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Decorated Dyck paths and the Delta conjecture

Michele D'Adderio*[1] and Anna Vanden Wyngaerd †[1]

[1] *Université Libre de Bruxelles (ULB), Département de Mathématique, Boulevard du Triomphe, B-1050 Bruxelles, Belgium*

Abstract. We discuss the combinatorics of the decorated Dyck paths appearing in the Delta conjecture framework in (Haglund et al 2015) and (Zabrocki 2016), by introducing two new statistics, bounce and bounce′. We then provide plethystic formulae for their q,t-enumerators, by proving in this way a decorated version of Haglund's q,t-Schröder theorem, answering a question in (Haglund et al 2015). In particular we provide both an algebraic and a combinatorial explanation of a symmetry conjecture in (Haglund et al 2015) and (Zabrocki 2016).

This is an extended abstract of (D'Adderio, Vanden Wyngaerd 2017).

Keywords: Macdonald Polynomials, Delta conjecture, decorated Dyck paths

1 Introduction

Motivated by the problem of proving the Schur positivity of the modified Macdonald polynomials, Garsia and Haiman introduced the $\mathfrak{S}_n$-module of diagonal harmonics, and they conjectured that its Frobenius characteristic is ∇e_n, where the operator ∇ is just a special case of a whole family of so called delta operators Δ_f, where f is a symmetric function.

In [10] a combinatorial formula for ∇e_n has been conjectured, in terms of labelled Dyck paths (i.e. parking functions). This formula has been known as the *shuffle conjecture*, and it has been recently proved by Carlsson and Mellit in [1], where they actually proved a refinement of this formula, known as the *compositional shuffle conjecture*, proposed in [9].

A generalization of the shuffle conjecture, known as the *Delta conjecture*, has been proposed in [10], in terms of decorated labelled Dyck paths. Some special cases and consequences of this conjecture have been proved: see [10] for a survey of these partial results. We should mention here also the work in [13], where the conjecture at $q = 1$ is proved, the work in [6], where the conjecture at $q = 0$ is proved, as they do not appear in [10]. Finally, in [2], relying on results in the present paper, the "$h_j h_{n-j}$" case of the Delta conjecture is proved. To this date, the general problem remains open.

*mdadderi@ulb.ac.be
†anvdwyng@ulb.ac.be

In [15], Zabrocki established one of the consequences of the Delta conjecture, the so called 4-*variable Catalan conjecture*. In fact, this conjecture, stated in [10], makes other predictions that have not been explained in [15]. One of them is the symmetry

$$\langle \Delta_{h_\ell} \nabla e_{n-\ell}, s_{k+1,1^{n-\ell-k-1}} \rangle = \langle \Delta_{h_k} \nabla e_{n-k}, s_{\ell+1,1^{n-k-\ell-1}} \rangle \qquad \text{for } n > k+\ell. \tag{1.1}$$

In [10] and [14], the authors asked for a proof of the decorated version of the famous q,t-Schröder theorem of Haglund [7]. This is a combinatorial interpretation of the formula

$$\langle \Delta'_{e_{a+b-k-1}} e_{a+b}, e_a h_b \rangle \tag{1.2}$$

as a sum of weights $q^{\mathsf{dinv}(D)} t^{\mathsf{area}(D)}$ of decorated Dyck paths, where dinv and area are certain functions from the space of decorated Dyck paths to the natural numbers.

In the present article we provide a solution of the last two problems. In particular we give both an algebraic and a combinatorial explanation of the symmetry in (1.1), and we prove three combinatorial interpretations of (1.2).

In fact, we introduce two more statistics on decorated Dyck paths, bounce and pbounce, which both extend Haglund's bounce statistic on Dyck paths. We prove that the zeta map of Haglund (also known as sweep map) switches the bistatistics $(\mathsf{dinv}, \mathsf{area})$ and $(\mathsf{area}, \mathsf{bounce})$ and the numbers of decorated rises and decorated peaks. Also, we provide a recursion for the q,t-enumerators of these two bistatistics, which matches a recursion in [14].

