ingalls and Thomas later showed that for finite dimensional algebras of Dynkin and affine type, the poset of semistable subcategories is isomorphic to the corresponding lattice of noncrossing partitions. We show that semistable subcategories defined by tiling algebras, introduced by Simões and Parsons, are in bijection with noncrossing tree partitions, introduced by the second author and McConville. Our work recovers that of Ingalls and Thomas in Dynkin type A.

Résumé. Les sous-catégories semi-stables ont été introduites dans le contexte du GIT de Mumford et interprétées par King en termes de théorie des représentations des algèbres de dimension finie. Ingalls et Thomas ont montré plus tard que pour les algèbres de dimension finie de type Dynkin et affines, l'ensemble ordonné des sous-catégories semi-stables est isomorphe au treillis des partitions non-croisées correspondant. Nous montrons que les sous-catégories semi-stables définies par des algèbres de pavage, introduites par Simões et Parsons, sont en bijection avec des partitions arborescentes non-croisées, introduites par le second auteur et McConville. Nous retrouvons des résultats d'Ingalls et Thomas pour Dynkin de type A.

Keywords: quiver, stability condition, noncrossing partition, wide subcategory

1 Introduction

Mumford's geometric invariant theory (GIT) provides a technique for taking the quotient of an algebraic variety by certain types of group actions in such a way that the resulting quotient is again an algebraic variety. Given a variety V and a reductive algebraic group G acting linearly on V, one replaces V by its "semistable points" and then forms the GIT quotient $V/\!/G$, which is an algebraic variety.

In [6], King interpreted this notion of semistable points in terms of representation theory of algebras as follows. Let $\Lambda = \Bbbk Q/I$ be the path algebra of a **quiver** Q (i.e., a directed graph) modulo an admissible ideal I and $\Bbbk$ is an algebraically closed field.

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Recall that the **path algebra** $\Bbbk Q$ consists of formal $\Bbbk$-linear combinations of paths in Q, and its multiplication is given by composition of paths. For such algebras, any Λ-module M may be regarded as a **representation** of Q (i.e., an assignment of a finite dimensional $\Bbbk$-vector space M_i to each vertex of Q and a $\Bbbk$-linear map to each arrow of Q). A representation M of Q naturally defines a **dimension vector**, denoted by $\mathbf{dim}(M) := (\dim_{\Bbbk} M_i)_{i=1}^n \in \mathbb{Z}_{\geq 0}^n$, where n will henceforth denote the number of vertices of Q.

Now let $V = \mathrm{mod}(\Lambda, \mathbf{d})$, the variety of finitely generated Λ-modules with dimension vector $\mathbf{d} = (d_1, \ldots, d_n)$, and let $G = \prod_{i=1}^n \mathrm{GL}_{d_i}(\Bbbk)$ act by base change at each vertex of Q. In [6], King showed that the semistable points of V, which from now on we call **semistable representations** (resp., **stable representations**), are the representations M where there exists a linear map $\theta \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$ satisfying

- $\theta(\mathbf{dim}(M)) = 0$, and
- for any subrepresentation $N \subset M$, one has $\theta(\mathbf{dim}(N)) \leq 0$ (resp., $\theta(\mathbf{dim}(N)) < 0$).

We refer to such linear maps $\theta \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$ as **stability conditions** on $\mathrm{mod}(\Lambda)$, the category of finitely generated Λ-modules. Any choice of stability condition θ defines a subcategory θ^{ss} of $\mathrm{mod}(\Lambda)$ consisting of the **θ-semistable representations**. We refer to θ^{ss} as a **semistable subcategory**. Note that two different stability conditions may define the same semistable subcategory.

We study the poset of all semistable subcategories of $\mathrm{mod}(\Lambda)$ ordered by inclusion, denoted Λ^{ss}. There are close connections between the theory of semistable subcategories and the combinatorics of Coxeter groups. If $\Lambda = \Bbbk Q$ where Q is an orientation of a simply-laced Dynkin or extended Dynkin diagram, it follows from [5, Theorem 1.1] that Λ^{ss} is isomorphic to the poset of noncrossing partitions associated with Q.

Other important examples of algebras Λ include cluster-tilted algebras [1], which appear in the context of cluster algebras, and also preprojective algebras. In the latter case, in [10] it is shown that Λ^{ss} is isomorphic to the shard intersection order of the Coxeter arrangement associated with Q (see [9] for more on the shard intersection order).

The purpose of this work is to combinatorially classify the semistable subcategories for the class of **tiling algebras**, introduced in [2] to study endomorphism algebras of maximal rigid objects in some negative Calabi–Yau categories. Following [3], these algebras, denoted Λ_T, are defined by the data of a tree T embedded in the disk D^2 whose interior vertices have degree at least 3 (see Figure 1). Examples of tiling algebras are given by the cluster-tilted algebras of cluster type A; the trees defining these algebras are those whose interior vertices are of degree 3.

The tree T defines a simplicial complex of noncrossing arcs on T called the noncrossing complex, denoted by $\Delta^{NC}(T)$ (see Section 2). Each facet of $\Delta^{NC}(T)$ consists of **red arcs**, **green arcs**, and **boundary arcs**. In [7], it is shown that if δ is a green or red arc in a facet of $\Delta^{NC}(T)$, it gives rise to a **g-vector**, denoted $\mathbf{g}(\delta) \in \mathbb{Z}^n$. Additionally, in [3], it is shown that the facets of $\Delta^{NC}(T)$ are in bijection with **wide subcategories** of $\mathrm{mod}(\Lambda_T)$. With these facts in mind, we arrive at our main theorem.

