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The unipotent modules of $\mathrm{GL}_n(\mathbb{F}_q)$ via tableaux

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Abstract. We construct the irreducible unipotent modules of the finite general linear groups from actions on tableaux. Our approach is analogous to that of James (1976) for the symmetric groups, answering an open question as to whether such a construction exists. We show that our modules are isomorphic to those previously constructed by James (1984), although the two presentations are quite different. Key to our construction are the generalized Gelfand–Graev representations of Kawanaka (1983).

Keywords: finite general linear group, unipotent representation, tableaux, generalized Gelfand–Graev representation

1 Introduction

A standard construction of the irreducible representations of the symmetric group S_n, initially due to James [7], uses the action of S_n on Young tableaux to define a module for each integer partition of n. In characteristic 0 these modules are (up to isomorphism) all of the irreducible S_n-modules; in other characteristics the irreducible modules appear as quotients of these modules.

There is a principle in representation theory that information about the finite general linear group $\mathrm{GL}_n(\mathbb{F}_q)$ can be related to that of S_n by "setting $q = 1$." In particular, there is a collection of irreducible representations of $\mathrm{GL}_n(\mathbb{F}_q)$, known as "unipotent representations," that one would expect to behave like the irreducible representations of S_n. In [5], James constructs the unipotent modules of $\mathrm{GL}_n(\mathbb{F}_q)$ over any field containing a nontrivial p^{th} root of unity (where p is the characteristic of $\mathbb{F}_q$). The construction is quite different from the tableaux approach for the symmetric group, and in particular the proofs do not translate to proofs for the symmetric group. James asks in the introduction to [5] whether an alternative construction exists that is more along the lines of that of the symmetric group.

Our approach is to label the boxes of Young diagrams with elements of $\mathbb{F}_q^n$ rather than by integers. The group $\mathrm{GL}_n(\mathbb{F}_q)$ acts on these objects; we use this action to define the irreducible unipotent modules. Our construction is a natural analogue to that of the symmetric group, providing a positive answer to James' question. A key tool in

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our approach is the use of the "generalized Gelfand–Graev representations," initially constructed by Kawanaka [8] and recently studied by Thiem and the author [1].

James shows in [4] that the irreducible module of S_n corresponding to the partition λ has a basis indexed by the standard Young tableaux of shape λ. It is still an open problem to determine bases for the unipotent irreducible modules of $\mathrm{GL}_n(\mathbb{F}_q)$ (see, for instance, [2]). In providing a new construction of the irreducible unipotent modules, we hope to shed some light on this question.

In Section 2, we cover necessary background material on partitions and the finite general linear groups. We provide motivation for our construction in Section 3, in particular explaining why the generalized Gelfand–Graev representations are the right tool for constructing unipotent modules. Our construction is in Section 4, and in Section 5 we look at some further directions for this line of research.

2 Preliminaries

2.1 Partitions and tableaux

Let n be a positive integer; a *partition* of n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_k = n$. We write $\lambda \vdash n$ to indicate that λ is a partition of n.

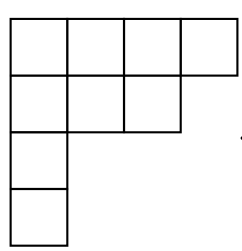
There is a partial order on the set of partitions of n with $\lambda \succeq \mu$ if and only if

$$\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i$$

for all k (setting $\lambda_i = 0$ if λ has fewer than i parts). This order is called the *dominance order* on partitions.

To each partition λ we associate a *Young diagram*, which is a left-justified array of blocks such that the number of blocks in the ith row is λ_i.

Example 2.1. Let $\lambda = (4, 3, 1, 1)$; then the Young diagram of shape λ is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The *conjugate* of a partition λ, denoted λ', is the partition defined by $\lambda'_i = |\{j \mid \lambda_j \geq i\}|$. Note that the Young diagram of λ' is obtained from that of λ by reflection about the diagonal.