We also prove a recursion for the q,t-enumerator of the bistatistic $(\mathsf{area}, \mathsf{pbounce})$, and we define a bijection ψ on decorated Dyck paths that preserves the bistatistic $(\mathsf{area}, \mathsf{pbounce})$ but switches the numbers of decorated rises and decorated peaks: this will eventually explain combinatorially the symmetry in (1.1).

This is an extended abstract of [3]: we refer to it for some definitions and full proofs.

2 Basics of symmetric functions and Macdonald polynomials

Given $f \in K[x_1, \dots, x_n]$ for some field K and $\sigma \in \mathfrak{S}_n$, we set

$$\sigma \cdot f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

We say that f is a *symmetric function* if for all $\sigma \in \mathfrak{S}_n$ we have $\sigma \cdot f = f$.

Let Λ be the ring of symmetric functions in $x_1, x_2, \dots$. As we are working with Macdonald symmetric functions involving two parameters q and t, we will consider this polynomial ring over the field $\mathbb{Q}(q,t)$. This ring has a grading $\Lambda = \bigoplus_{n \geq 0} \Lambda^{(n)}$ where Λ^n is the sub vector space of Λ consisting of symmetric functions of degree n. The standard basis of symmetric functions, indexed by partitions are

the complete homogeneous	$h_n = \sum_{i_1 \leq \ldots \leq i_n} x_{i_1} \ldots x_{i_n}$	$h_\lambda = h_{\lambda_1} \ldots h_{\lambda_n}$
the elementary	$e_n = \sum_{i_1 < \ldots < i_n} x_{i_1} \ldots x_{i_n}$	$e_\lambda = e_{\lambda_1} \ldots e_{\lambda_n}$
the power symmetric	$p_n = \sum_i x_i^n$	$p_\lambda = p_{\lambda_1} \ldots p_{\lambda_n}$
The Schur	$s_\lambda = \sum_{T \in SSYT(\lambda)} x^T.$	

In this last definition $SSYT(\lambda)$ are the semi-standard Young tableaux of shape λ and for such a tableau T we set $x^T = x_1^{t_1} x_2^{t_2} \ldots$ if the number i appears t_i times in T. We will use implicitly the usual convention that $e_0 = h_0 = 1$ and $e_k = h_k = 0$ for $k < 0$.

We will make extensive use of the *plethystic notation* (see [11] for a nice exposition). With this notation we will be able to add and subtract alphabets, which will be represented as sums of monomials $X = x_1 + x_2 + x_3 + \cdots$. Then, given a symmetric function f, and thinking of it as an element of Λ, we denote by $f[X]$ the expression f with p_k replaced by $x_1^k + x_2^k + x_3^k + \cdots$, for all k.

The *Kostka coefficients* $K_{\lambda,\mu}$ the coefficients appearing in

$$h_\mu = \sum_\lambda K_{\lambda,\mu} s_\lambda.$$

For a partition $\mu \vdash n$, we denote by

$$\widetilde{H}_\mu := \widetilde{H}_\mu[X] = \widetilde{H}_\mu[X; q, t] = \sum_{\lambda \vdash n} \widetilde{K}_{\lambda\mu}(q,t) s_\lambda \tag{2.1}$$

the *(modified) Macdonald polynomials*, where

$$\widetilde{K}_{\lambda\mu} := \widetilde{K}_{\lambda\mu}(q,t) = K_{\lambda\mu}(q, 1/t) t^{n(\mu)} \quad \text{with} \quad n(\mu) = \sum_{i \geq 1} \mu_i (i-1) \tag{2.2}$$

are the *(modified) Kostka coefficients* (see [8, Chapter 2] for more details).

The set $\{\widetilde{H}_\mu[X; q, t]\}_\mu$ is a basis of the ring of symmetric functions Λ with coefficients in $\mathbb{Q}(q,t)$. This is a modification of the basis introduced by Macdonald [12], and they are the Frobenius characteristic of the so called Garsia–Haiman modules (see [5]).

