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The totally nonnegative Grassmannian is a ball

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Abstract. We prove that three spaces of importance in topological combinatorics are homeomorphic to closed balls: the totally nonnegative Grassmannian, the compactification of the space of electrical networks, and the cyclically symmetric amplituhedron.

Résumé. Nous montrons que trois espaces d'importance en combinatoire topologique sont homéomorphes à des boules fermées: la grassmannienne totalement non négative, la compactification de l'espace des réseaux électriques et l'amplituèdre cycliquement symétrique.

Keywords: total positivity, Grassmannian, unipotent group, amplituhedron, electrical networks

1 Introduction

The prototypical example of a closed ball of interest in topological combinatorics is a convex polytope. Over the past couple of decades, an analogy between convex polytopes, and certain spaces appearing in total positivity and in electrical resistor networks, has been developed. One motivation for this analogy is that these latter spaces come equipped with cell decompositions whose face posets share a number of common features with the face posets of polytopes. A new motivation for this analogy comes from recent developments in high-energy physics, where physical significance is ascribed to certain differential forms on positive spaces which generalize convex polytopes. In this paper we show in several fundamental cases that this analogy holds at the topological level: the spaces themselves are closed balls.

The totally nonnegative Grassmannian. Let $\mathrm{Gr}_{\mathbb{R}}(k,n)$ denote the real Grassmannian of k-planes in $\mathbb{R}^n$. Postnikov [25] introduced the *totally nonnegative Grassmannian* $\mathrm{Gr}_{\geq 0}(k,n)$ as the set of $X \in \mathrm{Gr}_{\mathbb{R}}(k,n)$ whose Plücker coordinates are all nonnegative. The space

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$\mathrm{Gr}_{\geq 0}(k,n)$ is not a polytope, but Postnikov conjectured that it is the 'next best thing', namely a regular CW complex homeomorphic to a closed ball. He found a cell decomposition of $\mathrm{Gr}_{\geq 0}(k,n)$, where each cell is specified by requiring some subset of the Plücker coordinates to be strictly positive, and requiring the rest to equal zero.

Over the past decade, much work has been done towards Postnikov's conjecture. The face poset of the cell decomposition (described in [25, 27]) was shown to be shellable by Williams [32]. Postnikov, Speyer, and Williams [26] showed that the cell decomposition is a CW complex, and Rietsch and Williams [28] showed that the CW complex is regular up to homotopy (i.e. every cell closure is contractible). Our first main theorem is:

Theorem 1.1. *The space* $\mathrm{Gr}_{\geq 0}(k,n)$ *is homeomorphic to a* $k(n-k)$*-dimensional closed ball.*

It remains an open problem to establish Postnikov's conjecture, i.e. to address arbitrary cell closures in the cell decomposition of $\mathrm{Gr}_{\geq 0}(k,n)$. Each of Postnikov's cells determines a matroid known as a *positroid*, and Theorem 1.1 also reflects how positroids are related via specialization (see [1] for a related discussion about oriented matroids).

Separately, Lusztig [23] defined and studied the totally nonnegative part $(G/P)_{\geq 0}$ of a partial flag variety of a split real reductive group G. In the case $G/P = \mathrm{Gr}_{\mathbb{R}}(k,n)$, Rietsch showed that Lusztig's and Postnikov's definitions of the totally nonnegative part are the same (see e.g. [20, Remark 3.8] for a proof). Lusztig [22] showed that $(G/P)_{\geq 0}$ is contractible, and our approach to Theorem 1.1 is similar to his. We will return to $(G/P)_{\geq 0}$ in a separate work [11].

Our proof of Theorem 1.1 is based on a *cyclic shift vector field* S on $\mathrm{Gr}_{\geq 0}(k,n)$. It can be thought of as an affine (or loop group) analogue of Lusztig's flow, and is closely related to the whirl matrices of [21]. The flow defined by S contracts all of $\mathrm{Gr}_{\geq 0}(k,n)$ to a unique fixed point X_0. We construct a homeomorphism from $\mathrm{Gr}_{\geq 0}(k,n)$ to a closed ball $B \subset \mathrm{Gr}_{\geq 0}(k,n)$ centered at X_0, by mapping each trajectory in $\mathrm{Gr}_{\geq 0}(k,n)$ to its intersection with B. A feature of our construction is that we do not rely on any cell decomposition of the totally nonnegative Grassmannian.