If λ is a partition of n, a *tableau* of shape λ is a filling of the Young diagram of shape λ by the integers from 1 through n, each appearing exactly once. We say that a tableau is *standard* if the entries increase along rows and columns.

Example 2.2. Let $\lambda = (4, 3, 1, 1)$, and let

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 8 \\ \hline 3 & 4 & 6 \\ \cline{1-3} 7 \\ \cline{1-1} 9 \\ \cline{1-1}\end{array} \quad \text{and} \quad T' = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 & 8 & 7 \\ \cline{1-3} 9 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array};$$

then T and T' are both tableaux of shape λ, but T' is not standard as the pairs $(7, 8)$ and $(6, 9)$ violate the row-increasing and column-increasing conditions.

2.2 The finite general linear groups

Let q be a power of a prime, and let $\mathbb{F}_q$ be the finite field with q elements. We are interested in $G = \mathrm{GL}_n(\mathbb{F}_q)$, the group of invertible $n \times n$ matrices with entries in $\mathbb{F}_q$.

Let λ be a partition of n, and let T be the row-reading tableau of shape λ. We define

$$\begin{aligned} P_\lambda &= \{g \in G \mid g_{ij} = 0 \text{ if } i \text{ is strictly below } j \text{ in } T\} \text{ and} \\ U_\lambda &= \{g \in G \mid g_{ii} = 1 \text{ and } g_{ij} = 0 \text{ unless } i = j \text{ or } i \text{ is strictly above } j \text{ in } T\}. \end{aligned}$$

Example 2.3. Let $\lambda = (4, 3, 1, 1)$; then

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 \\ \cline{1-3} 8 \\ \cline{1-1} 9 \\ \cline{1-1}\end{array},$$

and we have

$$P_\lambda = \left\{ \begin{pmatrix} * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{pmatrix} \in \mathrm{GL}_n(\mathbb{F}_q) \right\}$$

and

$$U_\lambda = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 1 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 1 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_n(\mathbb{F}_q) \right\}.$$

We also define

$$P_\lambda^- = (P_\lambda)^t \quad \text{and} \quad U_\lambda^- = (U_\lambda)^t$$

to be the transposes of P_λ and U_λ.

Remark 2.4. The groups P_λ and P_λ^- are parabolic subgroups of G, with unipotent radicals U_λ and U_λ^-. In particular, the group of upper triangular matrices in G given by

$$B_n(\mathbb{F}_q) = P_{(1^n)}$$

is a Borel subgroup of G with unipotent radical

$$\mathrm{UT}_n(\mathbb{F}_q) = U_{(1^n)}.$$

We say that an irreducible module of G (over some field) is *unipotent* if it is a composition factor of $\mathrm{Ind}_{B_n(\mathbb{F}_q)}^G(\mathbb{1})$.

3 Motivation behind the construction

In this section we motivate our construction in the specific case where we are taking our representations over the field of complex numbers. None of the results of this section are necessary for the construction, however they provide insight as to why our approach makes sense. First, we recall some facts about the character theory of S_n and $\mathrm{GL}_n(\mathbb{F}_q)$ over $\mathbb{C}$ and the connection to symmetric functions.

The irreducible complex characters of S_n are indexed by the partitions of n; let ψ^λ denote the irreducible character of S_n corresponding to λ, as in [9]. Note that $\psi^{(n)}$ is the trivial character and $\psi^{(1^n)}$ is the sign character (denoted ϵ). The following proposition follows from the Pieri rules (see [9, p. I.5]).

Proposition 3.1. *Let λ be a partition of n, and let W_λ be a Young subgroup of S_n of shape λ. Then*

$$\mathrm{Ind}_{W_\lambda}^{S_n}(\mathbb{1}) = \sum_{\mu \succeq \lambda} K_{\mu\lambda}\psi^\mu \quad \textit{and} \quad \mathrm{Ind}_{W_\lambda}^{S_n}(\epsilon) = \sum_{\mu' \succeq \lambda} K_{\mu'\lambda}\psi^\mu,$$

where the $K_{\mu\lambda}$ are the Kostka numbers (see [9, p. I.6.4]).

