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# A combinatorial formula for Macdonald cumulants

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**Abstract.** Macdonald cumulants are symmetric functions that generalize Macdonald polynomials. We prove a combinatorial formula for them which extends the celebrated formula of Haglund for Macdonald polynomials. We also provide several applications of our formula – it gives a new, constructive proof of a strong factorization property of Macdonald polynomials and it proves that Macdonald cumulants are $q,t$–positive in the monomial and in the fundamental quasisymmetric bases. Furthermore, we use our formula to prove the recent higher-order Macdonald positivity conjecture for the coefficients of the Schur polynomials indexed by hooks. Our combinatorial formula links Macdonald cumulants to $G$–parking functions of Postnikov and Shapiro.

**Keywords:** Macdonald polynomials; Schur polynomials; Cumulants; Tutte polynomials; $G$–Parking functions; $q,t$-Kostka numbers

# 1 Macdonald cumulants

The main character in this paper is a certain symmetric function $\kappa(\lambda^1,\dots,\lambda^r)(\boldsymbol{x};q,t)$ indexed by a finite set of partitions $\{\lambda^1,\dots,\lambda^r\}$ and two additional parameters $q,t$. This function is called *Macdonald cumulant* and this extended abstract describes the main ideas presented in the sequence of papers [5, 3, 2] which initiated study of Macdonald cumulants.

## 1.1 Motivation

The initial motivation for studying cumulants $\kappa(\lambda_1,\dots,\lambda_r)$ comes from our attempts [5] on proving the $b$-conjecture – one of the major open problems in the theory of Jack symmetric functions posed by Goulden and Jackson [8]. The $b$-conjecture states that the coefficients of a certain multivariate generating function $\psi(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z};\beta)$ involving Jack

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symmetric functions can be interpreted as weighted generating functions of graphs embedded into surfaces. Except some special cases [11, 4] not much is known and the $b$-conjecture is still wide open. However, in our recent paper [5] the author and Féray were able to rewrite the function $\psi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}; \beta)$ as a linear combination of cumulants of Jack symmetric functions, which are specializations of $\kappa(\lambda_1, \dots, \lambda_r)$. In view of this result, understanding of the structure of Macdonald cumulants is of great interest as a potential tool for solving the $b$-conjecture.

Let $\mathcal{A}$ be a commutative ring with two different multiplicative structures $\cdot$ and $\oplus$ which define two (different) algebra structures on $\mathcal{A}$. Then, for any $X_1, \dots, X_r \in \mathcal{A}$ one can define a *conditional cumulant* $\kappa(X_1, \dots, X_r) \in \mathcal{A}$ by the following generating series

$$\kappa(X_1, \dots, X_r) := [t_1 \cdots t_r] \log \mathbb{E} e^{t_1 X_1 + \cdots t_r X_r}, \tag{1.1}$$

where $\mathbb{E} : (\mathcal{A}, \oplus) \to (\mathcal{A}, \cdot)$ is the identity map. This cumulant has the following interpretation: it measures the discrepancy between those two algebraic structures. Definition (1.1) can be transformed into an equivalent but more combinatorial definition:

$$\kappa(X_1, \dots, X_r) = \sum_{\pi \in \mathcal{P}([r])} (-1)^{\#\pi - 1} (\#\pi - 1)! \prod_{B \in \pi} u_B, \tag{1.2}$$

where $u_B := \mathbb{E}\left(\bigoplus_{b \in B} X_b\right) = \bigoplus_{b \in B} X_b$ and we sum over *set-partitions* of $[r] := \{1, 2, \dots, r\}$, that is all possible sets $\pi$ of nonempty subsets of $[r]$ such that every element $i \in [r]$ belongs to precisely one element of $\pi$; here $\#\pi$ denotes the number of elements of $\pi$.

