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Increasing Labelings, Generalized Promotion and Rowmotion

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Abstract. We generalize Bender–Knuth promotion on linear extensions to an analogous action on increasing labelings of any finite poset, in which the restrictions on the values of the labels satisfy a natural consistency condition. We give an equivariant bijection between such increasing labelings under this generalized promotion and order ideals in an associated poset under rowmotion. Additionally, we give a criterion for when certain kinds of toggle group actions on order ideals of a finite poset are conjugate to rowmotion. These results build upon work of O. Pechenik with the first two authors in the case of rectangular increasing tableaux and work of N. Williams with the second author relating promotion and rowmotion on ranked posets. We apply these results to posets embedded in the Cartesian product of ranked posets and increasing labelings with labels between 1 and q, in which case we obtain new instances of the resonance phenomenon.

1 Introduction

Promotion is a natural action defined by M.-P. Schützenberger on standard Young tableaux and, more generally, linear extensions of finite poset [5], arising from study of evacuation and the RSK correspondence. Promotion has many beautiful properties and significant applications in representation theory. In [6], R. Stanley surveys many of these properties of promotion on linear extensions.

In [7], N. Williams and the second author studied an action on order ideals they called *rowmotion*. Given a poset and an order ideal, rowmotion gives a new order ideal generated by the minimal elements of the poset not in the order ideal. In the case of a particular poset, they showed rowmotion is in equivariant bijection with Schützenberger promotion on two-row standard Young tableaux. They did this by constructing a *toggle group action* conjugate to rowmotion that corresponds to Schützenberger promotion in this special case; they named this toggle group action *promotion* because of this correspondence.

Toggles are defined as follows; toggle group actions are compositions of toggles.

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Definition 1.1. For any $e \in P$, P a poset, and $J(P)$ its lattice of order ideals, the *toggle* $t_e : J(P) \to J(P)$ is defined as follows:

$$t_e(I) = \begin{cases} I \cup \{e\} & \text{if } e \notin I \text{ and } I \cup \{e\} \in J(P) \\ I \setminus \{e\} & \text{if } e \in I \text{ and } I \setminus \{e\} \in J(P) \\ I & \text{otherwise.} \end{cases}$$

In [1], P. Cameron and D. Fon-der-Flaass showed rowmotion can be performed by toggling each element of a poset from top to bottom. In [7], N. Williams and the second author showed that there is an equivariant bijection between order ideals under promotion and rowmotion for any poset that can be projected into the two-dimensional lattice, not only those corresponding to two-row tableaux.

In [4], O. Pechenik generalized Schützenberger promotion on standard Young tableaux to *K-promotion* on *increasing tableaux*, using the *K-jeu de taquin* of H. Thomas and A. Yong [8]. In [2], O. Pechenik and the first and second authors built on this work to give a bijection between increasing tableaux of rectangular shape with a certain largest possible entry and order ideals in a product of three chains poset. While the bijection between the two is straightforward, the authors furthermore showed that *K-promotion* on increasing tableaux is carried equivariantly to the toggle group action *hyperplane promotion* on order ideals in the product of three chains poset. This was done by showing K-promotion can be written a product of *K-Bender–Knuth involutions*, which the authors defined and showed are equivalent to *hyperplane toggles* on the product of three chains poset. They also generalized the main result of [7] from two to n dimensions, showing that hyperplane promotion is conjugate to rowmotion for any poset that can be projected into the n-dimensional lattice. The authors also defined the notion of *resonance*, which occurs when a bijective action on a finite set projects to a cyclic action of small order. They found an instance of resonance on increasing tableaux under K-promotion.

Our main result is a generalization of the equivariant bijection in [2] between increasing tableaux under K-promotion and the product of three chains under rowmotion. In our generalization, increasing labelings of any finite poset play the role of increasing tableaux and we define a natural analogue of K-promotion on them. We construct an associated poset whose order ideals we show are in bijection with the increasing labelings. We then show that our generalized promotion on increasing labelings is conjugate to rowmotion on the order ideals of the associated poset. We also give a resonance result.

In Section 2, we generalize increasing tableaux to increasing labelings and construct the associated poset $\Gamma_1(P, R)$ needed for our main result; we prove increasing labelings are in bijection with order ideals of $\Gamma_1(P, R)$ in Theorem 2.6. In Section 3, we extend K-promotion for increasing tableaux to increasing labelings by defining analogues of both Bender–Knuth involutions and jeu-de-taquin and showing that either definition yields the same action. In Theorem 3.12, we extend the resonance result on increasing tableaux to increasing labelings. In Section 4, we show our main result, an equivariant bijection between increasing labelings under the analogue of K-promotion and the corresponding order ideals under rowmotion in Theorem 4.1; this completes the generalization of the equivariant bijection of [2].