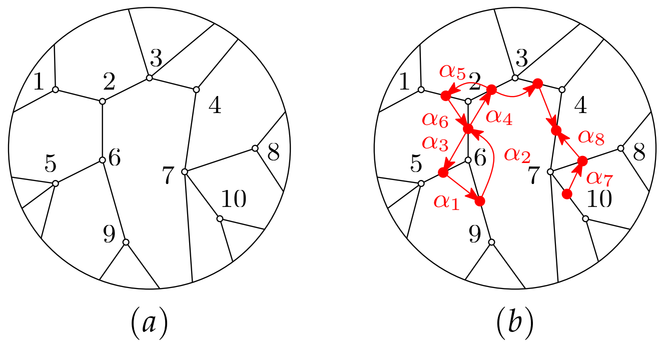

Figure 1: We show a tree T in (a) and the quiver Q_T it defines in (b). The associated tiling algebra is $\Lambda_T = \Bbbk Q_T / I_I$ where $I_T = \langle \alpha_2\alpha_1, \alpha_3\alpha_2, \alpha_1\alpha_3, \alpha_5\alpha_4, \alpha_6\alpha_5, \alpha_4\alpha_6, \alpha_8\alpha_7 \rangle$.

Theorem 1.1. *Let $\mathcal{W} \subset mod(\Lambda_T)$ be a wide subcategory, and let $\mathcal{F}_{\mathcal{W}}$ be the corresponding facet of $\Delta^{NC}(T)$ and $\mathcal{F}_{\mathcal{W}}^{gr}$ the set of green arcs. Then the* ***Kreweras stability condition*** *defined as*

$$\begin{array}{rcl} \theta_{\mathcal{F}_{\mathcal{W}}} : \mathbb{Z}^n & \longrightarrow & \mathbb{Z} \\ \boldsymbol{dim}(M) & \longmapsto & \sum_{\delta \in \mathcal{F}_{\mathcal{W}}^{gr}} \langle \mathbf{g}(\delta), \boldsymbol{dim}(M) \rangle, \end{array}$$

where $M \in mod(\Lambda_T)$ and $\langle -, - \rangle$ is the standard Euclidean inner product, satisfies $\theta_{\mathcal{F}_{\mathcal{W}}}^{ss} = \mathcal{W}$. Conversely, any semistable subcategory of $mod(\Lambda_T)$ is a wide subcategory of Λ_T.

The paper is organized as follows. In Section 2, we review the noncrossing complex of arcs on a tree. In Section 3, we associate **g**- and **c**-vectors to each facet of this complex, which are essential to our construction of semistable subcategories. In Section 4, we define the tiling algebras that we will study. In Section 5, we define noncrossing tree partitions, which will classify the semistable subcategories of mod(Λ_T). We also sketch an important step in the proof of Theorem 1.1. Lastly, in Section 6, we propose a natural extension of our work to general gentle algebras.

2 Noncrossing complex

A **tree** $T = (V_T, E_T)$ is a finite connected acyclic graph. Any tree may be embedded in the disk D^2 in such a way that a vertex is on the boundary if and only if it is a leaf. We will assume that any tree is accompanied by such an embedding in D^2. We say two trees T and T' are **equivalent** if there is an ambient isotopy between the spaces $D^2 \backslash T$ and $D^2 \backslash T'$. We consider trees up to equivalence. Additionally, we assume that the **interior vertices** of any tree T (i.e., the nonleaf vertices of T) have degree at least 3.

We say the closure of a connected component of $D^2 \backslash T$ is a **face** of T. A **corner** (v, F) of T is a pair consisting of an interior vertex v of T and a face F of T that contains v.

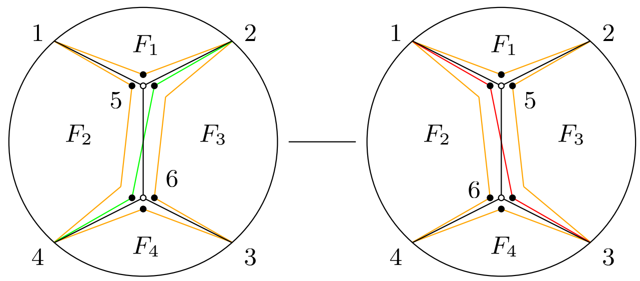

Figure 2: Both facets of this noncrossing complex contain 5 arcs. The boundary arcs are shown in gold. The marked corners of arcs are indicated by black dots. The faces are F_1, F_2, F_3, F_4.

An **acyclic path** supported by a tree T is a sequence $(v_0, v_1, ..., v_t)$ of pairwise distinct vertices of T such that v_i and v_j are adjacent if and only if $|i-j| = 1$. We will refer to v_0 and v_t as the **endpoints** of the acyclic path. Since T is acyclic, any acyclic path is determined by its endpoints, and we can therefore write $(v_0, v_1, ..., v_t) = [v_0, v_t]$.

An **arc** $\delta = (v_0, v_1, ..., v_t)$ is an acyclic path such that its endpoints are leaves and any two edges (v_{i-1}, v_i) and (v_i, v_{i+1}) are incident to a common face. We say δ **contains a corner** (v, F) if $v = v_i$ for some $i = 0, 1, \ldots, t$ and (v_{i-1}, v_i) and (v_i, v_{i+1}) are incident to F. We also note that δ divides D^2 into two **regions** composed of disjoint subsets of faces of T. We let $\mathrm{Reg}(\delta, F)$ denote the region defined by δ which contains F. We say that two arcs δ and δ' are **crossing** if given any regions R_δ and $\mathrm{R}_{\delta'}$ defined by δ and δ', respectively, then $\mathrm{R}_\delta \not\subset \mathrm{R}_{\delta'}$ or $\mathrm{R}_{\delta'} \not\subset \mathrm{R}_\delta$. Otherwise, we say δ and δ' are **noncrossing**.