The totally nonnegative part of the unipotent group. The interest in totally nonnegative spaces from the viewpoint of combinatorial topology dates at least back to Fomin and Shapiro [9]. Edelman [8] had shown that intervals in the poset formed by the symmetric group $\mathfrak{S}_n$ with Bruhat order are shellable, whence Björner's results [6] imply that there exists a regular CW complex homeomorphic to a ball whose face poset is isomorphic to $\mathfrak{S}_n$. Fomin and Shapiro [9] suggested that such a CW complex could be found naturally occurring in the theory of total positivity. Namely, let $U \subset \mathrm{GL}_n(\mathbb{R})$ be the subgroup of all upper-triangular unipotent matrices, and $U_{\geq 0}$ its totally nonnegative part, where all minors are nonnegative. Let $V_{\geq 0}$ denote the subset of $U_{\geq 0}$ of matrices whose superdiagonal entries sum to 1. The intersection of $V_{\geq 0}$ with the Bruhat stratification of U induces a decomposition of $V_{\geq 0}$ into cells, whose face poset is isomorphic to $\mathfrak{S}_n$ (see

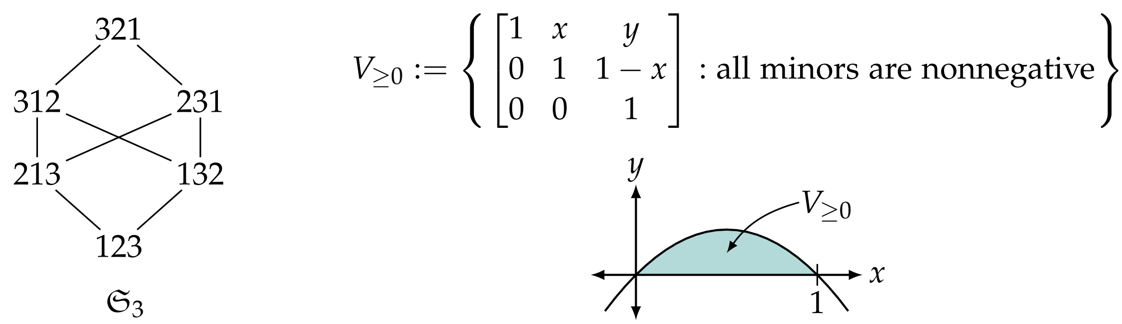

Figure 1: Bruhat order on $\mathfrak{S}_n$ is the face poset of $V_{\geq 0}$. In the case $n = 3$ shown here, $V_{\geq 0}$ is cut out by the nonnegativity of two minors: the 1×1 minor y and the 2×2 minor $x(1-x) - y$. It has one 2-cell, two edges, and two vertices.

Figure 1). Fomin and Shapiro conjectured that $V_{\geq 0}$ is a regular CW complex, which was proved by Hersh [13]. Applying her result to the cell of top dimension implies that $V_{\geq 0}$ is homeomorphic to an $\left(\binom{n}{2} - 1\right)$-dimensional closed ball. We discovered a new proof of this special case. We emphasize that our techniques in their present form are not able to address the other (lower-dimensional) cell closures in $V_{\geq 0}$, which appear in Hersh's result. In addition, Fomin and Shapiro's conjecture, as well as Hersh's theorem, hold in arbitrary Lie types, while we only consider type A.

The cyclically symmetric amplituhedron. A robust connection between the totally nonnegative Grassmannian and the physics of scattering amplitudes was developed in [5], which led Arkani-Hamed and Trnka [4] to define topological spaces called *amplituhedra*. A distinguishing feature that these topological spaces share (conjecturally) with convex polytopes is the existence of a canonical differential form [2]. This brings the analogy between totally nonnegative spaces and polytopes beyond the level of face posets.

Let k, m, n be nonnegative integers with $k + m \leq n$, and Z be a $(k+m) \times n$ matrix whose $(k+m) \times (k+m)$ minors are all positive. We regard Z as a linear map $\mathbb{R}^n \to \mathbb{R}^{k+m}$, which induces a map Z_{Gr} on $\mathrm{Gr}_{\mathbb{R}}(k, n)$ taking the subspace X to the subspace $\{Z(v) : v \in X\}$. The *(tree) amplituhedron* $\mathcal{A}_{n,k,m}(Z)$ is the image of $\mathrm{Gr}_{\geq 0}(k, n)$ in $\mathrm{Gr}(k, k+m)$ under the map Z_{Gr}. When $k = 1$, the totally nonnegative Grassmannian $\mathrm{Gr}_{\geq 0}(1, n)$ is a simplex in $\mathbb{RP}^{n-1}$, and the amplituhedron $\mathcal{A}_{n,1,m}(Z)$ is a *cyclic polytope* in $\mathbb{RP}^m$ [30]. Understanding the topology of amplituhedra, and more generally of *Grassmann polytopes* [20] (obtained by relaxing the positivity condition on Z), was one of the main motivations of our work.

We now take m to be even, and $Z = Z_0$ such that the rows of Z_0 span the unique element of $\mathrm{Gr}_{\geq 0}(k+m, n)$ invariant under $\mathbb{Z}/n\mathbb{Z}$-cyclic action (cf. [15]). We call $\mathcal{A}_{n,k,m}(Z_0)$ the *cyclically symmetric amplituhedron*. When $k = 1$ and $m = 2$, $\mathcal{A}_{n,1,2}(Z_0)$ is a regular n-gon in the plane. More generally, $\mathcal{A}_{n,1,m}(Z_0)$ is a polytope whose vertices are n regularly spaced points on the *trigonometric moment curve* in $\mathbb{RP}^m$.