The Kostka numbers $K_{\mu\lambda}$ satisfy that $K_{\lambda\lambda} = 1$ and $K_{\mu\lambda} = 0$ unless $\mu \succeq \lambda$. In particular, if W_λ is a Young subgroup of shape λ and $W_{\lambda'}$ is a Young subgroup of shape λ', we have

$$\langle \mathrm{Ind}_{W_\lambda}^{S_n}(\mathbb{1}), \mathrm{Ind}_{W_{\lambda'}}^{S_n}(\epsilon)\rangle = 1,$$

and the common irreducible constituent is ψ^λ. James [7] constructs an irreducible submodule V^λ of $\mathrm{Ind}_{W_\lambda}^{S_n}(\mathbb{1})$ in such a way that V^λ contains a one-dimensional subspace on which $W_{\lambda'}$ acts as ϵ. In other words,

$$\langle \mathrm{Ind}_{W_\lambda}^{S_n}(\mathbb{1}), V^\lambda\rangle \neq 0$$

and

$$\langle \mathrm{Ind}_{W_{\lambda'}}^{S_n}(\epsilon), V^\lambda\rangle = \langle \epsilon, \mathrm{Res}_{W_{\lambda'}}^{S_n}(V^\lambda)\rangle \neq 0.$$

This forces V^λ to be a module affording the character ψ^λ.

In order to apply a similar approach to construct the irreducible unipotent modules of $\mathrm{GL}_n(\mathbb{F}_q)$ we need to find modules that are analogous to $\mathrm{Ind}_{W_\lambda}^{S_n}(\mathbb{1})$ and $\mathrm{Ind}_{W_{\lambda'}}^{S_n}(\epsilon)$. The irreducible unipotent characters of $\mathrm{GL}_n(\mathbb{F}_q)$ are indexed by partitions of n; let χ^λ be the irreducible unipotent character corresponding to λ, as in [3, 9]. Note that $\chi^{(1^n)}$ is the trivial character and $\chi^{(n)}$ is the Steinberg character. As the trivial character is indexed by the transpose of (n) (rather than by (n) as with the symmetric group), we should expect our results to be transposed to some extent. The next proposition follows from the Pieri rules.

Proposition 3.2. *Let P_λ be the parabolic subgroup of G of shape λ; then we have*

$$\mathrm{Ind}_{P_\lambda}^{G}(\mathbb{1}) = \sum_{\mu' \succeq \lambda} K_{\mu'\lambda}\chi^\mu.$$

Both James' construction and our construction use $\mathrm{Ind}_{P_{\lambda'}}^{G}(\mathbb{1})$ as the analogue of $\mathrm{Ind}_{W_\lambda}^{S_n}(\mathbb{1})$.

Let Ψ^λ be the *degenerate Gelfand–Graev character* corresponding to the partition λ, as in [11, Section 12.1]. The character Ψ^λ is obtained by inducing a linear character of $\mathrm{UT}_n(\mathbb{F}_q)$ that is trivial on certain root subgroups determined by the partition λ.

Proposition 3.3 ([11, Section 12.1])**.** *If λ and μ are partitions of n, we have that*

$$\langle \Psi^\lambda, \chi^\mu\rangle = K_{\mu\lambda}.$$

It follows that $\langle \mathrm{Ind}_{P_{\lambda'}}^{G}(\mathbb{1}), \Psi^\lambda\rangle = 1$; the construction of James [5] uses Ψ^λ as the analogue of $\mathrm{Ind}_{W_{\lambda'}}^{S_n}(\epsilon)$.