We go back to the symmetric functions now. A classical problem in the symmetric functions theory is to understand the *structure constants* $a^{\lambda}_{\mu,\nu}$ of a given linear basis $\{s_\mu\}_\mu$:

$$s_\mu \cdot s_\nu = \sum_\lambda a^{\lambda}_{\mu,\nu}\, s_\lambda.$$

The celebrated basis of Macdonald polynomials $\left\{\tilde{H}_\mu(\boldsymbol{x}; q, t)\right\}_\mu$ (here, we use "transformed form" of Macdonald polynomials sometimes called "modified form"; see [10] for more details) is an interesting example. It was shown by Macdonald [12] (and it is straightforward from Haglund's formula (2.1)) that one has the following structure constants in the specialization $q = 1$:

$$\tilde{H}_\mu(\boldsymbol{x}; 1, t) \cdot \tilde{H}_\nu(\boldsymbol{x}; 1, t) = \tilde{H}_{\mu \oplus \nu}(\boldsymbol{x}; 1, t),$$

where for partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ we define a new partition $\lambda \oplus \mu := (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots)$ by adding coordinates of partitions $\lambda$ and $\mu$. Since Macdonald polynomials $\{\tilde{H}_\mu\}_\mu$ form a linear basis of the algebra $\Lambda$ of symmetric functions over $\mathbb{Q}(q, t)$, we can define a new multiplicative structure $(\Lambda, \oplus)$ by setting $\tilde{H}_\mu \oplus \tilde{H}_\nu := \tilde{H}_{\mu \oplus \nu}$ and extending it by linearity. Therefore $(\Lambda, \oplus)$ can be interpreted as an approximation of the algebra $(\Lambda, \cdot)$ of interest, as $q \to 1$ and it is natural to study cumulants of the form $\kappa(f_1, \dots, f_r)$, where $f_1, \dots, f_r \in \Lambda$, which describe this approximation on the "higher–order" level and may bring a better understanding of the structure constants for Macdonald polynomials. By multilinearity of cumulants it is enough to study $\kappa(\tilde{H}_{\lambda_1}, \dots, \tilde{H}_{\lambda_r})$ and it was first conjectured in [5], and then proved in [3], that this "higher–order" level approximation, is indeed of an expected order:

**Theorem 1.1** ([3]). *For any partitions $\lambda^1, \dots, \lambda^r$ one has*

$$\kappa(\tilde{H}_{\lambda^1}, \dots, \tilde{H}_{\lambda^r}) \in \mathbb{Z}[q,t]\{(q-1)^{r-1} m_\mu\}_\mu.$$

It is therefore natural to introduce the *Macdonald cumulant* $\kappa(\lambda^1, \dots, \lambda^r)(\boldsymbol{x}; q, t)$ as

$$\kappa(\lambda^1, \dots, \lambda^r)(\boldsymbol{x}; q, t) := \frac{\kappa(\tilde{H}_{\lambda_1}(\boldsymbol{x}; q, t), \dots, \tilde{H}_{\lambda_r}(\boldsymbol{x}; q, t))}{(q-1)^{r-1}}. \tag{1.3}$$

We finish this section by mentioning one more natural motivation for studying Macdonald cumulants. One of the most typical application of cumulants is to show that a certain family of random variables is asymptotically Gaussian. The main technique is to show that conditional cumulants of "observables" have a certain *small cumulant property* exactly of the same form as in Theorem 1.1; see [14, 6]. It is therefore natural to ask for a probabilistic interpretation of Theorem 1.1, which leads to some kind of a central limit theorem. The most natural framework to investigate this problem seems to be related with Macdonald processes introduced by Borodin and Corwin [1] and we leave this problem for the future research.