This is an extended abstract only, see [3] for the full version, including all proofs, more examples, and an extension to weakly increasing labelings.

2 Increasing labelings

In this section, we extend the ideas behind the relationship between increasing tableaux of square shape and order ideals in the product of three chains poset to increasing labelings of any poset and order ideals in an associated poset.

We say that a function $f : P \to \mathbb{Z}$ is an *increasing labeling* if $x <_P y$ implies that $f(x) < f(y)$ (with the usual total ordering on the integers). We will be interested in looking at sets of increasing labelings on P given certain restrictions.

Note all posets in this paper are finite.

2.1 Construction of Γ_1

In this subsection, we construct a poset, $\Gamma_1(P, R)$, whose order ideals are in bijection with increasing labelings of P with ranges restricted by R.

First, we consider increasing labelings of P where we may independently choose which labels each entry can attain. We use $R : P \mapsto \mathcal{P}(\mathbb{Z})$ to denote the function indicating which labels our increasing labeling f is allowed to attain. Let $\mathrm{Inc}^R(P)$ be the set of all increasing labelings f of P where $f(p) \in R(p)$. By convention, if $R(p) = \emptyset$ for any $p \in P$, then $\mathrm{Inc}^R(P)$ is also the empty set. For simplicity, we will assume $R(p)$ is finite for every $p \in P$. Also, let $R(p)^*$ be $R(p)$ with its largest element removed.

One natural restriction to place on R is to require that if $k \in R(p)$, then there must be some increasing labeling $f \in \mathrm{Inc}^R(P)$ with $f(p) = k$. Otherwise, we could remove k from the set of available labels for p and not change the set of allowable increasing labelings. We formalize this in the next definition and theorem.

Definition 2.1. Say that a labeling function $R : P \mapsto \mathcal{P}(\mathbb{Z})$ is *consistent* if for every covering relation $x \lessdot y$ in P, we have $\min(R(x)) < \min(R(y))$ and $\max(R(x)) < \max(R(y))$.

Theorem 2.2. *We have that R is consistent if and only if for every $p \in P$ and $k \in R(p)$, there is an increasing labeling $f \in \mathrm{Inc}^R(P)$ with $f(p) = k$.*

We now consider $\mathrm{Inc}^R(P)$ as a partially ordered set, where $f \leq g$ if and only if $f(p) \leq g(p)$ for all $p \in P$. Furthermore, it is a lattice, with meet given by $(f \wedge g)(p) = \min(f(p), g(p))$ and join given by $(f \vee g)(p) = \max(f(p), g(p))$. One can easily check that this lattice is distributive, so we may apply Birkhoff's Representation Theorem (also known as the fundamental theorem of finite distributive lattices).

Definition 2.3. Given a consistent labeling function R, let $R(p)_{>k}$ be the smallest label of $R(p)$ that is larger than k, and let $R(p)_{<k}$ be the largest label of $R(p)$ less than k.

Definition 2.4. Let P be a poset and R a consistent map of possible labels. Then define $\Gamma_1(P, R)$ to be the poset whose elements are (p, k) with $p \in P$ and $k \in R(p)^*$, and covering relations given by $(p_1, k_1) \lessdot (p_2, k_2)$ if and only if either

1. $p_1 = p_2$ and $R(p_1)_{>k_2} = k_1$ (i.e., k_1 is the next largest possible label after k_2), or
2. $p_1 \lessdot p_2$ (in P), $k_1 = R(p_1)_{<k_2} \neq \max(R(p_1))$, and no greater k in $R(p_2)$ has $k_1 = R(p_1)_{<k}$. That is to say, k_1 is the largest label of $R(p_1)$ less than k_2 ($k_1 \neq \max(R(p_1))$), and there is no greater $k \in R(p_2)$ having k_1 as the largest label of $R(p_1)$ less than k.

Remark 2.5. In $\Gamma_1(P, R)$, we lose the information about $\max(R(p))$ for each $p \in P$. So when we draw $\Gamma_1(P, R)$, we add a label $(p, \max(R(p)))$ underneath the chain of elements of the form (p, k) for $k \in R(p)^*$. This is a reminder that when an order ideal contains no elements of the form (p, k), in the corresponding increasing labeling, the element p is sent to $\max(R(p))$. See Figure 1.