In [8], Kawanaka constructs the *generalized Gelfand–Graev characters* of a reductive group over a finite field, with each character associated to a nilpotent orbit of the corresponding Lie algebra. In the case of $\mathrm{GL}_n(\mathbb{F}_q)$, the nilpotent orbits are indexed by the partitions of n; let Γ^λ be the generalized Gelfand–Graev character indexed by the partition λ. In [1], Thiem and the author show that Γ^λ can be obtained by inducing a linear character from $U_{\lambda'}$, and calculate the multiplicities of the unipotent characters in Γ^λ.

Theorem 3.4 ([1]). *Let λ and μ be partitions of n; then*

$$\langle \Gamma^\lambda, \chi^\mu \rangle = K_{\mu\lambda}(q),$$

where $K_{\mu\lambda}(q)$ is the Kostka polynomial (see [9, p. III.6]).

It follows that $\langle \mathrm{Ind}_{P_{\lambda'}}^{G}(\mathbb{1}), \Gamma^\lambda \rangle = 1$. Furthermore, note that $K_{\mu\lambda}(1) = K_{\mu\lambda}$; by setting $q = 1$ we see that Γ^λ is another analogue of $\mathrm{Ind}_{W_{\lambda'}}^{S_n}(\epsilon)$. We construct an irreducible submodule S^λ of $\mathrm{Ind}_{P_{\lambda'}}^{G}(\mathbb{1})$ that contains a one-dimensional submodule on which $U_{\lambda'}$ acts as the appropriate linear character. This forces S^λ to be a module affording the irreducible unipotent character χ^λ.

4 Construction of the modules

The construction in this section is motivated by the construction of the irreducible representations of the symmetric group (see [7, 4, 10]). Many of the results are similar to those found in [5], which is not surprising as our constructions produce isomorphic modules.

Let p be the characteristic of $\mathbb{F}_q$, and let $\mathbb{K}$ be a field that contains a nontrivial p^{th} root of unity (in particular, the characteristic of $\mathbb{K}$ cannot be p). For the remainder of the paper, fix a nontrivial homomorphism $\theta : \mathbb{F}_q^+ \to \mathbb{K}^\times$.

Definition 4.1. Let λ be a partition of n, and let T be a filling of the Young diagram of shape λ with linearly independent elements of $\mathbb{F}_q^n$. We call T an $\mathbb{F}_q^n$-*tableau*.

Note that G acts on the set of $\mathbb{F}_q^n$-tableaux by left multiplication of the entries (considered as column vectors). If T is an $\mathbb{F}_q^n$-tableau of shape λ, we obtain an ordered basis $\mathcal{B}(T)$ of $\mathbb{F}_q^n$ by numbering the entries of T from top to bottom, then left to right.

Example 4.2. Let $\lambda = (4, 2^2, 1)$; then the ordered basis $\mathcal{B}(T) = \{v_1, \ldots, v_9\}$ corresponds to the tableau

$$T = \begin{array}{|c|c|c|c|} \hline v_1 & v_5 & v_8 & v_9 \\ \hline v_2 & v_6 \\ \cline{1-2} v_3 & v_7 \\ \cline{1-2} v_4 \\ \cline{1-1} \end{array}\ .$$

To each $\mathbb{F}_q^n$-tableaux T we associate two subgroups of G, given by

$$U(T) = \{g \in G \mid g \cdot v_i - v_i \in \mathbb{F}_q\text{-span}\{v_j \mid v_j \text{ is strictly left of } v_i \text{ in } T\} \text{ for all } i\} \text{ and}$$
$$P(T) = \{g \in G \mid g \cdot v_i \in \mathbb{F}_q\text{-span}\{v_j \mid v_j \text{ is nonstrictly right of } v_i \text{ in } T\} \text{ for all } i\}.$$

If $\mathcal{B}(T)$ is the standard ordered basis of $\mathbb{F}_q^n$, then $U(T) = U_{\lambda'}$ and $P(T) = P_{\lambda'}^-$.

