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The decomposition of 0-Hecke modules associated to quasisymmetric Schur functions

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Abstract. Recently Tewari and van Willigenburg constructed modules of the 0-Hecke algebra that are mapped to the quasisymmetric Schur functions by the quasisymmetric characteristic. These modules have a natural decomposition into a direct sum of certain submodules. We show that the summands are indecomposable by determining their endomorphism rings.

Keywords: 0-Hecke algebra, composition tableau, quasisymmetric function, Schur function

1 Introduction

Since the 19th century mathematicians have been interested in the Schur functions s_λ and their various properties. For example, they form an orthonormal basis of *Sym*, the Hopf algebra of symmetric functions and are the images of the irreducible complex characters of the symmetric groups under the characteristic map [11]. The Hopf algebra *QSym* of quasisymmetric functions contains *Sym* and was defined in 1984 [6].

There is a representation theoretic interpretation of *QSym* as well. The 0-Hecke algebra $H_n(0)$ is a non-semisimple deformation of the group algebra $\mathbb{C}\mathfrak{S}_n$ obtained by replacing the generators $(i, i+1)$ of $\mathfrak{S}_n$ by projections π_i satisfying the same braid relations. Let $\mathcal{G}_0(H_n(0))$ denote the Grothendieck group of the finitely generated $H_n(0)$-modules and $\mathcal{G} := \bigoplus_{n\geq 0} \mathcal{G}_0(H_n(0))$. The connection to *QSym* is established by an algebra isomorphism $Ch\colon \mathcal{G} \to QSym$ called quasisymmetric characteristic [5, 9].

As *Sym* is contained in *QSym*, one may ask whether there are quasisymmetric analogues of the Schur functions. One such analogue is given by the quasisymmetric Schur functions $\mathcal{S}_\alpha$ [7]. They form a basis of *QSym* and nicely refine the Schur functions via $s_\lambda = \sum_{\widetilde{\alpha}=\lambda} \mathcal{S}_\alpha$ where λ is a partition and the sum runs over all compositions α that rearrange λ [7] (see Subsection 2.2 for definitions). In [3] skew quasisymmetric Schur functions $\mathcal{S}_{\alpha/\!/\beta}$ were defined and a Littlewood–Richardson rule for expressing them in the basis of quasisymmetric Schur functions was proved.

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Another basis of *QSym* sharing properties with the Schur functions is given by the dual immaculate functions [1]. Indecomposable 0-Hecke modules whose images under *Ch* are the dual immaculate functions were defined in [2].

In [12] Tewari and van Willigenburg constructed modules $\mathbf{S}_\alpha$ of the 0-Hecke algebra that are mapped to $\mathcal{S}_\alpha$ by *Ch*. Each $\mathbf{S}_\alpha$ has a $\mathbb{C}$-basis parametrized by a set of tableaux. By using an equivalence relation, they divided this set into equivalence classes, obtained a submodule $\mathbf{S}_{\alpha,E}$ of $\mathbf{S}_\alpha$ for each such equivalence class E and decomposed $\mathbf{S}_\alpha$ as $\mathbf{S}_\alpha = \bigoplus_E \mathbf{S}_{\alpha,E}$. Moreover, they gave a combinatorial formula for the expansion of the characteristic $\mathcal{S}_\alpha$ of $\mathbf{S}_\alpha$ in the fundamental basis of *QSym* or equivalently, the compositions factors of the module $\mathbf{S}_\alpha$; this formula can be carried over to the $\mathbf{S}_{\alpha,E}$. They defined and decomposed skew modules $\mathbf{S}_{\alpha/\!/\beta}$ whose image under *Ch* is $\mathcal{S}_{\alpha/\!/\beta}$ in the same way.

This article is mainly concerned with the modules $\mathbf{S}_\alpha$ and $\mathbf{S}_{\alpha,E}$. In [12] they considered a special equivalence class E_α and showed that $\mathbf{S}_{\alpha,E_\alpha}$ is indecomposable. Yet, the question of the indecomposability of the $\mathbf{S}_{\alpha,E}$ in general remained open. The goal of this paper is to answer this question. We show that for each $\mathbf{S}_{\alpha,E}$ the ring of $H_n(0)$-endomorphisms is $\mathbb{C}\,\mathrm{id}$ so that $\mathbf{S}_{\alpha,E}$ is indecomposable. As a consequence, $\mathbf{S}_\alpha = \bigoplus_E \mathbf{S}_{\alpha,E}$ is a decomposition into indecomposable submodules.

The structure of the abstract is as follows. In Section 2 we introduce the combinatorial and algebraic background and then review the modules $\mathbf{S}_{\alpha/\!/\beta}$ and $\mathbf{S}_{\alpha/\!/\beta,E}$. Section 3 is devoted to a related $H_n(0)$-operation on chains of a composition poset. From this we obtain an argument crucial for proving our main results; this is explained in Section 4. For a full version containing all the proofs see [8].

Acknowledgements

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2 Background

We set $\mathbb{N} := \{1, 2, \ldots\}$ and always assume that $n \in \mathbb{N}$. For $a, b \in \mathbb{Z}$ we define the *discrete interval* $[a, b] := \{c \in \mathbb{Z} \mid a \leq c \leq b\}$ and may use the shorthand $[a] := [1, a]$. For a set X $\mathrm{span}_{\mathbb{C}}\, X$ is the $\mathbb{C}$-vector space with basis X.