Theorem 2.6. *The poset $\Gamma_1(P, R)$ is isomorphic to the dual of the lattice of meet irreducibles of $\mathrm{Inc}^R(P)$. Therefore, order ideals of $\Gamma_1(P, R)$ are in bijection with $\mathrm{Inc}^R(P)$.*

2.2 Restricting the global set of labels

One special case of interest is when the only restriction we place is that the labels are in the bounded set $[q] = \{1, \dots, q\}$. We denote this set as $\mathrm{Inc}^q(P)$. For example, in K-theory of the Grassmannian, increasing tableaux that only use the labels between 1 and some fixed number q are of interest [2, 8].

In general, the range of possible values for a particular element is determined by a maximum length chain containing that element.

Definition 2.7. Given $p \in P$, let $\alpha(p)$ be the number of elements less than p in a maximum length chain containing p, and let $\beta(p)$ be the number of elements greater than p in a maximum length chain containing p.

Lemma 2.8. *If every chain in P has length at most q, then the map R taking p to $[1 + \alpha(p), q - \beta(p)]$ is consistent.*

From now on, assume the length of all maximal chains in P is at most q.

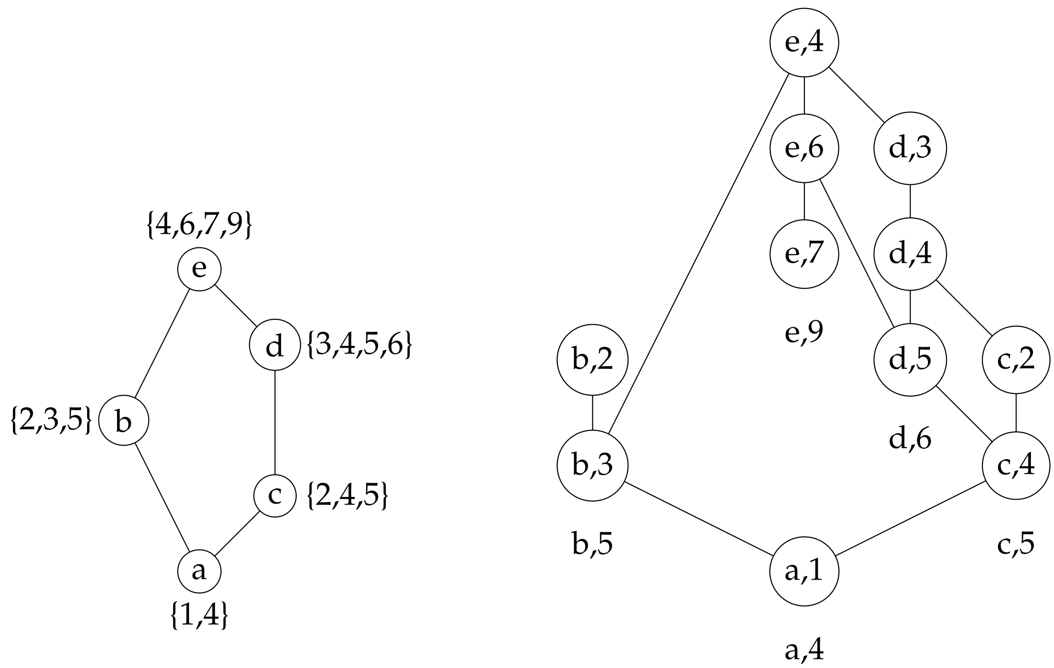

Figure 1: A poset P with restriction function R, and the associated poset $\Gamma_1(P, R)$.

Definition 2.9. Let $\Gamma_1(P, q)$ be the poset $\Gamma_1(P, R)$ for the restriction function given by $R(p) = [1 + \alpha(p), q - \beta(p)]$.

We obtain a simpler description of the covering relations in $\Gamma_1(P, q)$ than in the case of general ranges, because the range of each possible entry is an interval. See Figure 2.

Theorem 2.10. *Let R be a consistent restriction function for a poset P such that $R(p)$ is always a non-empty interval. Then $\Gamma_1(P, R)$ is the poset with elements $\{(p, k) \mid p \in P$ and $k \in R(p)^*\}$ and covering relations given by $(p_1, k_1) \lessdot (p_2, k_2)$ if and only if either*

1. $p_1 = p_2$ *and* $k_1 = k_2 + 1$, *or*
2. $p_1 \lessdot_P p_2$ *and* $k_1 + 1 = k_2$.