If we identify the partition μ with its Ferrers diagram, i.e. with the collection of cells $\{(i,j) \mid 1 \leq i \leq \mu_i, 1 \leq j \leq \ell(\mu)\}$, then for each cell $c \in \mu$ we refer to the *arm*, *leg*, *co-arm* and *co-leg* (denoted respectively as $a_\mu(c), l_\mu(c), a_\mu(c)', l_\mu(c)'$) as the number of cells in μ that are strictly to the right, above, to the left and below c in μ, respectively. We define for every partition μ

$$B_\mu := B_\mu(q,t) = \sum_{c \in \mu} q^{a'_\mu(c)} t^{l'_\mu(c)} \tag{2.3}$$

$$w_\mu := w_\mu(q,t) = \prod_{c \in \mu} (q^{a_\mu(c)} - t^{l_\mu(c)+1})(t^{l_\mu(c)} - q^{a_\mu(c)+1}). \tag{2.4}$$

3 Combinatorics of decorated Dyck paths

3.1 Basic definitions

We will use the notation $\mathsf{D}(n)$ to denote the set of Dyck paths of size n, i.e. the set of lattice paths starting from $(0,0)$ and ending at (n,n), using only north and east steps and staying weakly above the diagonal $x=y$, also called the *main diagonal*. Each Dyck path $D \in \mathsf{D}(n)$ can be described uniquely by a sequence $w_1(D)w_2(D)\cdots w_n(D)$ of n integers called the *area word* of D, where $w_i(D)$ is the number of whole squares on the i-th row between D and the main diagonal. A sequence $w_1w_2\cdots w_n$ of n nonnegative integers is an area word if and only if $w_1=0$ and $w_{i+1} \leq w_i+1$ for $i=1,\dots,n-1$.

The *rises* of a Dyck path D are the indices

$$\mathsf{Rise}(D) := \{2 \leq i \leq n \mid a_i(D) > a_{i-1}(D)\},$$

or the vertical steps that are directly preceded by another vertical step.

The *peaks* of a Dyck path D are the indices

$$\mathsf{Peak}(D) := \{1 \leq i \leq n \mid a_{i+1}(D) \leq a_i(D)\},$$

or the vertical steps that are followed by a horizontal step.

A *decorated Dyck path* is a Dyck path where certain peaks are decorated with a symbol $\circ$ (near the point joining the peak to the horizontal step following it) and certain rises are decorated with a symbol $*$. By $\mathsf{DD}(n)^{\circ a,*b}$ we denote the set of Dyck paths of size n with a decorated peaks and b decorated rises. By $\mathsf{DD}(n)^{\circ a,*b}$ we denote the set of Dyck paths of size n with a decorated peaks and b decorated rises. See Figure 1 for an example.

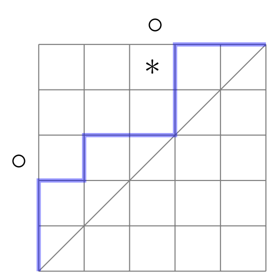

Figure 1: Example of an element in $\mathsf{DD}^{\circ 2,*1}(5)$

If a *fall* is a horizontal step that is followed by a second horizontal step, then there is a natural way of mapping rises into falls bijectively. Indeed the endpoint of a rise is a point where D vertically crosses a certain diagonal parallel to the main diagonal; since the path must end at the main diagonal it must cross the same diagonal horizontally with a fall at least once: choose the first occurrence and this yields a bijective map (see Figure 2). Therefore we can decorate rises or falls equivalently.