Define the **noncrossing complex** of T, denoted $\Delta^{NC}(T)$, to be the abstract simplicial complex of noncrossing arcs of T. By [4, Corollary 3.6], this is a **pure complex** (i.e., any two facets have the same cardinality). We will primarily work with the facets of $\Delta^{NC}(T)$.

Let $\mathcal{F}$ be any facet of $\Delta^{NC}(T)$. The arcs of $\mathcal{F}$ containing a corner (v, F) are linearly ordered: two arcs $\delta, \gamma \in \mathcal{F}$ containing (v, F) satisfy $\delta \leq_{(v,F)} \gamma$ if and only if $\mathrm{Reg}(\delta, F) \subset \mathrm{Reg}(\gamma, F)$. That such arcs are linearly ordered follows from the fact that they are pairwise noncrossing. We say that an arc δ of $\mathcal{F}$ is **marked** at corner (v, F) if δ contains (v, F) and it is the maximal such arc with respect to $\leq_{(v,F)}$. In [4, Proposition 3.5], it is shown that every $\delta \in \mathcal{F}$ is marked at either one or two corners. In the latter case, the two corners at which δ is marked belong to different regions defined by δ. We refer to the arcs marked at a single corner as **boundary arcs**, and we denote the set of boundary arcs of $\mathcal{F}$ by $\mathcal{F}^\partial$. We show an example of the noncrossing complex in Figure 2.

The arcs of $\mathcal{F}$ that are not boundary arcs come with extra data of a color as follows. A **flag** is a triple (v, e, F) of a vertex v incident to an edge e, which is incident to the face F. We say a flag is **green** if by orienting e away from v, face F is to the left of e. Otherwise, we say (v, e, F) is **red**. Let (v, F) and (u, G) be the two corners in which $\delta \in \mathcal{F}$ is marked, and let e and e' be edges of T contained in $[v,u]$ where the former is incident to v and the latter is incident to u. Both (v,e,F) and (u,e',G) have to be of the same color, as F and G belong to different regions determined by δ. We say δ is a **green arc** if it is marked at corners belonging to green flags, otherwise we say it is a **red arc**. Define $\mathcal{F}^{gr}$ (resp., $\mathcal{F}^{red}$) to be the set of green (resp., red) arcs of $\mathcal{F}$. Observe that $\mathcal{F} = \mathcal{F}^{red} \sqcup \mathcal{F}^{gr} \sqcup \mathcal{F}^{\partial}$. We show examples of red and green arcs in Figures 2 and 7.

3 Facets and their c- and g-vectors

In this section, we show how to associate a family of vectors in $\mathbb{Z}^n$ to each facet of the noncrossing complex where n denotes the number of edges of T connecting two interior vertices of T. We let $\mathrm{Int}(E_T)$ denote the set of such edges of T and $\{\mathbf{h}_e\}_{e\in\mathrm{Int}(E_T)}$ the canonical basis of $\mathbb{Z}^{|\mathrm{Int}(E_T)|} \cong \mathbb{Z}^n$. The definitions we present in this section are reformulations of the definitions presented in [7].

Now, fix a facet $\mathcal{F} \in \Delta^{NC}(T)$ and a red or green arc $\gamma = (v_0, v_1 \ldots, v_t) \in \mathcal{F}$. By choosing an orientation of γ, we define $\mathbf{g}(\gamma) := \sum_{e\in\mathrm{Int}(E_T)} g^e_\gamma \mathbf{h}_e \in \mathbb{Z}^n$ where for each $e = (v_i, v_{i+1}) \in \mathrm{Int}(E_T)$ we set

$$g^e_\gamma := \begin{cases} 1 & \text{if } \gamma \text{ turns left at } v_i \text{ and right at } v_{i+1}, \\ -1 & \text{if } \gamma \text{ turns right at } v_i \text{ and left at } v_{i+1}, \\ 0 & \text{if } \gamma \text{ does not change direction from } v_i \\ & \text{to } v_{i+1} \text{ or if } e \text{ is not an edge in } \gamma, \end{cases}$$

and we refer to $\mathbf{g}(\gamma)$ as the **g-vector** of γ (see Figure 3). Observe that $\mathbf{g}(\gamma)$ is independent of the choice of orientation of γ. We define the **zigzag** of γ to be the set $Z_\gamma = Z^+_\gamma \sqcup Z^-_\gamma \subset \mathrm{Int}(E_T)$ of edges e of T such that $g^e_\gamma \neq 0$, where Z^+_γ (resp., Z^-_γ) is the set of edges such that $g^e_\gamma = 1$ (resp., $g^e_\gamma = -1$). We also let $G(\mathcal{F}) := \{\mathbf{g}(\gamma)\}_{\gamma\in\mathcal{F}^{red}\sqcup\mathcal{F}^{gr}}$.