Corollary 2.11. *$\Gamma_1(P, q)$ is the poset with elements $\{(p, k) \mid p \in P$ and $k \in [1 + \alpha(p), q - \beta(p) - 1]\}$, and covering relations given by $(p_1, k_1) \lessdot (p_2, k_2)$ if and only if either*

1. $p_1 = p_2$ *and* $k_1 = k_2 + 1$, *or*
2. $p_1 \lessdot_P p_2$ *and* $k_1 + 1 = k_2$.

We are able to derive similar results for weakly increasing labelings, and in the case when a poset is ranked, one may pass between weakly increasing labelings and strictly increasing labelings. With this connection, the previous results of [2] can be recovered. See the full version on the arXiv [3] for more details.

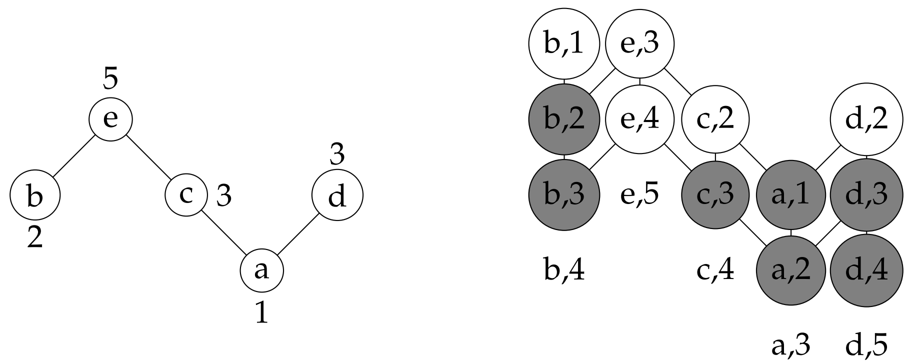

Figure 2: An increasing labeling of a poset P with largest possible entry 5, and the corresponding order ideal in $\Gamma_1(P, 5)$.

3 Promotion on $\mathrm{Inc}^R(P)$ and $\mathrm{Inc}^q(P)$

In this section, we generalize M.-P. Schützenberger's *promotion* operator to increasing labelings. We first work in $\mathrm{Inc}^q(P)$. We give two definitions in this setting: the first in terms of generalized Bender–Knuth involutions and the second in terms of generalized jeu de taquin slides. We prove the equivalence of these two definitions in Theorem 3.7. We then give a resonance result on this action. Finally, we generalize the Bender–Knuth involutions of Definition 3.1 to the case of general ranges $\mathrm{Inc}^R(P)$.

Definition 3.1. For each $i \in \mathbb{Z}$, define the ith Bender–Knuth involution $\rho_i : \mathrm{Inc}^q(P) \to \mathrm{Inc}^q(P)$ as follows. For $x \in P$, let

$$\rho_i(f)(x) = \begin{cases} i+1 & f(x) = i \text{ and the resulting labeling is still in } \mathrm{Inc}^q(P) \\ i & f(x) = i+1 \text{ and the resulting labeling is still in } \mathrm{Inc}^q(P) \\ f(x) & \text{otherwise.} \end{cases}$$

That is, ρ_i increments i and/or decrements $i+1$ *wherever possible*. Define *Bender–Knuth promotion* on f as the product $\mathrm{Pro}(f) = \rho_{q-1} \cdots \circ \rho_3 \circ \rho_2 \circ \rho_1(f)$.

We give another definition in terms of generalized jeu de taquin slides.

Definition 3.2. Let $\mathbb{Z}_\square(P)$ denote the set of labelings $g : P \to (\mathbb{Z} \cup \square)$. Define the ith *jeu de taquin slide* $\sigma_i : \mathbb{Z}_\square(P) \to \mathbb{Z}_\square(P)$ as follows:

$$\sigma_i(g)(x) = \begin{cases} i & g(x) = \square \text{ and } g(y) = i \text{ for some } y \gtrdot x \\ \square & g(x) = i \text{ and } g(z) = \square \text{ for some } z \lessdot x \\ g(x) & \text{otherwise.} \end{cases}$$

In words, $\sigma_i(g)(x)$ replaces a label $\square$ with i if i is the label of a cover of x, replaces a label i by $\square$ if x covers an element labeled by $\square$, and leaves all other labels unchanged.

Let $\sigma_{i\to j} : \mathbb{Z}_\square(P) \to \mathbb{Z}_\square(P)$ be defined as

$$\sigma_{i\to j}(g)(x) = \begin{cases} j & g(x) = i \\ g(x) & \text{otherwise.} \end{cases}$$

In words, $\sigma_{i\to j}(g)(x)$ replaces all labels i by j.