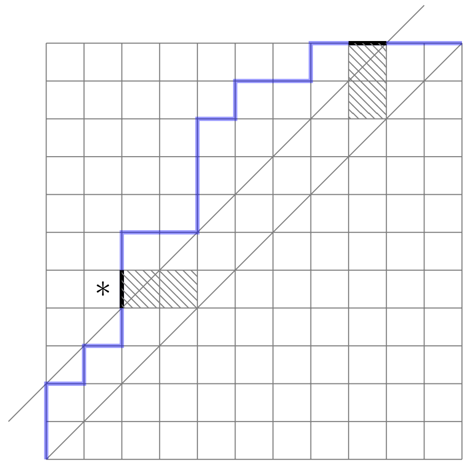

Figure 2: Correspondence between rises and falls

3.2 The statistics area, dinv, bounce and pbounce.

We introduce here four statistics on the set $\mathsf{DD}(n)^{\circ a,*b}$, all of which are generalisations of the usual statistics on undecorated Dyck paths. The area and dinv statistics are the "unlabelled" versions of the statistics defined on labelled decorated Dyck paths in [10] (cf. also [15]). The bounce and pbounce are new.

Take $D \in \mathsf{DD}(n)^{\circ a,*b}$, and let $w_1(D)w_2(D)\cdots w_n(D)$ be its area word.

Let $\mathsf{DRise}(D) \subseteq \mathsf{Rise}(D)$ be the set of indices such that $i \in \mathsf{DRise}(D)$ if the i-th vertical step of D is a decorated rise. We define the *area* of D as

$$\mathsf{area}(D) = \sum_{i \in \{1,\dots,n\} \setminus \mathsf{DRise}(D)} w_i(D). \tag{3.1}$$

For a more visual definition, the area is the number of whole squares that lie between the path and the main diagonal, except for the ones in the rows containing the vertical steps following a decorated rise.

Equivalently, we could define the area of D as the number of whole squares that lie between the path and the main diagonal, except the ones in the columns containing falls corresponding to decorated rises. For example, the path in Figure 3 has area equal to 6 (grey in the picture).

Similarly let $\mathsf{DPeak}(D) \subseteq \mathsf{Peak}(D)$ be the set of indices such that $i \in \mathsf{DPeak}(D)$ if the i-th vertical step of D is a decorated peak. We define the *dinv* of D, denoted $\mathsf{dinv}(D)$, to be the number of pairs $(i,j) \in \{1,\dots,n\} \times \{1,\dots,n\}$ with $i < j$ such that either

1. $w_i(D) = w_j(D)$ and $i \notin \mathsf{DPeak}(D)$

or,

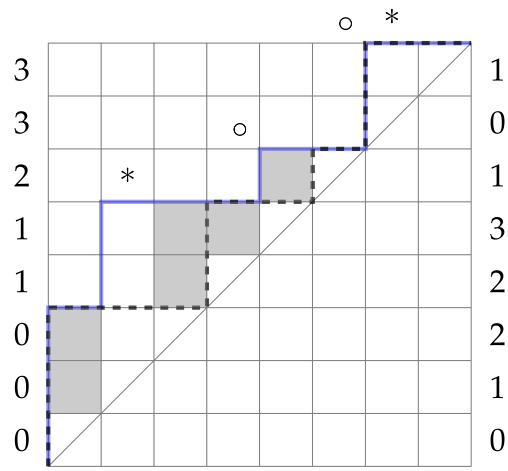

Figure 3: $D \in \mathsf{DD}^{\circ 2, *2}(8)$, its bounce word (left) and its area word (right).

2. $w_i(D) = w_j(D) + 1$ and $j \notin \mathsf{DPeak}(D)$.

We refer to the first and second kind of pairs as *primary* and *secondary dinv*, respectively. For example, the path in Figure 3 has dinv equal to 5: 4 primary and 1 secondary.

The definitions of our two bounce statistics both start from the usual definition of the bounce of undecorated Dyck paths, but they are modified (in two different ways) by the decorations on the peaks.