Theorem 1.2. *The cyclically symmetric amplituhedron $\mathcal{A}_{n,k,m}(Z_0)$ is homeomorphic to a km-dimensional closed ball.*

It is expected that every amplituhedron is homeomorphic to a closed ball. The topology of amplituhedra and Grassmann polytopes is not well understood in general; see [16, 3] for recent work.

The compactification of the space of planar electrical networks. Let Γ be an electrical network consisting only of resistors, modeled as an undirected graph whose edge weights (conductances) are positive real numbers. The electrical properties of Γ are encoded by the response matrix $\Lambda(\Gamma) : \mathbb{R}^n \to \mathbb{R}^n$, sending a vector of voltages at n distinguished boundary vertices to the vector of currents induced at the same vertices. The response matrix can be computed using (only) Kirchhoff's law and Ohm's law. Following Curtis, Ingerman, and Morrow [7] and Colin de Verdière, Gitler, and Vertigan [31], we consider the space of response matrices of planar electrical networks: those Γ embedded into a disk, with boundary vertices on the boundary of the disk. In [18], the third author defined a compactification E_n of this space. We may identify E_n with the intersection of $\mathrm{Gr}_{\geq 0}(n-1, 2n)$ (viewed as a subset of $\mathbb{P}^{\binom{n}{k}-1}$ under the Plücker embedding) and a certain linear subspace $\mathcal{H}$ of $\mathbb{P}^{\binom{n}{k}-1}$, spanned by vectors corresponding to noncrossing partitions of an n-element set. The subspace $\mathcal{H}$ is cyclically symmetric, which we exploit to prove the following theorem.

Theorem 1.3. *The space E_n is homeomorphic to an $\binom{n}{2}$-dimensional closed ball.*

A cell decomposition of E_n was defined in [18], extending earlier work in [7, 31]. The face poset of this cell decomposition had been defined and studied by Kenyon [17, Section 4.5.2]. Theorem 1.3 says that the closure of the unique cell of top dimension in E_n is homeomorphic to a closed ball. In [19], the third author showed that the face poset of the cell decomposition of E_n is Eulerian, and conjectured that it is shellable. Hersh and Kenyon recently proved this conjecture [14]. Björner's results [6] therefore imply that this poset is the face poset of some regular CW complex homeomorphic to a ball. We expect that E_n forms such a CW complex, so that the closure of every cell of E_n is homeomorphic to a closed ball. Proving this remains an open problem.

Outline. In Sections 2 to 4, we sketch the proof of Theorem 1.1, that $\mathrm{Gr}_{\geq 0}(k, n)$ is homeomorphic to a closed ball. The proofs for $V_{\geq 0}$, $\mathcal{A}_{n,k,m}(Z_0)$, and E_n are similar; see [10] for the details. In Section 5, we give examples in the case of $\mathrm{Gr}_{\geq 0}(1, 3)$ and $\mathrm{Gr}_{\geq 0}(2, 4)$.

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2 Global coordinates for $\mathrm{Gr}_{\geq 0}(k,n)$

Let $\mathrm{Gr}_{\mathbb{C}}(k,n)$ be the *complex Grassmannian* of all k-dimensional subspaces of $\mathbb{C}^n$. We set $[n] := \{1,2,\dots,n\}$, and let $\binom{[n]}{k}$ denote the set of k-element subsets of $[n]$. For $X \in \mathrm{Gr}_{\mathbb{C}}(k,n)$, we denote by $(\Delta_I(X))_{I \in \binom{[n]}{k}} \in \mathbb{CP}^{\binom{n}{k}-1}$ the *Plücker coordinates* of X: $\Delta_I(X)$ is the $k \times k$ minor of X (viewed as a $k \times n$ matrix modulo row operations) with column set I. We call the subspace $X \in \mathrm{Gr}_{\mathbb{C}}(k,n)$ *real* if X is closed under complex conjugation, or equivalently, if all Plücker coordinates of X are real (up to a common scalar). We regard the real Grassmannian $\mathrm{Gr}_{\mathbb{R}}(k,n)$ as the subset of $\mathrm{Gr}_{\mathbb{C}}(k,n)$ of real elements, endowed with the usual Euclidean topology. Recall that $\mathrm{Gr}_{\geq 0}(k,n)$ is the subset of $\mathrm{Gr}_{\mathbb{R}}(k,n)$ where all Plücker coordinates are nonnegative (up to a common scalar). We also define the *totally positive Grassmannian* $\mathrm{Gr}_{>0}(k,n)$ as the subset of $\mathrm{Gr}_{\geq 0}(k,n)$ where all Plücker coordinates are positive.