2.1 Symmetric groups and 0-Hecke algebras

The *symmetric group* $\mathfrak{S}_n$ is the group of all permutations of the set $[n]$. We proceed by reviewing $\mathfrak{S}_n$ as Coxeter group. More details can be found in [4].

The adjacent transpositions $s_i := (i, i+1)$, for $i = 1, \dots, n-1$, generate $\mathfrak{S}_n$ as a Coxeter group; they satisfy $s_i^2 = 1$ and the *braid relations* $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_i s_j = s_j s_i$ if $|i - j| \geq 2$.

Let $\sigma \in \mathfrak{S}_n$. We can write σ as a product $\sigma = s_{j_k} \cdots s_{j_1}$. If k is minimal among such expressions, $s_{j_k} \cdots s_{j_1}$ is a *reduced word* for σ and $\ell(\sigma) := k$ is the *length* of σ. We define the *support* of σ as $\mathcal{I}(\sigma) = \{i \in [n-1] \mid s_i \text{ appears in a reduced word of } \sigma\}$. By the *word property* [4, Theorem 3.3.1], the set of indices occurring in w is $\mathcal{I}(\sigma)$ for each reduced word w of σ. Let $\sigma, \tau \in \mathfrak{S}_n$. The *left weak order* $\leq_L$ is the partial order on $\mathfrak{S}_n$ given by

$$\sigma \leq_L \tau \iff \begin{array}{l} \tau = s_{i_k} \cdots s_{i_1} \sigma, \\ \ell(s_{i_r} \cdots s_{i_1} \sigma) = \ell(\sigma) + r \text{ for } r = 1, \dots, k. \end{array}$$

Theorem 2.1 ([4, Corollary 3.2.2]). *Let $\sigma, \tau \in \mathfrak{S}_n$. The interval in left weak order $[\sigma, \tau] := \{\rho \in \mathfrak{S}_n \mid \sigma \leq_L \rho \leq_L \tau\}$ is a graded lattice with rank function $\rho \mapsto \ell(\rho\sigma^{-1})$.*

Next, we define the 0-Hecke algebra $H_n(0)$. For details we refer to the more comprehensive introduction in [10, Chapter 1]. We use the presentation as in [12].

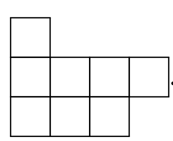
Definition 2.2. *The* 0-Hecke algebra $H_n(0)$ *is the unital associative $\mathbb{C}$-algebra generated by the elements $\pi_1, \pi_2, \dots, \pi_{n-1}$ subject to $\pi_i^2 = \pi_i$ and the braid relations $\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}$ and $\pi_i \pi_j = \pi_j \pi_i$ if $|i - j| \geq 2$.*

Note that the π_i are projections satisfying the same braid relations as the s_i. For $\sigma \in \mathfrak{S}_n$ we define $\pi_\sigma := \pi_{j_k} \cdots \pi_{j_1}$ where $s_{j_k} \cdots s_{j_1}$ is a reduced word for σ. The word property ensures that this is well defined. Moreover, $\{\pi_\sigma \mid \sigma \in \mathfrak{S}_n\}$ is a $\mathbb{C}$-basis of $H_n(0)$.

2.2 Compositions and composition tableaux

A *composition* $\alpha = (\alpha_1, \dots, \alpha_l)$ of n is a finite sequence of positive integers that sum up to n and denoted by $\alpha \vDash n$. The *length* of α is $\ell(\alpha) := l$, and $|\alpha| := \sum_{i=1}^{l} \alpha_i$ is the *size* of α. The α_i are called *parts* of α. A *partition* is a composition whose parts are weakly decreasing. We write $\lambda \vdash n$ if λ is a partition of n. For a composition α we denote the partition obtained by sorting the parts of α in decreasing order by $\widetilde{\alpha}$. The *empty composition* $\varnothing$ is the unique composition of length and size 0. For instance, $\alpha = (1, 4, 3) \vDash 8$ and $\widetilde{\alpha} = (4, 3, 1) \vdash 8$.

A *cell* (i, j) is an element of $\mathbb{N} \times \mathbb{N}$. A finite set of cells is called *diagram* and visualized in English notation. That is, for each cell (i, j) of a diagram we draw a box at position (i, j) in matrix coordinates. The *diagram* of $\alpha \vDash n$ is the set $\{(i, j) \in \mathbb{N} \times \mathbb{N} \mid i \leq \ell(\alpha), j \leq \alpha_i\}$. So, we display the diagram of α by putting α_i boxes in row i where the top row has index 1. We often identify α with its diagram. For example,

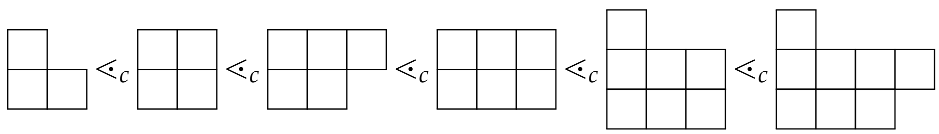
$(1, 4, 3) =$  .