The *bounce path* of a decorated Dyck path $D \in \mathsf{DD}(n)^{\circ a, *b}$ starts in $(0,0)$ and travels north until it encounters the beginning of an east step of D, then it turns east until it hits the main diagonal, then it turns north again, and so on. Thus it continues until it arrives at (n, n). We label the vertical steps of the bounce path as follows: all the steps before the first "bounce", i.e. the first time that the bounce path changes direction from north to east, are labelled with a 0; next the vertical steps are labelled with a 1, again until the path "bounces"; then with a 2, and so on. The labels of the bounce path from bottom to top create a sequence of integers $b_1(D)b_2(D)\cdots b_n(D)$ which we call the *bounce word*. We define the *bounce* of D as

$$\mathsf{bounce}(D) := \sum_{i \in \{1,\dots,n\} \setminus \mathsf{DPeak(D)}} b_i(D). \tag{3.2}$$

The path in Figure 3 has bounce equal to 5.

We now describe how to compute our *pbounce* statistic of a decorated Dyck path $D \in \mathsf{DD}(n)^{\circ a, *b}$. Delete all the horizontal steps that directly follow decorated peaks. When deleting the horizontal steps, move the rest of the path one square to the left, together with part of the main diagonal so that the number of squares between the path and the main diagonal on each row remains the same. In this way we end up with a path from $(0,0)$ to $(n-a,n)$ and some line segments that form the new *main diagonal*. This is what is called the corresponding *leaning stack* in [10]. See Figure 4 for an example.

Now we draw the *pbounce path* in this new object as described before: start from $(0,0)$, bounce at the beginning of horizontal steps of the path and against the main diagonal. Then label the vertical steps as before to get a *pbounce word* $b'_1(D)b'_2(D)\cdots b'_n(D)$. The *pbounce* of D is then defined as

$$\mathsf{pbounce}(D) := \sum_{i=1}^{n} b'_i(D). \tag{3.3}$$

In the example $D \in \mathcal{D}_{11}^{(2,0)}$ of Figure 4 its pbounce word is 00111111122, so that its pbounce is $\mathsf{pbounce}(D) = 11$.