Example 4.3. Let $\lambda = (4, 2^2, 1)$, and let $\mathcal{B}(T)$ be the standard ordered basis. Then

$$T = \begin{array}{|c|c|c|c|} \hline v_1 & v_5 & v_8 & v_9 \\ \hline v_2 & v_6 \\ \cline{1-2} v_3 & v_7 \\ \cline{1-2} v_4 \\ \cline{1-1} \end{array},$$

and we have

$$U(T) = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 1 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 1 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

and

$$P(T) = \left\{ \begin{pmatrix} * & * & * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 \\ * & * & * & * & * & * & * & * & * \end{pmatrix} \right\}.$$

If T is the $\mathbb{F}_q^n$-tableau corresponding to the ordered basis $\mathcal{B}(T) = \{v_1, v_2, \ldots, v_n\}$, let $X(T)$ be the set of pairs (i, j) such that v_i lies in the box directly to the left of v_j in T. In Example 4.3,

$$X(T) = \{(1,5), (2,6), (3,7), (5,8), (8,9)\}.$$

Define a linear character ψ_T of $U(T)$ by

$$\psi_T(u) = \theta\left(\sum_{(i,j)\in X(T)} \text{the coefficient of } v_i \text{ in } uv_j\right),$$

where θ is the fixed nontrivial homomorphism from $\mathbb{F}_q^+$ to $\mathbb{K}^\times$.

Remark 4.4. The groups $P(T)$ and $U(T)$ are analogous to the row-stabilizer R_T and the column-stabilizer C_T of the symmetric group; the linear character ψ_T is analogous to the sign character (see [7, 4, 10]).

Consider the permutation G-module

$$\mathbb{K}\text{-span}\{T \mid T \text{ is an } \mathbb{F}_q^n\text{-tableau of shape } \lambda\};$$

this module is isomorphic to the left regular module of G. We define

$$m_T = \sum_{p\in P(T)} pT \quad \text{and}$$
$$e_T = \sum_{u\in U(T)} \psi_T(u^{-1})m_{uT}.$$

The following proposition is easy to verify directly.

Lemma 4.5. *Let T be any $\mathbb{F}_q^n$-tableau.*

1. *For all $g \in G$, we have $U(gT) = gU(T)g^{-1}$ and $P(gT) = gP(T)g^{-1}$.*
2. *For all $g \in G$, we have $g \cdot m_T = m_{g\cdot T}$ and $g \cdot e_T = e_{gT}$.*
3. *For all $u \in U(T)$ and $g \in G$, we have $\psi_{gT}(gug^{-1}) = \psi_T(u)$.*
4. *For all $p \in P(T)$, we have $m_{pT} = m_T$.*
5. *For all $u \in U(T)$, we have $e_{uT} = \psi_T(u)e_T$.*

Let

$$M^\lambda = \mathbb{K}\text{-span}\{m_T \mid T \text{ is an } \mathbb{F}_q^n\text{-tableau of shape } \lambda\}$$

and

$$S^\lambda = \mathbb{K}\text{-span}\{e_T \mid T \text{ is an } \mathbb{F}_q^n\text{-tableau of shape } \lambda\};$$

by part (2) of Lemma 4.5, M^λ and S^λ are both G-modules.

Remark 4.6. The module M^λ is isomorphic to the permutation representation of G on the set of λ-flags. This means that our module M^λ is isomorphic to the module $M_{\lambda'}$ of James (as in [5, p. 10.1]). Note that

$$M^\lambda \cong \mathrm{Ind}_{P^-_{\lambda'}}^{G}(\mathbb{1})$$

and, in the case $\mathbb{K} = \mathbb{C}$, we have that

$$\mathrm{Ind}_{U(T)}^{\mathrm{GL}_n(\mathbb{F}_q)}(\psi_T) = \Gamma^\lambda$$

(this follows from the elementary construction of the generalized Gelfand–Graev representations in [1]). In particular, when $\mathbb{K} = \mathbb{C}$ the module S^λ satisfies the desired properties from Section 3.