Next, we will introduce standard composition tableaux and a related poset of compositions which arose in [3].

Definition 2.3. *The* composition poset $\mathcal{L}_c$ *is the set of all compositions together with the partial order* $\leq_c$ *given as the transitive closure of the following covering relation. For compositions* α *and* $\beta = (\beta_1, \dots, \beta_l)$

$$\beta \lessdot_c \alpha \iff \begin{array}{l} \alpha = (1, \beta_1, \dots, \beta_l) \text{ or} \\ \alpha = (\beta_1, \dots, \beta_k + 1, \dots, \beta_l) \text{ and } \beta_i \neq \beta_k \text{ for all } i < k. \end{array}$$

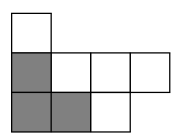
Example 2.4.



Let α and β be two compositions such that $\beta \leq_c \alpha$. In this situation we always assume that the diagram of β is moved to the bottom of the diagram of α, and we define the *skew composition diagram* (or *skew shape*) $\alpha /\!/ \beta$ to consist of all cells of α which are not contained in β. Moreover, we define $\operatorname{osh}(\alpha /\!/ \beta) = \alpha$ and $\operatorname{ish}(\alpha /\!/ \beta) = \beta$ as the *outer* and the *inner shape* of $\alpha /\!/ \beta$, respectively. The *size* of a skew shape is $|\alpha /\!/ \beta| := |\alpha| - |\beta|$. If $\beta = \emptyset$ then $\alpha /\!/ \beta = \alpha$ is an ordinary composition diagram and we call $\alpha /\!/ \beta$ *straight*.

Example 2.5. In the following the cells of the inner shape are gray.

$$(1,4,3) /\!/ (1,2) =$$



Let D be a diagram. A *tableau* T of shape D is a map $T\colon D \to \mathbb{N}$. It is visualized by filling each $(i,j) \in D$ with $T(i,j)$.

Definition 2.6. *Let* $\alpha /\!/ \beta$ *be a skew shape of size* n. *A* standard composition tableau *(SCT) of shape* $\alpha /\!/ \beta$ *is a bijective filling* $T\colon \alpha /\!/ \beta \to [n]$ *satisfying the following conditions:*

(1) The entries are decreasing in each row from left to right.
(2) The entries are increasing in the first column from top to bottom.
(3) (Triple rule). Set $T(i,j) := \infty$ *for all* $(i,j) \in \beta$. *If* $(j,k) \in \alpha /\!/ \beta$ *and* $(i,k-1) \in \alpha$ *such that* $j > i$ *and* $T(j,k) < T(i,k-1)$ *then* $(i,k) \in \alpha$ *and* $T(j,k) < T(i,k)$.

Let $a := T(j,k)$, $b := T(i,k-1)$ be two entries of an SCT T occurring in adjacent columns. Then the triple rule can be visualized as follows by considering the positions of entries in T:

$$\begin{array}{cc} \boxed{b} & \\ & \boxed{a} \end{array} \text{ and } a < b \quad \xrightarrow{\text{triple rule}} \quad \exists c \in T \text{ s.t. } \begin{array}{cc} \boxed{b} & \boxed{c} \\ & \boxed{a} \end{array} \text{ and } a < c.$$

The set of standard composition tableaux of shape $\alpha /\!/ \beta$ is denoted with $\mathrm{SCT}(\alpha /\!/ \beta)$. For an SCT T we write $\mathrm{sh}(T)$ for its shape. The notions of outer and inner shape are carried over from $\mathrm{sh}(T)$ to T. We call T *straight* if its shape is straight.

Example 2.7. We have $\mathrm{osh}(T) = (1,4,3)$ and $\mathrm{ish}(T) = (1,2)$ for the SCT

$$T = \begin{array}{|c|c|c|c|} \cline{1-1} 2 \\ \hline \cellcolor{gray} & 5 & 4 & 1 \\ \hline \cellcolor{gray} & \cellcolor{gray} & 3 \\ \cline{1-3} \end{array}.$$

Standard composition tableaux correspond to saturated chains of $\mathcal{L}_c$ in the following way.

Proposition 2.8 ([3, Proposition 2.11])**.** *Let $\alpha /\!/ \beta$ be a skew composition of size n. For $T \in \mathrm{SCT}(\alpha /\!/ \beta)$, a saturated chain in $\mathcal{L}_c$ is given by $\beta = \alpha^n \lessdot_c \alpha^{n-1} \lessdot_c \cdots \lessdot_c \alpha^0 = \alpha$ where*

$$\alpha^n = \beta, \quad \alpha^{k-1} = \alpha^k \cup T^{-1}(k) \quad \text{for} \quad k = 1, \ldots, n. \tag{2.1}$$

Moreover, we obtain a bijection from $\mathrm{SCT}(\alpha /\!/ \beta)$ to the set of saturated chains in $\mathcal{L}_c$ from β to α by mapping each tableau of $\mathrm{SCT}(\alpha /\!/ \beta)$ to its corresponding chain given by (2.1).

Example 2.9. The SCT from Example 2.7 corresponds to the chain from Example 2.4.