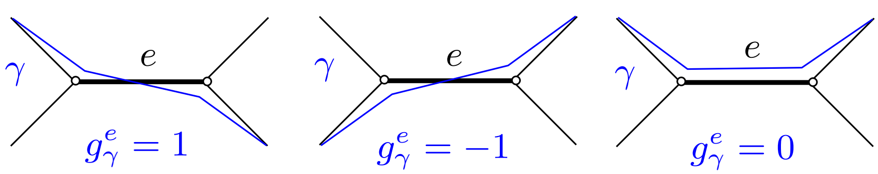

Figure 3: Different values for g^e_γ

Next, we let $s_{\gamma,\mathcal{F}} = [v,u]$ denote the acyclic path where (u,F) and (v,G) are the corners at which γ is marked in $\mathcal{F}$. We define the **c-vector** of γ with respect to $\mathcal{F}$ to be $\mathbf{c}_{\mathcal{F}}(\gamma) := \sum_{e\in s_{\gamma,\mathcal{F}}} \mathbf{h}_e \in \mathbb{Z}^n$ (resp., $\mathbf{c}_{\mathcal{F}}(\gamma) := -\sum_{e\in s_{\gamma,\mathcal{F}}} \mathbf{h}_e \in \mathbb{Z}^n$) if γ is green (resp., red). Note that the $\mathbf{c}$-vector of γ depends on the choice of facet $\mathcal{F}$ containing γ, whereas the $\mathbf{g}$-vector $\mathbf{g}(\gamma)$ is intrinsic to γ. We also let $C(\mathcal{F}) := \{\mathbf{c}_{\mathcal{F}}(\gamma)\}_{\gamma\in\mathcal{F}^{red}\sqcup\mathcal{F}^{gr}}$. As the following proposition shows, the $\mathbf{c}$- and $\mathbf{g}$-vectors defined by a given facet are dual bases of $\mathbb{R}^n$.

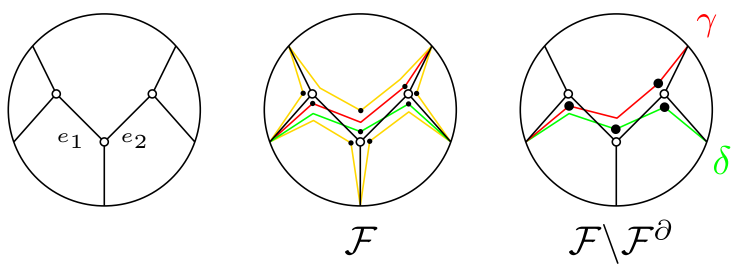

Figure 4: The tree and facet $\mathcal{F}$ of $\Delta^{NC}(T)$ from Example 3.2.

Proposition 3.1. *[7, Proposition 20] For any $\gamma, \delta \in \mathcal{F}$ we have $\langle \mathbf{g}(\delta), \mathbf{c}_{\mathcal{F}}(\gamma)\rangle \in \{0,1\}$ and equals 1 if and only if $\gamma = \delta$.*

Example 3.2. *Consider the tree in Figure 4 where $Int(E_T) = \{e_1, e_2\}$. The **g**- and **c**- vectors associated to the facet in this figure are as follows:*

$$\begin{array}{ll} \mathbf{g}(\gamma) = (-1,0) & \mathbf{c}_{\mathcal{F}}(\gamma) = (-1,-1) \\ \mathbf{g}(\delta) = (-1,1) & \mathbf{c}_{\mathcal{F}}(\delta) = (0,1). \end{array}$$

We end this section with a lemma that we interpret representation theoretically in the next section. For $s_{\gamma,\mathcal{F}} = (v_0, \ldots, v_t)$, let $C_{s_{\gamma,\mathcal{F}}}$ denote the set of acyclic paths $(v_i, \ldots, v_j)$ such that

- if $i > 0$ then s turns right at v_i, and
- if $j < l$ then s turns left at v_j.

One has that $C_{s_{\gamma,\mathcal{F}}} \backslash \{s_{\gamma,\mathcal{F}}\}$ is non-empty if and only if $s_{\gamma,\mathcal{F}}$ contains at least two edges.

Lemma 3.3. *Let $\mathcal{F}$ be a facet of $\Delta^{NC}(T)$ with at least one green arc, and let $\gamma \in \mathcal{F}^{red}$ be a red arc such that $s_{\gamma,\mathcal{F}}$ contains at least two edges of T. Then there exists a green arc $\mu \in \mathcal{F}^{gr}$ such that $|Z_\mu^- \cap \{\text{edges of } t\}| = |Z_\mu^+ \cap \{\text{edges of } t\}| + 1$ for any $t \in C_{s_{\gamma,\mathcal{F}}} \backslash \{s_{\gamma,\mathcal{F}}\}$. Moreover, for any green arc μ, $|Z_\mu^- \cap \{\text{edges of } t\}| \geq |Z_\mu^+ \cap \{\text{edges of } t\}|$.*

4 Tiling algebras

We now recall how a tree gives rise to a finite dimensional algebra. Given a tree T, let Q_T be the quiver whose vertex set is $\text{Int}(E_T)$ and where $e, e' \in \text{Int}(E_T)$ are connected by an arrow in Q_T if they meet in a corner of T. In this case, $e \xrightarrow{\alpha} e'$ if and only if e' is counter-clockwise from e. We define $I_T \subset \Bbbk Q_T$ to be the ideal generated by the relations $\alpha\beta$ where $\alpha : e_2 \to e_3$ defines the corner (v, F) and $\beta : e_1 \to e_2$ defines the corner (v, G).

We define the **tiling algebra** of T to be $\Lambda_T := \Bbbk Q_T / I_T$ where $\Bbbk$ is an algebraically closed field. We invite the reader to check that $\dim_{\Bbbk} \Lambda_T = 3$ (resp., 25) when T is the tree from Figure 4 (resp., Figure 1).