Lemma 4.7. *Let T and T' be $\mathbb{F}_q^n$-tableaux of shape λ. Then $U(T) \cap P(T') = \{1\}$ if and only if there exist $u \in U(T)$ and $p \in P(T')$ with $uT = pT'$.*

Lemma 4.8. *Suppose that T and T' are $\mathbb{F}_q^n$-tableaux of shape λ that satisfy $U(T) \cap P(T') \neq \{1\}$; then there exists $g \in U(T) \cap P(T')$ with $\psi_T(g) \neq 1$.*

For an $\mathbb{F}_q^n$-tableau T, define an element $k_T \in \mathbb{K}G$ by

$$k_T = \sum_{u \in U(T)} \psi_T(u^{-1})u.$$

The following two lemmas describe how k_T acts on $m_{T'}$ for certain $\mathbb{F}_q^n$-tableaux T'.

Lemma 4.9. *Let T and T' be $\mathbb{F}_q^n$-tableaux of the same shape; then $k_T m_{T'} \in \mathbb{K}e_T$.*

Lemma 4.10. *Let λ and μ be partitions of n. If T is an $\mathbb{F}_q^n$-tableaux of shape λ and T' is an $\mathbb{F}_q^n$-tableaux of shape μ, then $k_T m_{T'} = 0$ unless $\mu \succeq \lambda$.*

One consequence of Lemma 4.9 is the following proposition.

Proposition 4.11. *The module S^λ is indecomposable.*

There is an immediate corollary of Proposition 4.11.

Corollary 4.12. *If* $\mathrm{char}(\mathbb{K})$ *does not divide $|G|$, then S^λ is irreducible. In particular, if $\mathbb{K}$ has characteristic 0, then S^λ is irreducible.*

Given an $\mathbb{F}_q^n$-tableau T, let $\overline{T}$ be the $\mathbb{F}_q^n$-tableau obtained by replacing the entry v_i with $-v_i$ if v_i is in an odd column of T and fixing the entry v_i if v_i is in an even column of T. For example, if

$$T = \begin{array}{|c|c|c|c|} \hline v_1 & v_5 & v_8 & v_9 \\ \hline v_2 & v_6 & \multicolumn{2}{c}{} \\ \cline{1-2} v_3 & v_7 & \multicolumn{2}{c}{} \\ \cline{1-2} v_4 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}, \quad \text{then} \quad \overline{T} = \begin{array}{|c|c|c|c|} \hline -v_1 & v_5 & -v_8 & v_9 \\ \hline -v_2 & v_6 & \multicolumn{2}{c}{} \\ \cline{1-2} -v_3 & v_7 & \multicolumn{2}{c}{} \\ \cline{1-2} -v_4 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}.$$

Lemma 4.13. *For all $\mathbb{F}_q^n$-tableaux T, we have that $U(\overline{T}) = U(T)$, $\overline{T} \in P(T) \cdot T$, and $\psi_{\overline{T}}(u) = \psi_T(u^{-1})$.*

We define a G-invariant bilinear form on M^λ by

$$[m_T, m_{T'}] = \delta_{m_T, m_{T'}},$$

and extending by linearity. We remark that $\delta_{m_T, m_{T'}}$ is not the same as $\delta_{T,T'}$, as $m_T = m_{T'}$ exactly when $T' \in P(T) \cdot T$.

Proposition 4.14. *Let V be a submodule of M^λ; then either $S^\lambda \subseteq V$ or $V \subseteq (S^\lambda)^\perp$.*

When constructing the irreducible representations of the symmetric groups, it is possible to have $S^\lambda \subseteq (S^\lambda)^\perp$; that will not be the case, however, with the finite general linear groups.