Some of the following notions already played a role in [12]. Let T be an SCT and $i, j \in T$ be two entries. We say that *i attacks j* in T and write $i \rightsquigarrow_T j$ if $i \neq j$ and either they appear in adjacent columns of T such that i is located strictly above and left of j or they appear in the same column. Let J be a subset of the entries of T. If there is a $k \in J$ such that $i \rightsquigarrow_T k$, we say that *i attacks J* in T. If i is located strictly left of k for all $k \in J$, we say that i is strictly left of J. The notion of being (weakly) left is used analogously. We may omit the index T in $\rightsquigarrow_T$ if it is clear from context.

Example 2.10. In the tableau from Example 2.7 we have $2 \rightsquigarrow 5, 3 \rightsquigarrow 4, 4 \rightsquigarrow 3$, $5 \rightsquigarrow 3$ and $i \not\rightsquigarrow j$ for all other pairs of entries. Moreover, 3 is left of $\{1,4\}$ and $2 \rightsquigarrow \{3,5\}$.

Definition 2.11. *Let T be an SCT of size n.*

(1) $D(T) = \{i \in [n-1] \mid i$ is weakly left of $i+1\}$ is the descent set *of T.*
(2) $AD(T) = \{i \in D(T) \mid i \rightsquigarrow i+1\}$ is the set of attacking descents *of T.*
(3) $nAD(T) = \{i \in D(T) \mid i \notin AD(T)\}$ is the set of non-attacking descents *of T.*
(1') $D^c(T) = \{i \in [n-1] \mid i$ is strictly right of $i+1\} = [n-1] \setminus D(T)$ is the ascent set *of T.*
(2') $ND^c(T) = \{i \in D^c(T) \mid i$ is the right neighbor of $i+1\}$ is the set of neighborly ascents *of T.*

Example 2.12. Let T be the tableau from Example 2.7. Then $D(T) = \{2,3\}$, $AD(T) = \{3\}$, $D^c(T) = \{1,4\}$ and $ND^c(T) = \{4\}$.

2.3 0-Hecke modules of standard composition tableaux

In this subsection we introduce the skew 0-Hecke modules $\mathbf{S}_{\alpha/\!/\beta}$ and $\mathbf{S}_{\alpha/\!/\beta,E}$ and review related results from [12]. This also includes the special cases $\mathbf{S}_{\alpha}$ and $\mathbf{S}_{\alpha,E}$.

Theorem 2.13 ([12, Theorem 9.8]). *Let $\alpha/\!/\beta$ be a skew composition of size n. Then $\mathbf{S}_{\alpha/\!/\beta} := \operatorname{span}_{\mathbb{C}} \mathrm{SCT}(\alpha/\!/\beta)$ is an $H_n(0)$-module with respect to the following action. For $T \in \mathrm{SCT}(\alpha/\!/\beta)$ and $i = 1, \ldots, n-1$,*

$$\pi_i T = \begin{cases} T & \text{if } i \notin D(T) \\ 0 & \text{if } i \in AD(T) \\ s_i T & \text{if } i \in nAD(T) \end{cases}$$

where $s_i T$ is the tableau obtained from T by interchanging i and $i+1$.

The module $\mathbf{S}_{\alpha}$ is called *straight* if $\alpha = \alpha/\!/\beta$ is a composition. Even though the main results of this abstract only concern straight modules, we introduce the more general concept of skew modules here as they naturally arise in the context of the 0-Hecke action on chains of $\mathcal{L}_c$ in Section 3.

Example 2.14. Consider the SCT $T = \begin{array}{|c|c|c|c|}\hline 1 \\ \cline{1-1} 6 & 5 & 4 & 3 \\ \hline 8 & 7 & 2 \\ \cline{1-3}\end{array}$. Then $D(T) = \{1, 2, 6\}$,

$$\pi_i T = \begin{cases} T & \text{for } i = 3,4,5,7 \\ 0 & \text{for } i = 6 \\ s_i T & \text{for } i = 1,2, \end{cases} \qquad s_1 T = \begin{array}{|c|c|c|c|}\hline \mathbf{2} \\ \cline{1-1} 6 & 5 & 4 & 3 \\ \hline 8 & 7 & \mathbf{1} \\ \cline{1-3}\end{array} \quad \text{and} \quad s_2 T = \begin{array}{|c|c|c|c|}\hline 1 \\ \cline{1-1} 6 & 5 & 4 & \mathbf{2} \\ \hline 8 & 7 & \mathbf{3} \\ \cline{1-3}\end{array}.$$

Let $\alpha/\!/\beta$ be a skew composition of size n and $T_1, T_2 \in \mathrm{SCT}(\alpha/\!/\beta)$. An equivalence relation $\sim$ on $\mathrm{SCT}(\alpha/\!/\beta)$ is given by

$$T_1 \sim T_2 \iff \text{in each column the relative orders of entries in } T_1 \text{ and } T_2 \text{ coincide.}$$

We denote the set of equivalence classes under $\sim$ on $\mathrm{SCT}(\alpha/\!/\beta)$ by $\mathcal{E}(\alpha/\!/\beta)$. For example, the tableaux shown in Figure 1 form such an equivalence class.