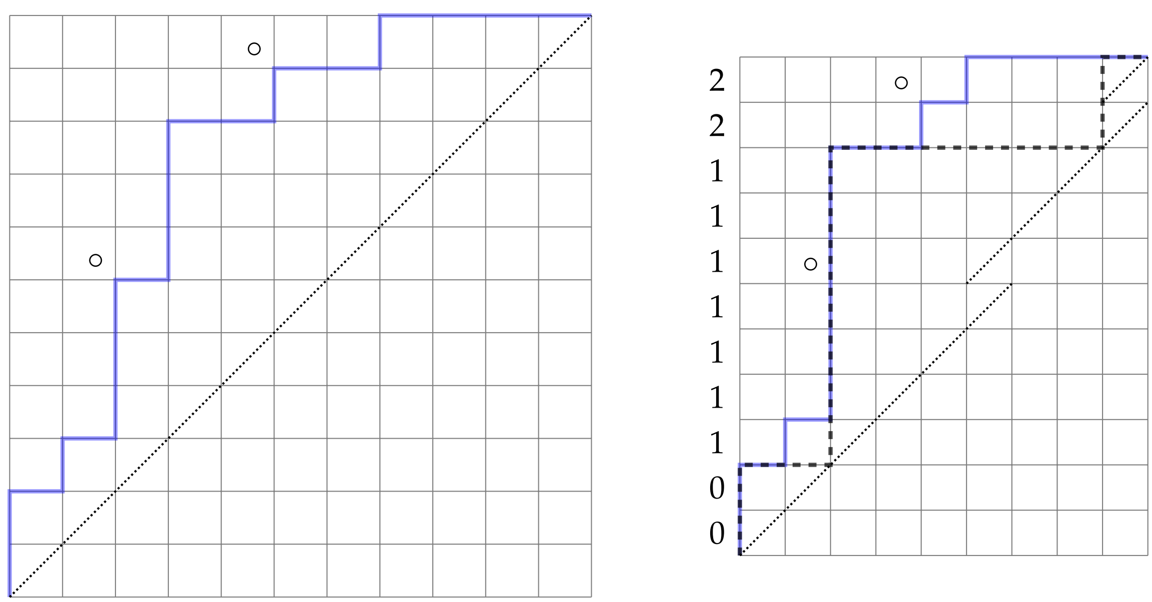

Figure 4: Construction of the pbounce path

Remark that a peak is a vertical step followed by a horizontal step, one could think of these two steps together as a diagonal step. So $\mathsf{DD}(n)^{\circ a,*b}$ can be interpreted as the set of *decorated Schröder paths* with a diagonal steps and b decorated rises. The definitions of *bounce* and *dinv* given in [8, Chapter 4] coincide with our definitions of pbounce and dinv. When $b=0$ the definition of the *area* in [8, Chapter 4] and our area also coincide.

Let us conclude this section by introducing and motivating some notation. For each of the three statistics that depend on the decorations on the peaks, i.e. dinv, bounce and pbounce, there is one specific peak that, when decorated, does not alter this statistic. Often, when dealing with one of these statistics, it will be useful to consider not the whole set of decorated Dyck paths, but rather the set of decorated Dyck paths where this specific peak is not decorated. We will use the following notations:

- $\mathsf{DDd}(n)^{\circ a,*b} \subseteq \mathsf{DD}(n)^{\circ a,*b}$ is the subset where the *rightmost highest peak*, i.e. the peak that lies the most to the right on the diagonal that is the farthest removed from the main diagonal, is never decorated, denoted as such because decorating this peak does not alter the dinv statistic;
- $\mathsf{DDb}(n)^{\circ a,*b} \subseteq \mathsf{DD}(n)^{\circ a,*b}$ is the subset where the *first peak*, i.e. the peak in the leftmost column, is never decorated, denoted as such because decorating this peak does not alter the bounce statistic;
- $\mathsf{DDp}(n)^{\circ a,*b} \subseteq \mathsf{DD}(n)^{\circ a,*b}$ is the subset where the *last peak*, i.e. the peak in the highest row is never decorated, denoted as such because decorating this peak does not alter the pbounce statistic.

3.3 Haglund's ζ (sweep) map

Theorem 3.1. *There exists a bijective map*

$$\zeta : \mathsf{DDd}(n)^{\circ a,*b} \to \mathsf{DDb}(n)^{\circ b,*a}$$

such that for all $D \in \mathsf{DDd}(n)^{\circ a,*b}$

$$\begin{aligned} \mathsf{area}(D) &= \mathsf{bounce}(\zeta(D)) \\ \mathsf{dinv}(D) &= \mathsf{area}(\zeta(D)). \end{aligned}$$

The map is essentially a decorated version of Haglund's ζ map on undecorated Dyck paths (see [4] or [8, Theorem 3.15]).

3.4 The ψ map exchanging peaks and falls

Theorem 3.2. *There exists a bijective map*

$$\psi : \mathsf{DDp}(n)^{\circ a,*b} \to \mathsf{DDp}(n)^{\circ b,*a}$$

such that for all $D \in \mathsf{DDp}(n)^{\circ a,*b}$

$$\begin{aligned} \mathsf{area}(D) &= \mathsf{area}(\psi(D)) \\ \mathsf{pbounce}(D) &= \mathsf{pbounce}(\psi(D)). \end{aligned}$$

Here we merely describe ψ. A detailed proof of its properties can be found in [3].

Let us first look at a simpler map, ψ_0, that transforms one decorated fall into a decorated peak. Call the endpoint of the decorated fall x. We travel southward from x until we hit the main diagonal, and then travel west until we hit the endpoint of a north step of the path. Call this point y. We delete the decorated east step and add an east

Figure 5: The map ψ_0

step right after the point y. Since y was the endpoint of a north step and we added an east step after y, we have created a peak, which we now decorate. See Figure 5 for an example.