For $f \in \mathrm{Inc}^q(P)$, let $jdt(f) = \sigma_{\square\to(q+1)}\sigma_q \circ \sigma_{q-1} \circ \cdots \circ \sigma_3 \circ \sigma_2 \circ \sigma_{1\to\square}(f)$. That is, first replace all 1 labels by $\square$. Then perform the ith jeu de taquin slide σ_i for all $2 \le i \le q$. Next, replace all labels $\square$ by $q+1$. Define *jeu de taquin promotion* on f as $\mathrm{Pro}(f)(x) = jdt(f)(x) - 1$.

Proposition 3.3. *For $f \in \mathrm{Inc}^q(P)$ and $\mathrm{Pro}(f)$ as in Definition 3.2, $\mathrm{Pro}(f) \in \mathrm{Inc}^q(P)$.*

Remark 3.4. Note that the above proposition would not hold if we used $\mathrm{Inc}^R(P)$ instead of $\mathrm{Inc}^q(P)$, since the result of the generalized jeu de taquin slides and then subtracting one would no longer be guaranteed to produce labels in the ranges required by R.

The next theorem justifies our use of the notation $\mathrm{Pro}(f)$ in both Definitions 3.1 and 3.2. We will need the following definition.

Definition 3.5. Define the *sliding subposet* of $\mathrm{Pro}(f)$ as the subposet $S(f) \subseteq P$ such that the label on x is $\square$ at some point during the algorithm of Definition 3.2.

Remark 3.6. The sliding subposet coincides with the *flow paths* of O. Pechenik in the case of increasing tableaux [4] and with the *jeu de taquin sliding path* or *promotion path* in the case of standard Young tableaux [6].

Theorem 3.7. *For $f \in \mathrm{Inc}^q(P)$, Bender–Knuth promotion on f equals jeu de taquin promotion on f, that is, Definitions 3.1 and 3.2 coincide.*

See Figure 3 for an example.

Remark 3.8. If we restrict to bijective labelings, this action reduces to promotion on linear extensions. If P is a partition shaped poset, this action is K-promotion on increasing tableaux.

Remark 3.9. J. Propp has defined a notion of promotion on P-partitions using local involutions. While Definition 3.1 can be viewed as defining a promotion on P-partitions using local involutions, the two notions are not the same.

We turn our attention to resonance, defined below.

Definition 3.10 ([2])**.** Suppose $G = \langle g \rangle$ is a cyclic group acting on a set X, $\mathcal{C}_\omega = \langle c \rangle$ a cyclic group of order ω acting nontrivially on a set Y, and $f : X \to Y$ a surjection. We say the triple (X, G, f) exhibits *resonance with frequency* ω if, for all $x \in X$, $c \cdot f(x) = f(g \cdot x)$.

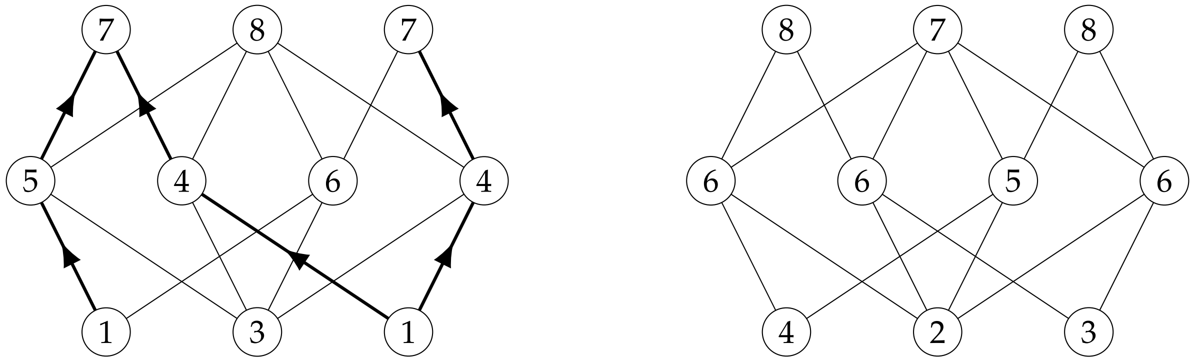

Figure 3: An increasing labeling of a poset with sliding poset indicated, and the resulting increasing labeling after promotion.

Definition 3.2 implies the following lemma, which we use to prove a new resonance statement in Theorem 3.12. Let the binary content of $f \in \mathrm{Inc}^q(P)$, denoted $\mathrm{Con}(f)$, be defined as the length q vector such that the ith digit of $\mathrm{Con}(f)$ is 1 if $f(x) = i$ for some $x \in P$ and 0 otherwise.