Lemma 4.15. *If T is any $\mathbb{F}_q^n$-tableaux, then $e_T \notin (S^\lambda)^\perp$, hence $S^\lambda \not\subseteq (S^\lambda)^\perp$.*

The following corollary is an immediate consequence of Proposition 4.14.

Corollary 4.16. *We have the following.*

1. *$S^\lambda \cap (S^\lambda)^\perp$ is the unique maximal submodule of S^λ.*

2. *The G-module $S^\lambda/(S^\lambda \cap (S^\lambda)^\perp)$ is irreducible.*

Define $D^\lambda = S^\lambda/(S^\lambda \cap (S^\lambda)^\perp)$.

Proposition 4.17. *Let λ and μ be partitions of n; then we have the following.*

1. *If D^λ is a composition factor of M^μ, then $\lambda \preceq \mu$.*

2. *D^λ is a composition factor of M^λ.*

3. *If $D^\lambda \cong D^\mu$, then $\lambda = \mu$.*

In [5], James constructs a collection of irreducible modules of G, one for each partition of n. James denotes these modules by D_λ, and shows the following.

1. The modules D_λ are the unipotent modules of G. In other words, these modules are exactly the composition factors of $\mathrm{Ind}_B^G(\mathbb{1})$, up to isomorphism.

2. If $D_\lambda \cong D_\mu$, then $\lambda = \mu$.

3. The module D_λ is a composition factor of $\mathrm{Ind}_{P_\lambda}^G(\mathbb{1})$.

4. Every composition factor of $\mathrm{Ind}_{P_\mu}^G(\mathbb{1})$ is isomorphic to D_λ for some $\lambda \succeq \mu$.

Note that the modules D_λ are uniquely characterized (up to isomorphism) by properties (2)–(4). As $M^\lambda \cong \mathrm{Ind}_{P^-_{\lambda'}}^G(\mathbb{1})$ and $\mathrm{Ind}_{P^-_{\lambda'}}^G(\mathbb{1}) \cong \mathrm{Ind}_{P_{\lambda'}}^G(\mathbb{1})$ (see [5, p. 14.7]), by Proposition 4.17 we have the following.

Corollary 4.18. *We have that* $D^\lambda \cong D_{\lambda'}$*. In particular, the* D^λ *are the irreducible unipotent modules of* G*.*

Remark 4.19. The indexing of our modules and those of James differs by the transpose of the partition. Our indexing is chosen to match the convention of Green [3] and MacDonald [9].

5 Further directions

There are a number of questions raised by our construction; a few of particular interest are listed below.

- Even in the case that $\mathbb{K} = \mathbb{C}$ it is still an open problem to find bases for the irreducible unipotent modules of $\mathrm{GL}_n(\mathbb{F}_q)$. James addresses this question in [5], and a number of papers (for example [2]) contribute partial results. For the symmetric group, the bases for the irreducible modules are indexed by the standard Young tableaux. The number of standard Young tableaux of shape λ is given by the *hook length formula* (see, for example, [10, Theorem 3.10.2]). The dimension of the unipotent module of $\mathrm{GL}_n(\mathbb{F}_q)$ of shape λ is given by a q-analogue of the hook length formula (see [9, III.6 Example 2]). This suggests that bases might be constructed where each standard Young tableau corresponds to a collection of basis elements, the number of which is given by a polynomial in q.

- The finite orthogonal, symplectic, and unitary groups also have unipotent representations, however explicit constructions of the modules are not known. For these groups there are not enough degenerate Gelfand–Graev representations to distinguish unipotent representations, however by instead using the generalized Gelfand–Graev representations it might be possible to construct the unipotent modules.

- Most of the irreducible representations of $\mathrm{GL}_n(\mathbb{F}_q)$ are not unipotent. In [6], James uses the degenerate Gelfand–Graev representations to construct all of the irreducible modules of $\mathrm{GL}_n(\mathbb{F}_q)$. By modifying our construction, we might be able to obtain more elementary constructions of these modules using the generalized Gelfand–Graev representations.

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