Let $R := \{\zeta \in \mathbb{C} : \zeta^n = (-1)^{k-1}\}$, and for $\zeta \in R$ let $v_\zeta := (1, \zeta, \zeta^2, \dots, \zeta^{n-1}) \in \mathbb{C}^n$. Let R_0 denote the set of k elements of R with greatest real part, and let $X_0 \in \mathrm{Gr}_{\mathbb{C}}(k,n)$ be the span of v_ζ for $\zeta \in R_0$. Then X_0 is totally positive: using Vandermonde's determinantal formula, we find that

$$\Delta_I(X_0) = \prod_{i,j \in I,\, i<j} \sin\left(\tfrac{j-i}{n}\pi\right) > 0 \quad \text{for all } I \in \tbinom{[n]}{k}. \tag{2.1}$$

(This result is essentially due to Scott [29]; see [15] for more details.)

Now let $M \simeq \mathbb{C}^{k(n-k)}$ denote the space of complex $k \times (n-k)$ matrices A with rows indexed by R_0 and columns indexed by $R \setminus R_0$. Define $\phi : M \to \mathrm{Gr}_{\mathbb{C}}(k,n)$ by

$$\phi(A) := \mathrm{span}(v_\zeta + \textstyle\sum_{\omega \in R \setminus R_0} A_{\zeta,\omega} v_\omega : \zeta \in R_0). \tag{2.2}$$

Note that $\phi(0) = X_0$. Also, ϕ is an embedding, and its image is the Schubert cell

$$\phi(M) = \{X \in \mathrm{Gr}_{\mathbb{C}}(k,n) : X \cap \mathrm{span}(v_\omega : \omega \in R \setminus R_0) = 0\}.$$

Proposition 2.1. *The image $\phi(M)$ contains $\mathrm{Gr}_{\geq 0}(k,n)$.*

Proof. Let $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ denote the inner product $\langle v, w \rangle := \sum_{j=1}^n v_j \overline{w_j}$, and for $X \subset \mathbb{C}^n$ let $X^\perp := \{v \in \mathbb{C}^n : \langle v, w \rangle = 0 \text{ for all } w \in X\}$ denote the subspace orthogonal to X.

Note that $\{v_\zeta : \zeta \in R\}$ is an orthogonal basis of $\mathbb{C}^n$, so $X_0^\perp = \operatorname{span}(v_\omega : \omega \in R \setminus R_0)$ and thus $\phi(M) = \{X \in \mathrm{Gr}_\mathbb{C}(k,n) : X \cap X_0^\perp = 0\}$. We now show that given $X \in \mathrm{Gr}_{\geq 0}(k,n)$, we have $X \cap X_0^\perp = 0$. Since X and X_0 are both real, it suffices to show that $X \cap X_0^\perp \cap \mathbb{R}^n = 0$. (Indeed, if $v \in X \cap X_0^\perp$ is nonzero, then $\mathrm{Re}(v) = \frac{v+\overline{v}}{2}$ and $\mathrm{Im}(v) = \frac{v-\overline{v}}{2i}$ are both in $X \cap X_0^\perp \cap \mathbb{R}^n$, and at least one of them is nonzero.) This follows from a general result of Gantmakher and Krein [12, Theorems V.3 and V.6]: for any $Y \in \mathrm{Gr}_\mathbb{R}(k,n)$, Y is totally nonnegative if and only if all vectors in $Y \cap \mathbb{R}^n$ change sign at most $k-1$ times, and Y is totally positive if and only if all nonzero vectors in $Y^\perp \cap \mathbb{R}^n$ change sign at least k times. (By definition, the number of sign changes of $(v_1, \dots, v_n) \in \mathbb{R}^n$ is the number of times the sequence $v_1, \dots, v_n$ changes sign, ignoring any zero entries.) □

Note that for $A \in M$, the element $\phi(A)$ of $\mathrm{Gr}_\mathbb{C}(k,n)$ is real if and only if $\overline{A_{\zeta,\omega}} = A_{\overline{\zeta},\overline{\omega}}$ for all $\zeta \in R_0$, $\omega \in R \setminus R_0$. Let $M_\mathbb{R} \simeq \mathbb{R}^{k(n-k)}$ denote the subset of M of matrices A satisfying this property. Then Proposition 2.1 implies that $\phi|_{M_\mathbb{R}} : M_\mathbb{R} \hookrightarrow \mathrm{Gr}_\mathbb{R}(k,n)$ is an embedding whose image contains $\mathrm{Gr}_{\geq 0}(k,n)$. See Section 5 for examples in the case of $\mathrm{Gr}_\mathbb{R}(1,3)$ and $\mathrm{Gr}_\mathbb{R}(2,4)$.