Lemma 3.11. *Promotion on* $\mathrm{Inc}^q(P)$ *rotates the binary content vector* $Con(f)$.

This lemma yields the following resonance statement, which is an analogue of [2, Theorem 2.2] in the case of increasing tableaux.

Theorem 3.12. $(\mathrm{Inc}^q(P), \langle \mathrm{Pro} \rangle, \mathrm{Con})$ *exhibits resonance with frequency* q.

Finally, in this section we generalize Definition 3.1 to $\mathrm{Inc}^R(P)$. We will use this definition in Theorem 4.17 to show this generalized promotion on order ideals equivariantly takes generalized Bender–Knuth promotion to a certain toggle group action on $\Gamma_1(P, R)$.

Definition 3.13. Suppose $R : P \mapsto \mathcal{P}(\mathbb{Z})$ is a consistent map of possible labels. Recall $R(x)_{>i}$ denotes the smallest element in $R(x)$ greater than i. For each $i \in \mathbb{Z}$, define the ith Bender–Knuth involution $\rho_i : \mathrm{Inc}^R(P) \to \mathrm{Inc}^R(P)$ as follows. For $x \in P$, let

$$\rho_i(f)(x) = \begin{cases} R(x)_{>i} & f(x) = i \text{ and the resulting labeling is still in } \mathrm{Inc}^R(P) \\ i & f(x) = R(x)_{>i} \text{ and the resulting labeling is still in } \mathrm{Inc}^R(P) \\ f(x) & \text{otherwise.} \end{cases}$$

That is, ρ_i changes i to $R(x)_{>i}$ and/or $R(x)_{>i}$ to i *wherever possible*. Define *Bender–Knuth promotion* on f as the product $\mathrm{Pro}(f) = \cdots \circ \rho_3 \circ \rho_2 \circ \rho_1 \circ \cdots (f)$.

Note since each $R(x)$ is finite, the infinite product of the ρ_i reduces to a finite product.

Remark 3.14. If $R(x) = [1 + \alpha(x), q - \beta(x)]$ for all $x \in P$, where α and β are as in Definition 2.7, the definition above reduces to Definition 3.1.

4 Equivariance of the bijection

The purpose of this section will be to prove the following theorem and corollary.

Theorem 4.1. *When H_{Γ_1} is a column toggle order, there is an equivariant bijection between* $\mathrm{Inc}^R(P)$ *under promotion and order ideals in* $\Gamma_1(P,R)$ *under rowmotion.*

Corollary 4.2. *There is an equivariant bijection between* $\mathrm{Inc}^q(P)$ *under promotion and order ideals in* $\Gamma_1(P,q)$ *under rowmotion.*

4.1 Toggle-promotion is conjugate to rowmotion

In this section, we define a toggle group action that toggles every element of the poset exactly once using a *toggle order*. A toggle order does not specify a total ordering on the poset in which elements must be toggled, but it allows elements that are not part of a covering relation to be toggled simultaneously. In the specific case where this toggle order is a *column toggle order*, we will show this toggle group action is conjugate to rowmotion.

Note that as opposed to previous results establishing the conjugacy of rowmotion and various promotion toggle group actions in [7] and [2], we do not require P to be ranked, and our constructions do not rely on any kind of geometric embedding.

Definition 4.3. We say that a function $H : P \to \mathbb{Z}$ is a *toggle order* if $p_1 \lessdot p_2$ implies $H(p_1) \neq H(p_2)$. Given a toggle order H, define T_H^i to be the toggle group element that is the product of all t_p for $p \in P$ such that $H(p) = i$.

Definition 4.4. We say that *toggle-promotion* with respect to H, denoted Pro_H, is the toggle group element given by $\ldots T_H^{-2} T_H^{-1} T_H^{0} T_H^{1} T_H^{2} \ldots$

Note every element of P gets toggled exactly once in Pro_H. Now, consider a special toggle order.

Definition 4.5. We say that a function $H : P \to \mathbb{Z}$ is a *column toggle order* if whenever $p_1 \lessdot p_2$ in P, then $H(p_1) = H(p_2) \pm 1$.

We call this a column toggle order because it implies that our poset elements can be partitioned into subsets we call *columns* whose elements have covering relations only with elements in adjacent columns. We can also think of it as inducing a bipartite coloring of the Hasse diagram of P.