Clearly, ψ_0 is invertible. Note that ψ_0^{-1} does not yield the desired result when the decoration is on the highest peak: the added horizontal step is the last of the path, so not a fall. This explains why ψ is defined on ψ on $\mathsf{DDp}(n)^{\circ a,*b}$.

The general map ψ relies on ψ_0 as follows: apply ψ_0 to all the decorated falls and ψ_0^{-1} to all the decorated peaks, one by one. However, we must be careful, because the order in which we transform the decorations matters, see Figure 6 for an example. We use the following order:

1. transform *all* the decorated falls, from left to right. We end up with a path all of whose decorations are on peaks.

2. Transform *some* of the decorated peaks, starting from the top row, down to the to bottom row. Indeed, decorated peaks that came from decorated falls in step (1) must not be transformed again.

4 Decorated q, t-Schröder

We provide three combinatorial interpretations of the formula $\langle \Delta'_{e_{a+b-k-1}} e_{a+b}, e_a h_b \rangle$.

Theorem 4.1. *For $a, b, k \in \mathbb{N} \cup \{0\}$, $a \geq 1$, $a + b \geq k + 1$, we have*

$$\langle \Delta'_{e_{a+b-k-1}} e_{a+b}, s_{b+1,1^{a-1}} \rangle = \mathsf{DDd}_{q,t}(a+b)^{\circ b,*k} = \mathsf{DDp}_{q,t}(a+b)^{\circ b,*k} = \mathsf{DDb}_{q,t}(a+b)^{\circ k,*b}.$$