3 The cyclic shift S

For $g \in \mathrm{GL}_n(\mathbb{C})$, we let g act on $\mathrm{Gr}_\mathbb{C}(k,n)$ by taking the subspace X to $g \cdot X := \{g(v) : v \in X\}$. We let $1 \in \mathrm{GL}_n(\mathbb{C})$ denote the identity, and for $x \in \mathfrak{gl}_n(\mathbb{C}) = \mathrm{End}(\mathbb{C}^n)$ we let $\exp(x) := \sum_{j=0}^\infty \frac{x^j}{j!} \in \mathrm{GL}_n(\mathbb{C})$ denote the matrix exponential of x.

Define the *cyclic shift* $S \in \mathfrak{gl}_n(\mathbb{C})$ by $S(v_1, \dots, v_n) := (v_2, \dots, v_n, (-1)^{k-1}v_1)$. We examine the action of $\exp(tS)$ on $\mathrm{Gr}_\mathbb{C}(k,n)$.

Lemma 3.1. *Let $X \in \mathrm{Gr}_{\geq 0}(k,n)$. Then $\exp(tS) \cdot X \in \mathrm{Gr}_{>0}(k,n)$ for all $t > 0$.*

Proof. We will make use of the operator $1 + tS$, which belongs to $\mathrm{GL}_n(\mathbb{C})$ for $|t| < 1$. The Plücker coordinates of $(1+tS) \cdot X$ are

$$\Delta_I((1+tS)\cdot X) = \sum_{\epsilon \in \{0,1\}^k} t^{\epsilon_1 + \cdots + \epsilon_k} \Delta_{\{i_1+\epsilon_1, \dots, i_k+\epsilon_k\}}(X) \quad \text{for } I = \{i_1, \dots, i_k\} \subset [n],$$

where $i_1 + \epsilon_1, \dots, i_k + \epsilon_k$ are taken modulo n. Therefore $(1+tS) \cdot X \in \mathrm{Gr}_{\geq 0}(k,n)$ for $t \in [0,1)$. Since $\exp(tS) = \lim_{j\to\infty}\left(1 + \frac{tS}{j}\right)^j$, we get $\exp(tS) \cdot X \in \mathrm{Gr}_{\geq 0}(k,n)$ for $t \geq 0$.

Now we must show that $X' := \exp(t_0 S) \cdot X \in \mathrm{Gr}_{>0}(k,n)$ for any $t_0 > 0$. Suppose otherwise that $X' \in \mathrm{Gr}_{\geq 0}(k,n) \setminus \mathrm{Gr}_{>0}(k,n)$. We will show that $\exp(tS) \cdot X' \notin \mathrm{Gr}_{\geq 0}(k,n)$ for all $t < 0$ sufficiently close to 0. (Since $\exp(tS) \cdot X' = \exp((t_0 + t)S) \cdot X$, this contradicts the conclusion of the previous paragraph.) From $\exp(tS) = 1 + tS + O(t^2)$, we obtain

$$\Delta_I(\exp(tS) \cdot X') = \Delta_I(X') + t \sum_{I'} \Delta_{I'}(X') + O(t^2) \quad \text{for } I \in \tbinom{[n]}{k},$$

where the sum is over all $I' \in \binom{[n]}{k}$ obtained from I by increasing exactly one element by 1 modulo n. If we can find such I and I' with $\Delta_I(X') = 0$ and $\Delta_{I'}(X') > 0$, then $\Delta_I(\exp(tS) \cdot X') < 0$ for all $t < 0$ sufficiently close to zero. In order to do this, we introduce the directed graph D with vertex set $\binom{[n]}{k}$, where $J \to J'$ is an edge of D if and only if we can obtain J' from J by increasing exactly one element by 1 modulo n. Note that for every pair of vertices K and K' of D, there exists a directed path from K to K':

- we can get from $[k]$ to any $\{i_1 < \cdots < i_k\}$ by shifting k to i_k, $k-1$ to i_{k-1}, etc.;
- similarly, we can get from any $\{i_1 < \cdots < i_k\}$ to $\{n-k+1, n-k+2, \ldots, n\}$;
- we can get from $\{n-k+1, \ldots, n\}$ to $[k]$ by shifting n to k, $n-1$ to $k-1$, etc.

Take $K, K' \in \binom{[n]}{k}$ with $\Delta_K(X') = 0$ and $\Delta_{K'}(X') > 0$, and consider a directed path from K to K'. It goes through an edge $I \to I'$ with $\Delta_I(X') = 0$ and $\Delta_{I'}(X') > 0$, as desired. $\square$

Let us see how $\exp(tS)$ acts on matrices $A \in M$. Note that $S(v_\zeta) = \zeta v_\zeta$ for $\zeta \in R$, so $\exp(tS)(v_\zeta) = e^{t\zeta} v_\zeta$. Therefore $\exp(tS)$ acts on the basis of $\phi(A)$ in (2.2) by

$$\exp(tS)(v_\zeta + \textstyle\sum_{\omega \in R \setminus R_0} A_{\zeta,\omega} v_\omega) = e^{t\zeta}(v_\zeta + \textstyle\sum_{\omega \in R \setminus R_0} e^{t(\omega-\zeta)} A_{\zeta,\omega} v_\omega)$$

for all $\zeta \in R_0$. Thus $\exp(tS) \cdot \phi(A) = \phi(\exp(tS) \cdot A)$, where by definition

$$(\exp(tS) \cdot A)_{\zeta,\omega} := e^{t(\omega-\zeta)} A_{\zeta,\omega}. \tag{3.1}$$

Note that the action of $\exp(tS)$ on M preserves $M_{\mathbb{R}}$.