Remark 4.6. Definition 4.5 generalizes the columns of rc-posets from [7] and hyperplane toggles from [2], as these are both examples of column toggle orders. Therefore, Theorem 4.7 below is a generalization of the promotion and rowmotion theorems of [7] and [2]. Note that Theorem 4.7 applies to non-ranked posets, while in the previous cases, the posets were required to be ranked.

Theorem 4.7. *Let P be a poset and H a column toggle order of P. Then the toggle group action* Pro_H *is conjugate to rowmotion.*

4.2 Applications of the conjugacy of toggle-promotion and rowmotion

As our first application of Theorem 4.7, we consider $\Gamma_1(P,R)$.

Definition 4.8. Let $H_{\Gamma_1} : \Gamma_1(P,R) \to \mathbb{Z}$ denote the map taking (p,k) to k.

Lemma 4.9. *For any $\Gamma_1(P,R)$, H_{Γ_1} defines a toggle order.*

Since the construction of Γ_1 gives a natural toggle order, we may define toggle-promotion with respect to this toggle order. We give this the following notation.

Definition 4.10. Let Pro_{Γ_1} denote $\mathrm{Pro}_{H_{\Gamma_1}}$.

Lemma 4.9 and Theorem 4.7 yield the following corollary.

Corollary 4.11. *If H_{Γ_1} is a column toggle order, then toggle-promotion Pro_{Γ_1} on $\Gamma_1(P,R)$ is conjugate to rowmotion.*

In Theorem 4.17, we show that Pro_{Γ_1} on $\Gamma_1(P,R)$ exactly corresponds to Bender–Knuth promotion on $\mathrm{Inc}^R(P)$.

We obtain a stronger result when we look at the case where the range of values for each entry is an interval. Note that $\Gamma_1(P,q)$ is one such example.

Lemma 4.12. *If a consistent restriction function R always has $R(p)$ a non-empty interval, then for $\Gamma_1(P,R)$, the map H_{Γ_1} defines a column toggle order.*

This lemma and Corollary 4.11 yield the following.

Corollary 4.13. *If a consistent restriction function R always has $R(p)$ a non-empty interval, then toggle-promotion Pro_{Γ_1} on $\Gamma_1(P,R)$ is conjugate to rowmotion.*

A second application comes from Cartesian products.

Definition 4.14. We say that a Cartesian embedding of a ranked poset P into an ordered pair of ranked posets (P_1,P_2) is an order and rank preserving map from P into the Cartesian product $P_1 \times P_2$.

Lemma 4.15. *Let P be a ranked poset with an order and rank preserving map W to the Cartesian product of two ranked posets, $P_1 \times P_2$. Then if $W(p) = (x,y) \in P_1 \times P_2$, the map $H : p \mapsto \mathrm{rk}_{P_1}(x) - \mathrm{rk}_{P_2}(y)$ defines a column toggle order, and thus Pro_H on P is conjugate to rowmotion.*

Remark 4.16. Hyperplane promotion $\mathrm{Pro}_{\pi,v}$ of [2] with respect to a lattice embedding π can be thought of a special case of this. In particular, the 2^n choices of hyperplanes correspond to the 2^n ways that we can choose a subset $S \subseteq [n]$ and define a Cartesian embedding from $\mathbb{Z}^n$ to $\mathbb{Z}^{|S|} \times \mathbb{Z}^{n-|S|}$ by permuting coordinates so coordinates in S go to one of the first $|S|$ copies of $\mathbb{Z}$, and coordinates not in S get permuted to the last $n-|S|$ copies of $\mathbb{Z}$.

4.3 Toggle-promotion is Bender–Knuth promotion

In this subsection, we show the following theorem, which is the final ingredient in the proof of Theorem 4.1.

Theorem 4.17. *The map from* $\mathrm{Inc}^R(P)$ *to order ideals of* $\Gamma_1(P,R)$ *equivariantly takes Bender–Knuth promotion on* $\mathrm{Inc}^R(P)$ *to* Pro_{Γ_1} *on* $J(\Gamma_1(P,R))$.

This follows from the lemma below.

Lemma 4.18. *The map from* $\mathrm{Inc}^R(P)$ *to order ideals in* $\Gamma_1(P,R)$ *equivariantly takes the operator* ρ_k *to the toggle operator* $T^k_{H_{\Gamma_1}}$.