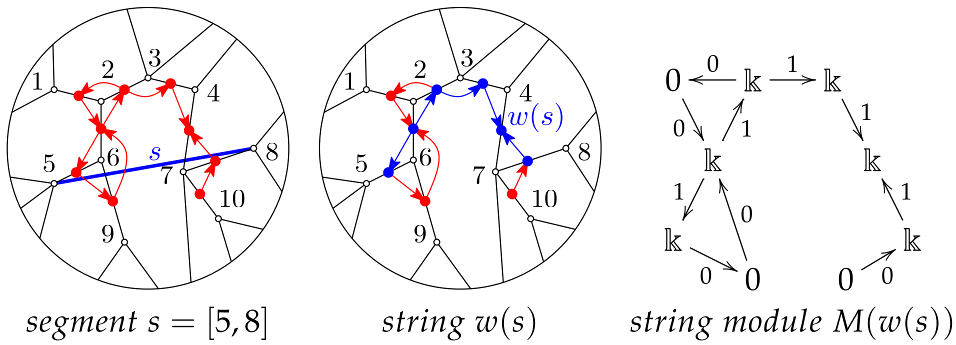

Figure 5: A segment $s = [5, 8]$, the string $w(s)$ that it defines, and the corresponding string module $M(w(s))$.

The category of finitely generated left modules over Λ_T is equivalent to category of finite dimensional representations of Q_T over $\Bbbk$ that are **compatible** with the relations from I_T (i.e., a representation $V = ((V_i)_{i \in Q_{T0}}, (\varphi_\alpha)_{\alpha \in Q_{T1}})$ of Q_T where $\varphi_\alpha \varphi_\beta = 0$ for all $\alpha\beta \in I_T$).[1] We also know that the indecomposable Λ_T-modules are **string modules**, denoted $M(w)$, which was first observed in [2, Proposition 3.2]. A **string** $w = w_1 \stackrel{\alpha_1}{\longleftrightarrow} w_2 \stackrel{\alpha_2}{\longleftrightarrow} \cdots \stackrel{\alpha_{k-1}}{\longleftrightarrow} w_k$ is an irredundant walk in Q_T where for any $i \in \{1, \ldots, k-1\}$ if $\alpha_i \alpha_{i+1} \neq 0$ in $\Bbbk Q_T$, then $\alpha_i \alpha_{i+1} \notin I_T$. In other words, a string is an irredundant walk in Q_T that obeys the relations in I_T. In the setting of tiling algebras, all strings are supported on connected acyclic subgraphs of Q_T, but this is not the case in general. The string module $M(w)$ is the representation of Q_T obtained by assigning the vector space $\Bbbk$ to each vertex in the string w and identity morphisms to each arrow in w.[2]

Using these facts, we obtain that the indecomposable Λ_T-modules are parameterized by **segments** of T (i.e., acyclic paths $s = (v_0, v_1, ..., v_t)$ whose endpoints are interior vertices of T and any two edges (v_{i-1}, v_i) and (v_i, v_{i+1}) are incident to a common face) [3, Corollary 4.3]. We show an example of this bijection in Figure 5. Using this bijection, one obtains the following lemma.

Lemma 4.1. *Given a facet $\mathcal{F}$ of $\Delta^{NC}(T)$ and $\gamma \in \mathcal{F}$, the set map $C_{s_{\gamma,\mathcal{F}}} \to mod(\Lambda_T)$ defined by $t \mapsto M(w(t))$ induces a bijection from $C_{s_{\gamma,\mathcal{F}}}$ to the indecomposable submodules of $M(w(s_{\gamma,\mathcal{F}}))$.*

It follows from this and Lemma 3.3 that for fixed facet $\mathcal{F}$, any red arc $\gamma \in \mathcal{F}$ satisfies $\theta_{\mathcal{F}}(M(w(s_{\gamma,\mathcal{F}}))) = 0$ and $\theta_{\mathcal{F}}(M(w(t))) < 0$ for any indecomposable submodule $M(w(t))$ of $M(w(s_{\gamma,\mathcal{F}}))$. That is, $M(w(s_{\gamma,\mathcal{F}}))$ is a $\theta_{\mathcal{F}}$-stable object. On the other hand,

[1]For a general finite dimensional $\Bbbk$-algebra $\Lambda = \Bbbk Q/I$ where I is an admissible ideal, one can also equivalently describe modules over Λ as representations of Q compatible with I.

[2]For simplicity, we have given the definition of $M(w)$ only in the generality of tiling algebras.

any green arc $\gamma \in \mathcal{F}$ satisfies $\theta_{\mathcal{F}}(M(w(s_{\gamma,\mathcal{F}}))) = 1$, which means that no indecomposable module coming from a green arc is $\theta_{\mathcal{F}}$-semistable.

5 Noncrossing tree partitions

Now let V° denote the set of interior vertices of T, choose an ϵ so that the balls of radius ϵ around each vertex in V° do not intersect each other and are contained in D^2. For each corner (v, F), fix a point $z(v, F)$ in the interior of F and such that $d(z(v, F), v) = \epsilon$. Let (v, e, F) and (u, e', G) be two green (resp., red) flags, a **green** (resp., **red**) admissible curve for $[v, u]$ is a simple curve $\sigma : [0, 1] \to D^2 \backslash V$ for which $\sigma(0) = z(v, F)$, $\sigma(1) = z(u, G)$ and $\sigma([0, 1])$ crosses an edge h of T only if h is an edge in $[v, u]$.[3] Two segments are **noncrossing** if they admit admissible curves that do not intersect each other, otherwise they are **crossing**. A segment is **green** (resp., **red**) if it is represented by a green (resp., red) admissible curve. For $B \subseteq V^\circ$, let $\mathrm{Seg}_r(B)$ be the set of inclusion-minimal red segments whose endpoints lie in B. A **noncrossing tree partition** $\mathbf{B} = \{B_1, B_2, ..., B_\ell\}$ is a set partition of V° such that any two segments of $\mathrm{Seg}_r(\mathbf{B}) = \cup_{i=1}^{l} \mathrm{Seg}_r(B_i)$ are noncrossing and each block of $\mathbf{B}$ is **segment-connected** (i.e., for any two vertices in B_i there exists a sequence of segments in $\mathrm{Seg}_r(B_i)$ that joins them). Let NCP(T) be the poset of noncrossing tree partitions of T ordered by refinement.