Now let us show that for any $t > 0$, $\exp(tS)$ is a Lipschitz map on M, with respect to the L^2-norm $\|\cdot\|$ on M defined by $\|A\|^2 := \sum_{\zeta,\omega} |A_{\zeta,\omega}|^2$.

Lemma 3.2. *Define $C := e^{\cos(\frac{k-1}{n}\pi) - \cos(\frac{k+1}{n}\pi)} > 1$. Then $\|\exp(tS) \cdot A\| \le C^{-t}\|A\|$ for all $A \in M$ and $t \ge 0$. Equivalently, $\|\exp(tS) \cdot A\| \ge C^{-t}\|A\|$ for all $A \in M$ and $t \le 0$.*

Proof. Consider a matrix $A \in M$. Then for any $\zeta \in R_0$, $\omega \in R \setminus R_0$, and $t \ge 0$, we have

$$|(\exp(tS) \cdot A)_{\zeta,\omega}| = |e^{t(\omega-\zeta)} A_{\zeta,\omega}| = e^{-t\,\mathrm{Re}(\zeta-\omega)}|A_{\zeta,\omega}| \le C^{-t}|A_{\zeta,\omega}|.$$

Here we used the fact that $\mathrm{Re}(\zeta) \ge \cos\left(\frac{k-1}{n}\pi\right)$ and $\mathrm{Re}(\omega) \le \cos\left(\frac{k+1}{n}\pi\right)$. $\square$

4 Trajectories of $\exp(tS)$ and the proof of Theorem 1.1

Let $\Pi \subset M_{\mathbb{R}}$ be the preimage of $\mathrm{Gr}_{\ge 0}(k,n)$ under ϕ, so that $\phi|_\Pi$ gives a homeomorphism $\Pi \to \mathrm{Gr}_{\ge 0}(k,n)$. Since $\mathrm{Gr}_{\ge 0}(k,n)$ is a closed subspace of the projective space $\mathbb{RP}^{\binom{n}{k}-1}$, it is compact. Hence Π is also compact. Also recall that $\phi(0) = X_0 \in \mathrm{Gr}_{>0}(k,n)$, and $\mathrm{Gr}_{>0}(k,n)$ is open in $\mathrm{Gr}_{\mathbb{R}}(k,n)$, so $0 \in \mathrm{int}(\Pi)$.

For $r > 0$, we denote by B_r the closed ball in $M_{\mathbb{R}}$ of radius r in the L^2-norm centered at 0. Consider the curve $t \mapsto \exp(tS) \cdot A$ starting at any point $A \in M_{\mathbb{R}} \setminus \{0\}$. By Lemma 3.2, for any $r > 0$ this curve intersects the sphere ∂B_r for a unique t, which we denote by $t_r(A)$. We also claim that this curve $t \mapsto \exp(tS) \cdot A$ intersects the boundary $\partial\Pi$ for a unique t, which we will denote by $t_\partial(A)$. Indeed, let us consider the set $T := \{t \in \mathbb{R} : \exp(tS) \cdot A \in \Pi\}$. By Lemma 3.2, $t \in T$ for $t \gg 0$, and T is bounded from below. By Lemma 3.1, if $t \in T$ then $[t, \infty) \subset T$, and moreover $\exp(t'S) \cdot A \in \operatorname{int}(\Pi)$ for $t' > t$. Also, T is closed since it is the preimage of Π under the continuous map $t \mapsto \exp(tS) \cdot A$. Therefore T is an interval of the form $[t_0, \infty)$ for some t_0, and we set $t_\partial(x) := t_0$.

Lemma 4.1. *The functions t_r (for any $r > 0$) and t_∂ are continuous on $M_{\mathbb{R}} \setminus \{0\}$.*