Proof. Recall that the column toggle order H_{Γ_1} maps $(p,k) \in \Gamma_1(P,R)$ to k, so $T^k_{H_{\Gamma_1}}$ toggles all elements in $\Gamma_1(P,R)$ of the form (p,k). Suppose (p,k) can be toggled out. It can only be toggled out if it is a maximal element of the order ideal, which means that the corresponding increasing labeling gives the label k to p. When we toggle (p,k) out of I, the corresponding increasing labeling now gives the label $R(p)_{>k}$ to p, and the result is an increasing labeling. This is exactly the effect of ρ_i in this case. Now suppose (p,k) can be toggled in. This either means that $(p,R(p)_{>k})$ is in I, or no (p,k') is in I. In both cases, the corresponding increasing labeling starts with p being labeled with $R(p)_{>k}$ and getting reduced to k. This is exactly the effect of ρ_k in this case. Finally, suppose (p,k) can neither be toggled in nor out of I. This means that changing p to $R(P)_{>k}$ (or vice versa) does not result in an increasing labeling. $\square$

Proof of Theorem 4.1. By Theorem 4.17, we know that the bijection between $\mathrm{Inc}^R(P)$ and $\Gamma_1(P,R)$ carries Bender–Knuth promotion on $\mathrm{Inc}^R(P)$ to Pro_{Γ_1} on $J(\Gamma_1(P,R))$. Then by Corollary 4.11, if H_{Γ_1} is a column toggle order, then Pro_{Γ_1} on $\Gamma_1(P,R)$ is conjugate to rowmotion. $\square$

Proof of Corollary 4.2. By Lemma 4.12, H_{Γ_1} is a column toggle order for $\Gamma_1(P,q)$. $\square$

Remark 4.19. In this paper, we have shown the two components of the proof both hold more generally; Theorem 4.1 is a particular case where both components hold. Theorem 4.17 shows Bender–Knuth promotion on increasing labelings corresponds to toggle-promotion for a generic consistent restriction function R, not only the ones for which H_{Γ_1} is a column toggle order. Similarly, Theorem 4.7 shows toggle-promotion is conjugate to rowmotion not only for $\Gamma_1(P,q)$, but for any poset which can be given a column toggle order.

Finally, we obtain as a corollary of Corollary 4.2 and Theorem 3.12 the following resonance result on order ideals in $\Gamma_1(P,q)$ under rowmotion.

Corollary 4.20. *Let* φ *be the map from an order ideal in* $\Gamma_1(P,q)$ *to the corresponding increasing labeling on* P. *Then* $(J(\Gamma_1(P,q)), \langle\mathrm{Row}\rangle, \mathrm{Con}\circ\varphi)$ *exhibits resonance with frequency* q.

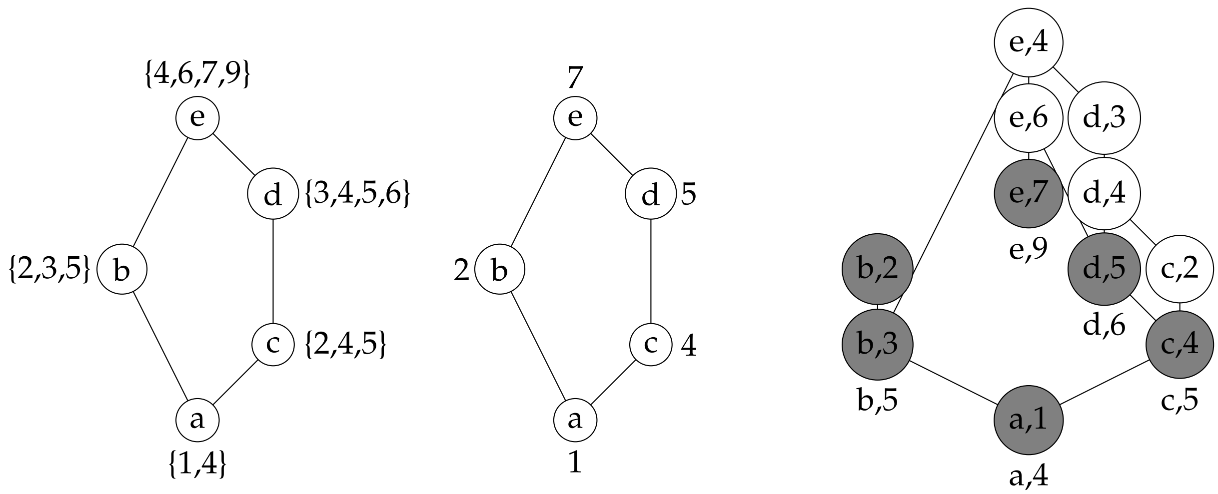

Figure 4: Left: A poset P with restriction function R; Middle: An increasing labeling of P; Right: The associated poset $\Gamma_1(P, R)$ and corresponding order ideal.

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