Returning to tiling algebras, a full, additive subcategory $\mathcal{W} \subset \mathrm{mod}(\Lambda_T)$ is a **wide subcategory** if it is abelian and for any short exact sequence $0 \to X \to Z \to Y \to 0$ with $X, Y \in \mathcal{W}$, one has that $Z \in \mathcal{W}$. Let $\mathrm{wide}(\Lambda_T)$ denote the poset of wide subcategories of $\mathrm{mod}(\Lambda_T)$ ordered by inclusion. The intersection of two wide subcategories is a wide subcategory, and the zero subcategory (resp., $\mathrm{mod}(\Lambda_T)$) is the bottom (resp., top) element in $\mathrm{wide}(\Lambda_T)$. As Λ_T is representation-finite, the poset $\mathrm{wide}(\Lambda_T)$ is a lattice. In [3, Theorem 7.1], the second author and McConville obtained a poset isomorphism between the lattice of noncrossing tree partitions and the lattice of wide subcategories given by

$$\begin{array}{rcl} \phi : \mathrm{NCP}(T) & \longrightarrow & \mathrm{wide}(\Lambda_T) \\ \mathbf{B} & \longmapsto & \mathrm{add}\left(\oplus_{w(s)} M(w(s)) \mid s \in \overline{\mathrm{Seg}(\mathbf{B})}\right) \end{array}$$

where $\overline{\mathrm{Seg}(\mathbf{B})} \subset \mathrm{Seg}(T)$ is the smallest set containing $\mathrm{Seg}(\mathbf{B})$ such that $s = (v_0, v_1, \ldots, v_k)$, $t = (v_k, v_{k+1} \ldots, v_\ell) \in \mathrm{Seg}(\mathbf{B})$ and $s \circ t := (v_0, v_1 \ldots, v_k, v_{k+1}, \ldots, v_\ell) \in \mathrm{Seg}(T)$ implies $s \circ t \in \overline{\mathrm{Seg}(\mathbf{B})}$. Here $\mathrm{add}(M)$ where $M \in \mathrm{mod}(\Lambda_T)$ is defined as the smallest full, additive subcategory of $\mathrm{mod}(\Lambda_T)$ that contains M and is closed under direct summands. We show in Figure 6 an example of this isomorphism.

Now fix a facet $\mathcal{F} \in \Delta^{NC}(T)$. For each $\delta \in \mathcal{F} \backslash \mathcal{F}^\partial$, let σ be an admissible curve for $s_{\delta,\mathcal{F}} = [v, u]$ where v and u are the two vertices corresponding to the corners (v, F) and

[3] Up to coloring-preserving isotopy relative to $z(v, F)$ and $z(u, G)$, there is a unique green (resp., red) admissible curve for $[v, u]$.

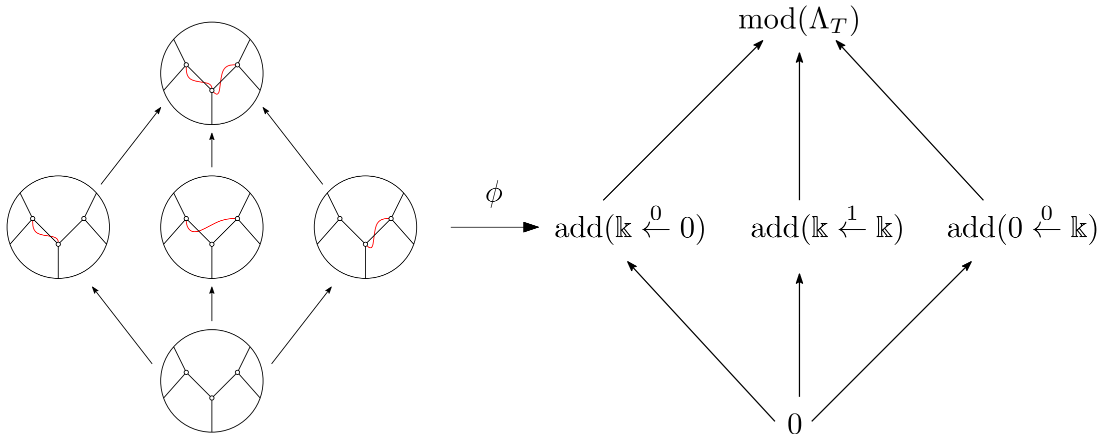

Figure 6: The lattice of wide subcategories of $\mathrm{mod}(\Lambda_T)$ and its corresponding lattice of noncrossing tree partitions, which we show realized as sets of red admissible curves.

(u, G) where δ is marked so that σ has the same color as δ. We define $\psi_r(\mathcal{F})$ (resp., $\psi_g(\mathcal{F})$) to be the set partition associated to set of red (resp., green) admissible curves only (see Figure 7). The map ψ_r is a bijection between the facets of $\Delta^{NC}(T)$ and $\mathrm{NCP}(T)$. In fact, it induces a bijection $\mathrm{Kr} : \mathrm{NCP}(T) \to \mathrm{NCP}(T)$ defined by $\mathrm{Kr}(\psi_r(\mathcal{F})) := \psi_g(\mathcal{F})$. We refer to $\mathrm{Kr}(\mathbf{B})$ with $\mathbf{B} \in \mathrm{NCP}(T)$ as the **Kreweras complement** of $\mathbf{B}$. We remark that using the map ϕ, we can regard the Kreweras complementation map as a cyclic action on wide subcategories of $\mathrm{mod}(\Lambda_T)$.