Let $E \in \mathcal{E}(\alpha/\!/\beta)$ and define $\mathbf{S}_{\alpha/\!/\beta,E} := \operatorname{span}_{\mathbb{C}} E$. Observe that the definition of the 0-Hecke action on SCTx in Theorem 2.13 implies that $\mathbf{S}_{\alpha/\!/\beta,E}$ is an $H_n(0)$-submodule of $\mathbf{S}_{\alpha/\!/\beta}$. Thus we have the following.

Proposition 2.15 ([12, Lemma 6.6]). *Let $\alpha/\!/\beta$ be a skew composition. Then we have $\mathbf{S}_{\alpha/\!/\beta} = \bigoplus_{E \in \mathcal{E}(\alpha/\!/\beta)} \mathbf{S}_{\alpha/\!/\beta,E}$ as $H_n(0)$-modules.*

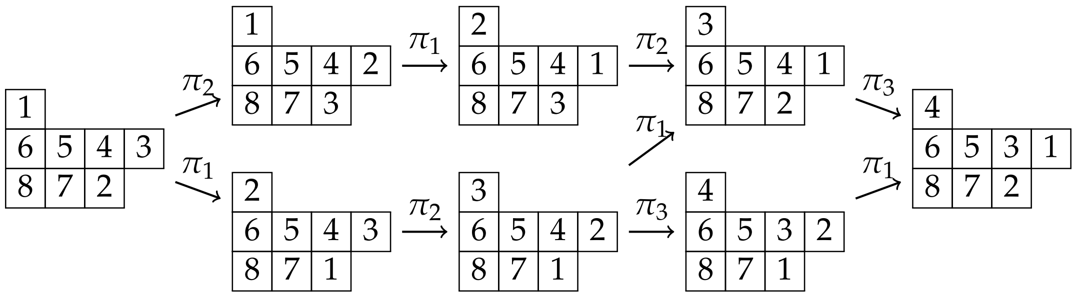


Figure 1: A poset $(E, \preceq)$. Each covering relation is labeled with the 0-Hecke generator π_i realizing it. The leftmost tableau is the source tableau, the rightmost is the sink tableau of E.

The main result of this abstract is that the $H_n(0)$-endomorphism ring of each straight module $\mathbf{S}_{\alpha,E}$ is $\mathbb{C}\,\mathrm{id}$ and, therefore, we obtain a decomposition of $\mathbf{S}_\alpha$ into indecomposable submodules from Proposition 2.15.

We continue by studying the $\mathbf{S}_{\alpha/\!/\beta,E}$ and their equivalence classes more deeply. Let $\alpha/\!/\beta$ be a skew composition of size n, $E \in \mathcal{E}(\alpha/\!/\beta)$ and $T_1, T_2 \in E$. In [12, Section 4] it is shown that a partial order $\preceq$ on E is given by

$$T_1 \preceq T_2 \iff \exists \sigma \in \mathfrak{S}_n \text{ such that } \pi_\sigma T_1 = T_2.$$

We refer to the poset $(E, \preceq)$ simply by E. An example is shown in Figure 1.

The following theorem summarizes results of [12, Section 6].

Theorem 2.16. *Let $\alpha/\!/\beta$ be a skew composition and $E \in \mathcal{E}(\alpha/\!/\beta)$.*

(1) There is a unique tableau $T_{0,E} \in E$ such that $D^c(T_{0,E}) = ND^c(T_{0,E})$. This tableau is the least element of E and is called source tableau *of E.*

(2) There is a unique tableau $T_{1,E} \in E$ such that $D(T_{1,E}) = AD(T_{1,E})$. This tableau is the greatest element of E and is called sink tableau *of E.*

In particular, $\mathbf{S}_{\alpha/\!/\beta,E}$ is a cyclic module generated by $T_{0,E}$.

For the equivalence class in Figure 1, its source and sink appear as the leftmost and rightmost tableau, respectively. Next, we establish a connection between E and an interval in left weak order. To do this we introduce the notion of *column words*. Given $T \in \mathrm{SCT}(\alpha/\!/\beta)$ of size n and $j \geq 1$, let w_j be the word obtained by reading the entries in the jth column of T from top to bottom. Then $\mathrm{col}_T = w_1 w_2 \cdots$ is the *column word* of T. Clearly, col_T can be regarded as an element of $\mathfrak{S}_n$ (in one-line notation).

Example 2.17. We have $\mathrm{col}_T = 16857423$ for the tableau T from Example 2.14.

Lemma 2.18 ([12, Lemma 4.4])**.** *Let T_1 be an SCT, $i \in nAD(T_1)$ and $T_2 = \pi_i T_1$. Then $\mathrm{col}_{T_2} = s_i\,\mathrm{col}_{T_1}$ and $\ell(\mathrm{col}_{T_2}) = \ell(\mathrm{col}_{T_1}) + 1$. That is, col_{T_2} covers col_{T_1} in left weak order.*

Theorem 2.19 ([12, Theorem 6.18]). *Let $\alpha /\!/ \beta$ be a skew composition, $E \in \mathcal{E}(\alpha /\!/ \beta)$ and $I = [\mathrm{col}_{T_{0,E}}, \mathrm{col}_{T_{1,E}}]$ be an interval in left weak order. The map $\mathrm{col}\colon E \to I, T \mapsto \mathrm{col}_T$ is a poset isomorphism. In particular, E is a graded lattice with rank function $\delta\colon T \mapsto \ell(\mathrm{col}_T \mathrm{col}_{T_{0,E}}^{-1})$.*

Let $T_1 \preceq T_2$ be two SCTx. In the poset of the left weak order, saturated chains from col_{T_1} to col_{T_2} correspond to reduced words for $\mathrm{col}_{T_2} \mathrm{col}_{T_1}^{-1}$ [4, Proposition 3.1.2]. By Theorem 2.19 these reduced words also correspond to chains from T_1 to T_2. From this and Lemma 2.18 one obtains the following.