Proof. Let us fix $r > 0$ and show that t_r is continuous. It suffices to show that the preimage of any open interval $I \subset \mathbb{R}$ is open. To this end, let $A \in t_r^{-1}(I)$. Take $t_1, t_2 \in I$ with $t_1 < t_r(A) < t_2$. By Lemma 3.2, we have $\|\exp(t_1 S) \cdot A\| > r > \|\exp(t_2 S) \cdot A\|$. Now note that the map $\gamma_1 : M_{\mathbb{R}} \to M_{\mathbb{R}}, A' \mapsto \exp(t_1 S) \cdot A'$ is continuous and $M_{\mathbb{R}} \setminus B_r$ is open, so $\gamma_1^{-1}(M_{\mathbb{R}} \setminus B_r)$ is an open neighborhood of A. Similarly, defining $\gamma_2 : M_{\mathbb{R}} \to M_{\mathbb{R}}, A' \mapsto \exp(t_2 S) \cdot A'$, we have that $\gamma_2^{-1}(\operatorname{int}(B_r))$ is an open neighborhood of A. Therefore $\gamma_1^{-1}(M_{\mathbb{R}} \setminus B_r) \cap \gamma_2^{-1}(\operatorname{int}(B_r))$ is an open neighborhood of A, whose image under t_r is contained in $(t_1, t_2) \subseteq I$. Thus $t_r^{-1}(I)$ contains an open neighborhood of A, for all $A \in t_r^{-1}(I)$.

The proof that t_∂ is continuous is very similar, where we replace B_r by Π. Then $\exp(t_1 S) \cdot A$ is contained in $M_{\mathbb{R}} \setminus \Pi$ (which is open), and $\exp(t_2 S) \cdot A$ is contained $\operatorname{int}(\Pi)$ (which is the preimage of $\operatorname{Gr}_{>0}(k,n)$ under ϕ, and hence is open because $\operatorname{Gr}_{>0}(k,n)$ is open in $\operatorname{Gr}_{\mathbb{R}}(k,n)$). □

Proof of Theorem 1.1. We show that Π is homeomorphic to a closed ball. Fix $r > 0$ such that $B_r \subset \Pi$, and define the maps $\alpha : \Pi \to B_r$, $\beta : B_r \to \Pi$ by

$$\alpha(A) := \exp((t_r(A) - t_\partial(A))S) \cdot A, \quad \beta(A) := \exp((t_\partial(A) - t_r(A))S) \cdot A$$

for $A \neq 0$, and $\alpha(0) := 0$, $\beta(0) := 0$. We claim that α and β are inverse homeomorphisms.

First let us show that α and β are inverse maps. Given $A \in \Pi \setminus \{0\}$, let $A' := \alpha(A) \in B_r$. Then $t_\partial(A') = 2t_\partial(A) - t_r(A)$ and $t_r(A') = t_\partial(A)$, so

$$\beta(A') = \exp(((2t_\partial(A) - t_r(A)) - t_\partial(A))S) \cdot (\exp((t_r(A) - t_\partial(A))S) \cdot A) = A.$$

Therefore $\beta \circ \alpha = \mathrm{id}_\Pi$. We can verify that $\alpha \circ \beta = \mathrm{id}_{B_r}$ by a similar argument.

Now α is continuous everywhere except possibly at 0 by Lemma 4.1. By Lemma 3.2, we have $\alpha(B_s) \subset B_s$ for all $s < r$, so α is also continuous at 0. Thus α is a continuous bijection from a compact space to a Hausdorff space, so it is a homeomorphism. □

Remark 4.2. It follows from results of Rietsch (see [15]) that the point X_0 of the flow $\exp(tS)$ is, rather surprisingly, also the unique totally nonnegative critical point of the $q = 1$ specialization of the *superpotential* of the Grassmannian [24, Section 6]. However, the superpotential is not defined on the boundary of $\mathrm{Gr}_{\geq 0}(k, n)$. Its exact relation to the cyclic shift flow is somewhat mysterious.

5 Examples: $\mathrm{Gr}_{\mathbb{R}}(1,3)$ and $\mathrm{Gr}_{\mathbb{R}}(2,4)$

The case $\mathrm{Gr}_{\mathbb{R}}(1,3)$**.** The map $\phi : \mathbb{R}^2 \simeq M_{\mathbb{R}} \to \mathrm{Gr}_{\mathbb{R}}(1,3) = \mathbb{RP}^2$ is

$$A = \begin{array}{c c} & \begin{array}{cc} e^{2\pi i/3} & e^{-2\pi i/3} \end{array} \\ 1 & \begin{bmatrix} a+bi & a-bi \end{bmatrix} \end{array} \quad \overset{\phi}{\longmapsto} \quad (1+2a : 1-a-\sqrt{3}b : 1-a+\sqrt{3}b) \in \mathbb{RP}^2.$$

Therefore $\Pi \subseteq \mathbb{R}^2$ is the equilateral triangle with vertices $(a,b) = (1,0)$, $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$. The trajectories of $\exp(tS)$ are shown in Figure 2. We remark that this

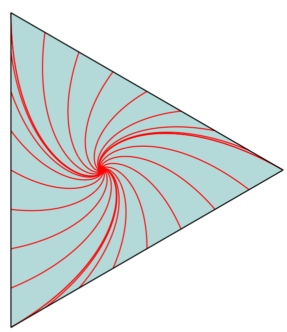

Figure 2: The trajectories of the cyclic flow S on $\mathrm{Gr}_{\geq 0}(1,3)$, regarded as an equilateral triangle with vertices $(1,0)$, $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$. The cyclic shift map itself acts by rotating the triangle counterclockwise by $120°$.

figure is related to an old puzzle: if we place a dog at each vertex of the equilateral triangle, and each runs towards the dog at the next (counterclockwise) vertex at constant speed 1, how long does it take for the dogs to catch each other? The answer is time $\frac{2}{\sqrt{3}}$, which we can see by fixing one of the dogs and viewing everything as taking place in this dog's frame of reference. The trajectories themselves are given in Figure 2.