In [4, Theorem 5.11], it is shown that given a facet $\mathcal{F}$, the union of the sets $S_r = \{\text{Red curves associated to } \mathcal{F}\}$ and $S_g = \{\text{Green curves associated to } \mathcal{F}\}$ define a new tree $\mathcal{T}_{\mathcal{F}}$, that we call the **red-green tree**, whose vertex set is V^o and whose edge set is $S_r \sqcup S_g$. The red-green tree $\mathcal{T}_{\mathcal{F}}$ allows us to evaluate $\theta_{\mathcal{F}}$ on any indecomposable module.

Lemma 5.1. *For any facet $\mathcal{F}$, the set $\{\boldsymbol{dim}(M(w(s_{\gamma,\mathcal{F}}))) \mid \gamma \in \mathcal{F}\}$ is obtained by taking absolute values of the entries of all **c**-vectors in $C(\mathcal{F})$. Thus, this set is a basis of $\mathbb{Z}^{|Int(E_T)|}$.* [4]

With Lemma 5.1 in mind, we show how to express the dimension vector of any indecomposable Λ_T-module as a linear combination of elements of $\{\mathbf{dim}(M(w(s_{\gamma,\mathcal{F}})))| \gamma \in \mathcal{F}\}$. Let $s = [v, u]$ be a segment in T and $\varsigma = (\sigma_1, ..., \sigma_\ell)$ the shortest sequence of admissible curves in $\mathcal{T}_{\mathcal{F}}$ joining u and v where σ_i and σ_j with $i < j$ share an endpoint only if $j = i + 1$. By abuse of notation, we identify admissible curves with the segments associated to them. Writing $s = (v_0, v_1, \ldots, v_k)$ with $v = v_0$ and $u = v_k$, let $(v_0, v_{i_1}, \ldots, v_{i_r})$ with $u = v_{i_r}$ denote the vertices in s that are endpoints of the curves in ς.

Next, consider the subsequence $(\sigma_1, \ldots, \sigma_{i_1})$ of ς where σ_{i_1} is the first admissible curve in ς that has v_{i_1} as an endpoint, and suppose $\sigma_i = [e, g]$ and $\sigma_{i+1} = [f, g]$ are two admissi-

[4] We can rephrase this lemma by saying that this set generates the Grothendieck group of $\mathrm{mod}(\Lambda_T)$.

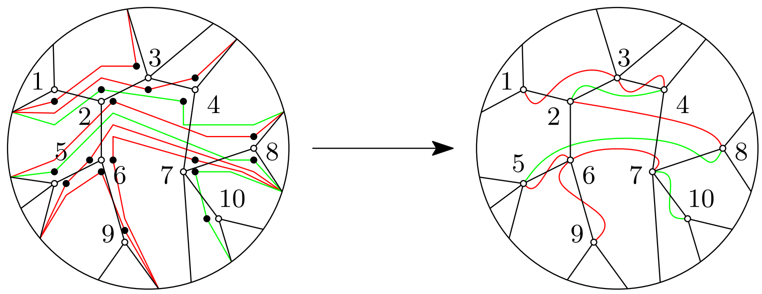

Figure 7: We show the red-green tree corresponding to a facet of $\Delta^{NC}(T)$. We do not show the boundary arcs of this facet. Here $\psi_r(\mathcal{F}) = \{\{1,3,4\},\{2,8\},\{5,6,7,9\},\{10\}\}$ and $\mathrm{Kr}(\psi_r(\mathcal{F})) = \psi_g(\mathcal{F}) = \{\{1\},\{2,4\},\{3\},\{5,8\},\{6\},\{7,10\},\{9\}\}$.

ble curves in this sequence with a common subsegment. As shown in [3, Section 5], they must be different colors and agree along the shorter of the two segments. Without loss of generality suppose that $[f,g]$ is the shorter one. Construct a new admissible curve σ' for the segment $[e,f]$. The color of σ' is that of the admissible curve with longest associated segment. Note that $\mathbf{dim}(M(w(\sigma'))) = \mathbf{dim}(M(w(\sigma_i))) - \mathbf{dim}(M(w(\sigma_{i+1})))$. Replace $(\sigma_1,\ldots,\sigma_{i_1})$ with the sequence of admissible curves $(\sigma_1,...,\sigma_{i-1},\sigma',\sigma_{i+2},...,\sigma_{i_1})$ that also joins u with v.

Alternatively, suppose σ_i and σ_{i+1} are two admissible curves of the same color. Replace $(\sigma_1,\ldots,\sigma_{i_1})$ with the sequence $(\sigma_1,...,\sigma_{i-1},\sigma'',\sigma_{i+2},...,\sigma_{i_1})$ where σ'' is the admissible curve with the same color as σ_i and whose segment satisfies $\mathbf{dim}(M(w(\sigma''))) = \mathbf{dim}(M(w(\sigma_i))) + \mathbf{dim}(M(w(\sigma_{i+1})))$.

We repeatedly apply these operations to obtain a sequence consisting of a single admissible curve σ_1' connecting v_0 and v_{i_1}. Now apply the same process to the remaining sequences $(\sigma_{i_j},\ldots,\sigma_{i_{j+1}})$ to obtain admissible curves σ_{j+1}' connecting v_{i_j} and $v_{i_{j+1}}$.