Corollary 2.20. *Let T_1 and T_2 be two SCTx and $\sigma \in \mathfrak{S}_n$ such that $T_2 = \pi_\sigma T_1$. Then T_1 and T_2 belong to the same equivalence class under $\sim$. Let δ be the rank function of that class. Then*

(1) $\delta(T_2) - \delta(T_1) = \ell(\mathrm{col}_{T_2} \mathrm{col}_{T_1}^{-1})$,

(2) $\delta(T_2) - \delta(T_1) \leq \ell(\sigma)$ where we have equality if and only if $\sigma = \mathrm{col}_{T_2} \mathrm{col}_{T_1}^{-1}$.

By using Lemma 2.18 inductively, we obtain the following.

Proposition 2.21. *Let T be an SCT, $i, j \in T$ be such that $i < j$ and $\square = T^{-1}(i)$. If in T i is located left of $[i+1, j]$ and does not attack $[i+1, j]$ then*

(1) $T' := \pi_{j-1} \cdots \pi_{i+1} \pi_i T \in \mathrm{SCT}$,

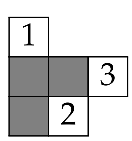
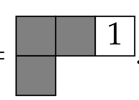
(2) $s_{j-1} \cdots s_{i+1} s_i$ is a reduced word for $\mathrm{col}_{T'} \mathrm{col}_T^{-1}$,

(3) $T'(\square) = j$.

3 A 0-Hecke action on chains of the composition poset

In Proposition 2.8 a bijection between saturated chains in the composition poset $\mathcal{L}_c$ and standard composition tableaux is given. In this section we study the 0-Hecke action on these chains induced by this bijection. The main purpose of this section is to present Proposition 3.2. This proposition is an important ingredient for the proof of the main result described in Section 4. We begin with some notation.

Let T be an SCT of shape $\alpha /\!/ \beta$ and size n, $\beta = \alpha^n \lessdot_c \alpha^{n-1} \lessdot_c \cdots \lessdot_c \alpha^0 = \alpha$ the chain in $\mathcal{L}_c$ corresponding to T and $m \in [0, n]$. The SCT of shape $\alpha^m /\!/ \beta$ corresponding to the chain $\alpha^n \lessdot_c \alpha^{n-1} \lessdot_c \cdots \lessdot_c \alpha^m$ is denoted by $T^{>m}$. For example, let

$T =$  then $T^{>2} =$ .

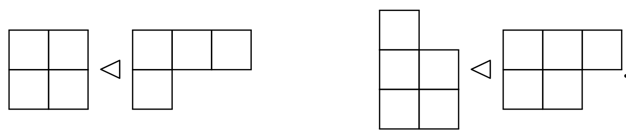
By definition, $\alpha^m = \mathrm{osh}(T^{>m})$. We may use this notation to access the compositions within the chain of an SCT. We use a preorder $\trianglelefteq$ to describe how the 0-Hecke action affects the compositions within chains of $\mathcal{L}_c$. For $\alpha = (\alpha_1, \ldots, \alpha_l) \vDash n$ and $j \in \mathbb{N}$ define

$|\alpha|_j = \#\{i \in [l] \mid \alpha_i \geq j\}$, the number of cells in the jth column of the diagram of α. On the set of compositions of size n we define

$$\alpha \trianglelefteq \beta \iff \sum_{j=1}^{k} |\beta|_j \leq \sum_{j=1}^{k} |\alpha|_j \text{ for all } k \geq 1.$$

In addition, set $\alpha \triangleleft \beta \iff \alpha \trianglelefteq \beta$ and $\alpha \neq \beta$.

Obviously $\trianglelefteq$ is reflexive and transitive. It is not antisymmetric since for example $(2,1) \trianglelefteq (1,2)$ and $(1,2) \trianglelefteq (2,1)$. If we restrict $\trianglelefteq$ to partitions, we obtain the classical dominance order appearing, for example, in [11]. For instance,



Comparing two chains whose SCTx are related by a covering relation, one can show:

Lemma 3.1. *Let $\alpha /\!/ \beta$ be a skew composition of size n and $T_1, T_2 \in SCT(\alpha /\!/ \beta)$ be such that $T_2 = \pi_i T_1$ for an $i \in nAD(T_1)$. Then*

$$\operatorname{osh}(T_2^{>i}) \triangleleft \operatorname{osh}(T_1^{>i}) \quad \text{and} \quad \operatorname{osh}(T_2^{>m}) = \operatorname{osh}(T_1^{>m}) \text{ for } m \in [0,n], m \neq i.$$

Let $\alpha /\!/ \beta$ be a skew composition, $E \in \mathcal{E}(\alpha /\!/ \beta)$ and $T_1, T_2 \in E$ be such that $T_1 \preceq T_2$. Recall that for each saturated chain from T_1 to T_2 in E the index set of the 0-Hecke operators establishing the covering relations within the chain is $\mathcal{I}(\operatorname{col}_{T_2} \operatorname{col}_{T_1}^{-1})$. By applying Lemma 3.1 on each such covering relation, we obtain the following.