The case $\mathrm{Gr}_{\mathbb{R}}(2,4)$. The map $\phi : \mathbb{R}^4 \simeq M_{\mathbb{R}} \to \mathrm{Gr}_{\mathbb{R}}(2,4)$ is

$$A = \begin{array}{cc} & \begin{array}{cc} e^{3\pi i/4} & e^{-3\pi i/4} \end{array} \\ \begin{array}{c} e^{\pi i/4} \\ \\ e^{-\pi i/4} \end{array} & \begin{bmatrix} \frac{a+bi}{\sqrt{2}} & \frac{c+di}{\sqrt{2}} \\ \frac{c-di}{\sqrt{2}} & \frac{a-bi}{\sqrt{2}} \end{bmatrix} \end{array} \quad \overset{\phi}{\longmapsto} \quad \begin{aligned} \Delta_{\{1,2\}} &= 2 + a^2 + b^2 - c^2 - d^2 + 2\sqrt{2}(a-b-d), \\ \Delta_{\{2,3\}} &= 2 + a^2 + b^2 - c^2 - d^2 + 2\sqrt{2}(d-a-b), \\ \Delta_{\{3,4\}} &= 2 + a^2 + b^2 - c^2 - d^2 + 2\sqrt{2}(b-d-a), \\ \Delta_{\{1,4\}} &= 2 + a^2 + b^2 - c^2 - d^2 + 2\sqrt{2}(a+b+d), \\ \Delta_{\{1,3\}} &= 2\sqrt{2} - \sqrt{2}(a^2 + b^2 - c^2 - d^2) + 4c, \\ \Delta_{\{2,4\}} &= 2\sqrt{2} - \sqrt{2}(a^2 + b^2 - c^2 - d^2) - 4c. \end{aligned}$$

(We have chosen the entries of A so that the sum of the squares of the absolute values of its entries equals $a^2 + b^2 + c^2 + d^2 = \|(a,b,c,d)\|^2$.) The inverse map is given by

$$\begin{aligned} a &= (\Delta_{\{1,2\}} - \Delta_{\{2,3\}} - \Delta_{\{3,4\}} + \Delta_{\{1,4\}})/\delta, \\ b &= (-\Delta_{\{1,2\}} - \Delta_{\{2,3\}} + \Delta_{\{3,4\}} + \Delta_{\{1,4\}})/\delta, \qquad c = \sqrt{2}(\Delta_{\{1,3\}} - \Delta_{\{2,4\}})/\delta, \\ d &= (-\Delta_{\{1,2\}} + \Delta_{\{2,3\}} - \Delta_{\{3,4\}} + \Delta_{\{1,4\}})/\delta, \end{aligned}$$

where $\delta = \Delta_{\{1,3\}} + \Delta_{\{2,4\}} + \frac{1}{\sqrt{2}}(\Delta_{\{1,2\}} + \Delta_{\{2,3\}} + \Delta_{\{3,4\}} + \Delta_{\{1,4\}})$. The image $\phi(M_{\mathbb{R}})$ is the subset of $\mathrm{Gr}_{\mathbb{R}}(2,4)$ where $\delta \neq 0$, which we see includes $\mathrm{Gr}_{\geq 0}(2,4)$, verifying Proposition 2.1 in this case.

Our results imply that $\mathrm{Gr}_{\geq 0}(2,4)$ is homeomorphic to the subset of $\mathbb{R}^4$ where the 6 polynomials $\Delta_{\{i,j\}}$ (for $1 \leq i < j \leq 4$) in a,b,c,d are nonnegative. The closures of cells in the cell decomposition of $\mathrm{Gr}_{\geq 0}(2,4)$ are obtained by intersecting with the zero locus of some subset of these 6 polynomials. The 0-dimensional cells (i.e. the points of $\mathrm{Gr}_{\geq 0}(2,4)$ with only one nonzero Plücker coordinate) are

$$\sqrt{2}(1,-1,0,-1),\ \sqrt{2}(-1,-1,0,1),\ \sqrt{2}(-1,1,0,-1),\ \sqrt{2}(1,1,0,1),\ \sqrt{2}(0,0,1,0),\ \sqrt{2}(0,0,-1,0).$$

In general, using ϕ we can describe $\mathrm{Gr}_{\geq 0}(k,n)$ as the subset of $\mathbb{R}^{k(n-k)}$ where some $\binom{n}{k}$ polynomials of degree at most k are nonnegative.

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