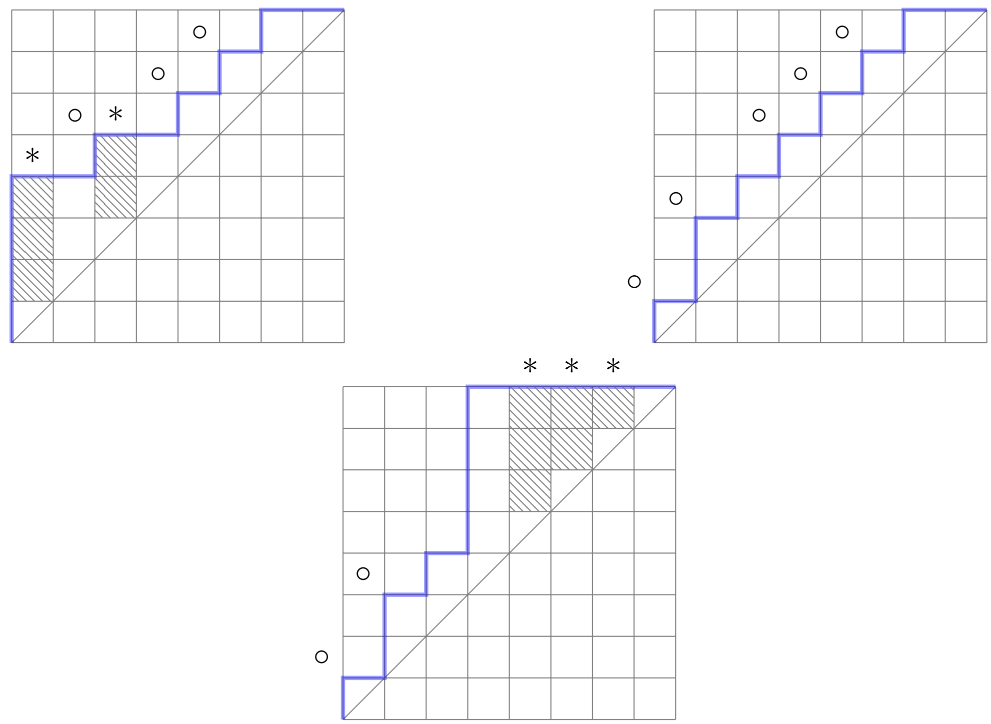

Figure 6: The two steps in the definition of the map ψ

so that, for $a+b\geq k+1$, we have

$$\begin{aligned}\langle \Delta'_{e_{a+b-k-1}} e_{a+b}, e_a h_b\rangle &= \sum_{D\in \mathsf{DD}(a+b)^{\circ b,*k}} q^{\mathsf{dinv}(D)} t^{\mathsf{area}(D)} \\ &= \sum_{D\in \mathsf{DD}(a+b)^{\circ b,*k}} q^{\mathsf{area}(D)} t^{\mathsf{bounce}(D)} \\ &= \sum_{D\in \mathsf{DD}(a+b)^{\circ b,*k}} q^{\mathsf{area}(D)} t^{\mathsf{pbounce}(D)}.\end{aligned}$$

The proof of Theorem 4.1 relies on recursions that can be shown to be satisfied by both the combinatorial objects and the symmetric functions. These recursions are crucial also in the proof of the main results in [2], and, together with them, they provide a complete solution of Problem 8.1 in [10].

For both the statements and the proofs of these recursions, we refer to [3]. Here we would like to mention only that the most technical part of these results is the following theorem, that generalizes a result of Haglund.

Theorem 4.2. *For $m,k\geq 1$ and $\ell\geq 0$, we have*

$$\sum_{\gamma\vdash m} \frac{\widetilde{H}_\gamma[X]}{w_\gamma} h_k[(1-t)B_\gamma] e_\ell[B_\gamma] = \tag{4.1}$$

$$= \sum_{j=0}^{\ell} t^{\ell-j} \sum_{s=0}^{k} q^{\binom{s}{2}} \begin{bmatrix} s+j \\ s \end{bmatrix}_q \begin{bmatrix} k+j-1 \\ s+j-1 \end{bmatrix}_q h_{s+j} \left[\frac{X}{1-q}\right] h_{\ell-j} \left[\frac{X}{M}\right] e_{m-s-\ell} \left[\frac{X}{M}\right].$$

5 A symmetry result

In [10], the following symmetry is conjectured.

$$\langle \Delta_{h_\ell} \nabla e_{n-\ell}, s_{k+1,1^{n-\ell-k-1}} \rangle = \langle \Delta_{h_k} \nabla e_{n-k}, s_{\ell+1,1^{n-k-\ell-1}} \rangle \qquad \text{for } n > k+\ell. \tag{5.1}$$

We provide both an algebraic and combinatorial proof of this. The combinatorial proof relies on the ψ-map of Theorem 3.2, which in fact has the property that it preserves the bistatistic $(\mathsf{area}, \mathsf{pbounce})$ (see [3] for proofs). The algebraic one uses the following theorem.

Theorem 5.1. *For all $a, b, k \in \mathbb{N}$, with $a \geq 1$, $b \geq 1$ and $1 \leq k \leq a$, we have*

$$\langle \Delta'_{e_a} \Delta'_{e_{a+b-k-1}} e_{a+b}, h_{a+b} \rangle = \langle \Delta_{h_k} \Delta'_{e_{a-k}} e_{a+b-k}, e_{a+b-k} \rangle. \tag{5.2}$$

Theorem 5.2. *For $n > k+\ell$, the following expression is symmetric in k and ℓ:*

$$\langle \Delta_{h_k} \Delta'_{e_{n-k-\ell-1}} e_{n-k}, e_{n-k} \rangle. \tag{5.3}$$

Algebraic proof of Theorem 5.2. Using (5.2), with $n = a+b$ and $\ell = b-1$, we have

$$\langle \Delta_{h_k} \Delta'_{e_{n-k-\ell-1}} e_{n-k}, e_{n-k} \rangle = \langle \Delta'_{e_{n-\ell-1}} \Delta'_{e_{n-k-1}} e_n, h_n \rangle, \tag{5.4}$$

which is obviously symmetric in k and ℓ. □

Notice that the expressions in (5.1) and in Theorem 5.2 can easily be shown to be the same.

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