Proposition 3.2. *Let $\alpha /\!/ \beta$ be a skew composition of size n, $i \in [n-1], E \in \mathcal{E}(\alpha /\!/ \beta)$ and $T_1, T_2 \in E$ be such that $T_1 \preceq T_2$. Then $i \in \mathcal{I}(\operatorname{col}_{T_2} \operatorname{col}_{T_1}^{-1})$ if and only if $\operatorname{sh}(T_2^{>i}) \neq \operatorname{sh}(T_1^{>i})$.*

4 The endomorphism ring of $S_{\alpha,E}$

For each $\alpha \vDash n$ there is an equivalence class $E_\alpha \in \mathcal{E}(\alpha)$ such that for all $T \in E_\alpha$ the entries increase in each column from top to bottom [12, Section 8]. In [12], Tewari and van Willigenburg showed that S_{α,E_α} is indecomposable.

In this section, we show for all $E \in \mathcal{E}(\alpha)$ that $\operatorname{End}_{H_n(0)}(S_{\alpha,E}) = \mathbb{C}\operatorname{id}$ and hence $S_{\alpha,E}$ is indecomposable; this extends the result of Tewari and van Willigenburg to the general case. By Proposition 2.15 we then have our main result, the desired decomposition of S_α. We fix some notation that we use in the entire section unless stated otherwise. Let $\alpha \vDash n$, $E \in \mathcal{E}(\alpha)$ and $T_0 := T_{0,E}$ be the source tableau of E. Moreover, let $f \in \operatorname{End}_{H_n(0)}(S_{\alpha,E})$, $v := f(T_0)$ and $v = \sum_{T \in E} a_T T$ be the expansion of v in the $\mathbb{C}$-basis E. Since $S_{\alpha,E}$ is cyclically generated by T_0, f is already determined by v. The *support* of v is given by

$\operatorname{supp}(v) := \{T \in E \mid a_T \neq 0\}$. Our goal is to show that T_0 is the only tableau that may occur in $\operatorname{supp}(v)$ since then $f = a_{T_0}\,\mathrm{id} \in \mathbb{C}\,\mathrm{id}$. We begin with an easy consequence of $v = f(T_0)$ and the definition of the 0-Hecke action.

Lemma 4.1. *If $T \in \operatorname{supp}(v)$ then $D(T) \subseteq D(T_0)$.*

Let $T \in E$ be such that $T \neq T_0$ and $D(T) \subseteq D(T_0)$. Thanks to Lemma 4.1 it remains to show $a_T = 0$. To do this we use a 0-Hecke operator π_σ where $\sigma = s_{j-1} \cdots s_i$ and i and j are given by

$$\begin{aligned} i &= \max\left\{k \in [n] \mid T^{-1}(k) \neq T_0^{-1}(k)\right\}, \\ j &= \min\left\{k \in [n] \mid k > i \text{ and } i \rightsquigarrow_{T_0} k\right\}. \end{aligned} \tag{4.1}$$

That is, i is the greatest entry whose position in T differs from that in T_0 and j is the smallest entry in T_0 which is greater than i and attacked by i in T_0. At this point it is not clear that j is well defined since the defining set could be empty. However, the next two lemmas will show that there always exists an element in this set.

Example 4.2. Consider the equivalence class E from Figure 1 and let T_0 be the source tableau of E. There is exactly one other $T \in E$ with a descent set contained in $D(T_0)$:

$$T_0 = \begin{array}{|c|c|c|c|} \cline{1-1} 1 \\ \hline 6 & 5 & 4 & 3 \\ \hline 8 & 7 & 2 \\ \cline{1-3} \end{array} \xrightarrow{\pi_1} T = \begin{array}{|c|c|c|c|} \cline{1-1} 2 \\ \hline 6 & 5 & 4 & 3 \\ \hline 8 & 7 & 1 \\ \cline{1-3} \end{array}$$

Defining i and j for T as in (4.1) we obtain $i = 2$ and $j = 4$. Note that $2 \in D(T_0)$. This property holds in general by the following result.

Lemma 4.3. *Let T and i be as in (4.1). Then $i \in D(T_0)$.*

In the proof of Lemma 4.3, $i \neq n$ follows directly from the fact that T_0 and T are straight. The reasoning for $i \notin D^c(T_0)$ is more complicated but mainly uses that $D^c(T_0) = ND^c(T_0)$ by Theorem 2.16. The next result depends on the assumption that T_0 is straight.

Lemma 4.4. *For each $d \in D(T_0)$ there exists $k \in T_0$ such that $k > d$ and $d \rightsquigarrow_{T_0} k$.*

Let T, i and j be as in (4.1). By Lemma 4.3 and Lemma 4.4, j is well defined. We proceed by considering the relative positions of i and $[i+1, j]$ in T_0 and in T. This will allow us to deduce useful properties of the operator π_σ to be defined in Lemma 4.7. By definition, $i \rightsquigarrow j$ in T_0. In contrast, the next lemma shows $i \not\rightsquigarrow j$ in T.