Definition 5.2. *We define the* ***simple red-green path*** *of s in $\mathcal{T}_{\mathcal{F}}$ to be the resulting sequence $\varsigma_{\mathcal{F}}(s) = (\sigma_1',...,\sigma_r')$ of red and green admissible curves where σ_j' is an admissible curve for $[v_{i_{j-1}},v_{i_j}]$. An example of this construction is shown in Figure 8.*

Proposition 5.3. *Let $\mathcal{F}$ be a facet and let $s \in Seg(T)$. Then $\theta_{\mathcal{F}}(M(w(\sigma'))) \leq 0$ (resp., $\theta_{\mathcal{F}}(M(w(\sigma'))) > 0$) for any red (resp., green) admissible curve σ' in $\varsigma_{\mathcal{F}}(s)$.*

We can now show that if $\theta_{\mathcal{F}}(M(w(s))) = 0$ and $s \neq s_{\gamma,\mathcal{F}}$ for any arc $\gamma \in \mathcal{F}$, then its simple red-green path must contain at least two curves of different color. In addition, we can show that any segment $t = [v_{i_{j-1}},v_{i_j}] \circ \cdots \circ [v_{i_{k-1}},v_{i_k}]$ associated to a maximal subsequence of $\varsigma_{\mathcal{F}}(s)$ of consecutive green admissible curves $(\sigma_j',...,\sigma_k')$ has $t \in C_s$. That is, $M(w(t))$ is a submodule of $M(w(s))$. By Proposition 5.3, $M(w(s))$ is not a semistable representation since $\theta_{\mathcal{F}}(M(w(t))) = \sum_{i=1}^{l} \theta_{\mathcal{F}}(M(w(\sigma_i'))) > 0$. This proves Theorem 1.1.

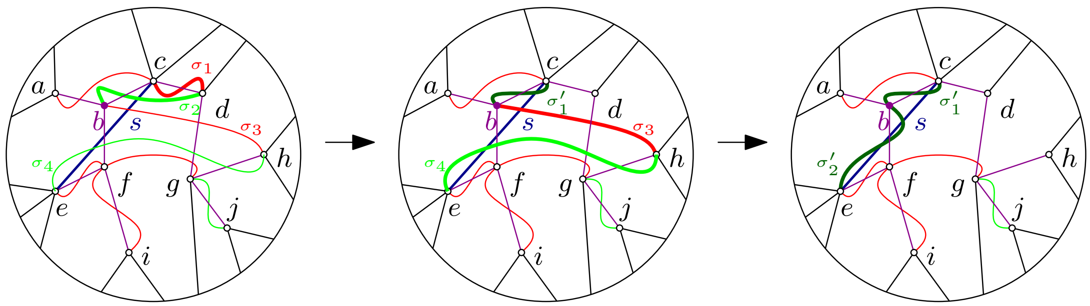

Figure 8: The simple red-green path for the segment $s = [c, e]$ is (σ_1', σ_2'). Here $\mathbf{dim}(M(w(s))) = \mathbf{dim}(M(w(\sigma_1'))) + \mathbf{dim}(M(w(\sigma_2'))) = [\mathbf{dim}(M(w(\sigma_2))) - \mathbf{dim}(M(w(\sigma_1)))] + [\mathbf{dim}(M(w(\sigma_4))) - \mathbf{dim}(M(w(\sigma_3)))]$.

Remark 5.4. *If T is a tree all of whose interior vertices have degree 3 and has no subconfiguration of the form shown in Figure 9, then Q_T is a type A Dynkin quiver. Theorem 1.1 therefore recovers Ingalls' and Thomas' bijection in [5, Theorem 1] between wide subcategories and semistable subcategories in type A.*

Remark 5.5. *The assertion in Theorem 1.1 that all wide subcategories of $\mathrm{mod}(\Lambda_T)$ are semistable can be deduced from [11, Theorem 1.2]. Our work differs from that of Yurikusa in that we have found a canonical choice of stability condition realizing a wide subcategory as a semistable subcategory.*

From Theorem 1.1 and [3, Theorem 6.23] it follows that the poset of semistable subcategories of $\mathrm{mod}(\Lambda_T)$ is isomorphic to the lattice of noncrossing tree partitions of T as the following corollary shows. In particular, we obtain a combinatorial classification of the semistable subcategories of $\mathrm{mod}(\Lambda_T)$.

Corollary 5.6. *For any tree T, the map $\phi : NCP(T) \to \Lambda_T^{ss}$ is an isomorphism of posets.*

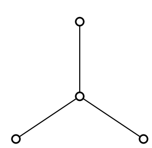

Figure 9: The forbidden subconfigurations from Remark 5.4.

6 Additional questions

A crucial step in proving Theorem 1.1 was the use of the combinatorics of the red-green tree $\mathcal{T}_{\mathcal{F}}$ to evaluate $\theta_{\mathcal{F}}$ on any indecomposable Λ_T-module. A second crucial step in the proof was the fact that for any facet $\mathcal{F}$ of the noncrossing complex, the $\mathbf{g}$-vectors in $G(\mathcal{F})$ and the **c**-vectors in $C(\mathcal{F})$ are dual bases with respect to $\langle -, - \rangle$. This fact has already been established for general **gentle algebras** of which tiling algebras are examples (see [8, Proposition 4.16]). There the role of the noncrossing complex is played by the **blossoming complex**. However, we are not aware of a notion of the red-green tree associated to a facet of this complex. We propose the following problem.

Problem 6.1. *Find a combinatorial description of the Kreweras complement $Kr : wide(\Lambda) \to wide(\Lambda)$ where Λ is a gentle algebra, and use this to determine when the Kreweras stability condition $\theta_{\mathcal{F}_\mathcal{W}}$ satisfies $\theta^{ss}_{\mathcal{F}_\mathcal{W}} = \mathcal{W}$ for all facets $\mathcal{F}_\mathcal{W}$ of the blossoming complex.*

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