Lemma 4.5. *Let T, i and j be as in (4.1). Then j is well defined and we have the following.*
(1) In T_0, i is located strictly left of $[i+1, j-1]$ and does not attack $[i+1, j-1]$.
(2) In T, i appears strictly left of $[i+1, j]$ and does not attack $[i+1, j]$.

The proof of Lemma 4.5 is somewhat technical. In part (1), $i \in D(T_0)$ by Lemma 4.3 and $D^c(T_0) = ND^c(T_0)$ is used. The main idea of part (2) is that from the choice of i and Proposition 3.2 one can conclude that the position of i in T is strictly left of the position of i in T_0. This can be observed in the following example.

Example 4.6. For

$$T_0 = \begin{array}{|c|c|c|c|} \cline{1-1} 1 \\ \hline 6 & 5 & 4 & 3 \\ \hline 8 & 7 & 2 \\ \cline{1-3} \end{array} \quad \text{and} \quad T = \begin{array}{|c|c|c|c|} \cline{1-1} 2 \\ \hline 6 & 5 & 4 & 3 \\ \hline 8 & 7 & 1 \\ \cline{1-3} \end{array}$$

from above we have $i = 2$ and $j = 4$. Note $2 \rightsquigarrow_{T_0} 4$ but $2 \not\rightsquigarrow_T 4$.

Lemma 4.7. *Let T, i, j and $\sigma = s_{j-1} \cdots s_{i+1} s_i$ be as in (4.1). Then*

(1) $\pi_\sigma T_0 = 0$,
(2) $\pi_\sigma T \in E$,
(3) $\sigma = \operatorname{col}_{\pi_\sigma T} \operatorname{col}_T^{-1}$.

By combining Proposition 2.21 and Lemma 4.5, one can describe how π_σ acts on T_0 and T. This implies Lemma 4.7.

Example 4.8. Continuing our running example, we obtain $\sigma = s_3 s_2$. Applying π_σ yields

$$T_0 = \begin{array}{|c|c|c|c|} \cline{1-1} 1 \\ \hline 6 & 5 & 4 & 3 \\ \hline 8 & 7 & 2 \\ \cline{1-3} \end{array} \xrightarrow{\pi_2} \begin{array}{|c|c|c|c|} \cline{1-1} 1 \\ \hline 6 & 5 & 4 & 2 \\ \hline 8 & 7 & 3 \\ \cline{1-3} \end{array} \xrightarrow{\pi_3} 0,$$

$$T = \begin{array}{|c|c|c|c|} \cline{1-1} 2 \\ \hline 6 & 5 & 4 & 3 \\ \hline 8 & 7 & 1 \\ \cline{1-3} \end{array} \xrightarrow{\pi_2} \begin{array}{|c|c|c|c|} \cline{1-1} 3 \\ \hline 6 & 5 & 4 & 2 \\ \hline 8 & 7 & 1 \\ \cline{1-3} \end{array} \xrightarrow{\pi_3} \begin{array}{|c|c|c|c|} \cline{1-1} 4 \\ \hline 6 & 5 & 3 & 2 \\ \hline 8 & 7 & 1 \\ \cline{1-3} \end{array}.$$

We now come to our main result.

Theorem 4.9. *Let $\alpha \vDash n$ and $E \in \mathcal{E}(\alpha)$. Then $\operatorname{End}_{H_n(0)}(\mathbf{S}_{\alpha,E}) = \mathbb{C}\,\mathrm{id}$. In particular, $\mathbf{S}_{\alpha,E}$ is an indecomposable $H_n(0)$-module.*

Sketch of proof. Let $T_* \in \operatorname{supp}(v)$ be of maximal rank in E and assume $T_* \neq T_0$. By Lemma 4.1, $D(T_*) \subseteq D(T_0)$, so Lemma 4.7 provides an element π_σ for T_*. Using $\sigma = \operatorname{col}_{\pi_\sigma T_*} \operatorname{col}_{T_*}^{-1}$, the maximality of T_* and Corollary 2.20, one can show that the coefficient of $\pi_\sigma T_*$ in $\pi_\sigma v$ equals a_{T_*}. On the other hand, property (1) of π_σ yields $\pi_\sigma v = f(\pi_\sigma T_0) = 0$. Hence $a_{T_*} = 0$, contradicting the assumption $T_* \in \operatorname{supp}(v)$. Thus, $\operatorname{End}_{H_n(0)}(\mathbf{S}_{\alpha,E}) = \mathbb{C}\,\mathrm{id}$. In particular, this implies that $\mathbf{S}_{\alpha,E}$ is indecomposable. $\square$

Combining Theorem 4.9 with Proposition 2.15 we obtain the decomposition of $\mathbf{S}_\alpha$.

Corollary 4.10. *Let $\alpha \vDash n$. Then $\mathbf{S}_\alpha = \bigoplus_{E \in \mathcal{E}(\alpha)} \mathbf{S}_{\alpha,E}$ is a decomposition into indecomposable submodules.*

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