## 1.2 Schur–positivity problem

Our personal favorite reason for studying cumulants $\kappa(\lambda^1, \dots, \lambda^r)$ is their beautiful and mysterious combinatorial structure. We recall that monomial symmetric functions have integer coefficients in the Schur basis expansion, thus one can reformulate Theorem 1.1 as follows: for any partitions $\lambda^1, \dots, \lambda^r$ one has the following expansion

$$\kappa(\lambda^1, \dots, \lambda^r) \in \mathbb{Z}[q,t]\{s_\mu\}_\mu.$$

Remarkably, extensive computer simulations suggest that Macdonald cumulants are, in fact, Schur–positive, which we conjectured in our recent paper [3]:

**Conjecture 1.2** (Higher–order Macdonald positivity conjecture [3]). *Let $\lambda^1, \dots, \lambda^r$ be partitions. Then, for any partition $\mu$, the* multivariate $q,t$-Kostka number $\tilde{K}^{(q,t)}_{\mu;\lambda^1,\dots,\lambda^r}$ *defined by the following expansion*

$$\kappa(\lambda^1, \dots, \lambda^r) := \sum_\mu \tilde{K}^{(q,t)}_{\mu;\lambda^1,\dots,\lambda^r}\, s_\mu$$

*is a polynomial in $q,t$ with* ***nonnegative integer*** *coefficients.*

Note that the special case $r = 1$ of Conjecture 1.2 corresponds to the Macdonald positivity ex-conjecture [12] which says that the coefficients $\tilde{K}_{\lambda,\mu}(q,t)$ in the following expansion:

$$\tilde{H}_\mu(\boldsymbol{x};q,t) = \sum_\lambda \tilde{K}_{\lambda,\mu}(q,t)\, s_\lambda(\boldsymbol{x})$$

are polynomials in $q,t$ with nonnegative integer coefficients. It took more than ten years to Haiman to prove it [10] by connecting their conjectural representation theoretic interpretation with the problem from algebraic geometry and solving it. This result is considered as a great breakthrough in the symmetric functions theory and it initiated very active research in the remarkable algebraic combinatorics of the Macdonald polynomials. Thus, our conjecture generalizes Macdonald positivity ex-conjecture from the cumulant of order 1 to cumulants of higher order.

The main result of this paper is an explicit combinatorial formula for Macdonald cumulants $\kappa(\lambda^1,\dots,\lambda^r)$ and its applications. Moreover, it will be shortly clear that our formula connects Macdonald cumulants with some seemingly unrelated topics from algebraic combinatorics. We believe that these links enrich the world of combinatorics of Macdonald polynomials and give one more motivation for studying the new interesting family of the symmetric functions $\{\kappa(\lambda^1,\lambda^2,\dots)\}_{\lambda^1,\lambda^2,\dots}$.

## 2 $G$–parking functions and fillings of the Young diagrams

Before we describe our formula, let us briefly recall the celebrated combinatorial formula for Macdonald polynomials, which coincides with our formula for a trivial cumulant $\kappa(\lambda) = \tilde{H}_{\lambda_r}(\boldsymbol{x};q,t)$ and which was a great source of inspiration for our work.

**Theorem 2.1** ([9]). *Let $\lambda$ be a partition. Then*

$$\tilde{H}_\lambda(\boldsymbol{x};q,t) = \sum_{\sigma:\lambda\to\mathbb{N}_+} q^{\mathrm{inv}(\sigma)}\, t^{\mathrm{maj}(\sigma)}\, \boldsymbol{x}^\sigma, \tag{2.1}$$

*where we sum over all possible fillings of $\lambda$ by nonnegative integers, and where $\boldsymbol{x}^\sigma := \prod_{\square\in\lambda} x_{\sigma(\square)}$.*

With each filling $\sigma : \lambda \to \mathbb{N}_+$ one can associate certain statistics $\mathrm{inv}(\sigma)$ and $\mathrm{maj}(\sigma)$ which play crucial role in (2.1) and we are going to describe them in the following.

### 2.1 Fillings of Young diagrams and their statistics

For any partition $\lambda \vdash n$ let $\sigma : \lambda \to \mathbb{N}_+$ be a filling of the boxes of the diagram $\lambda$ by positive integers. A *descent* of $\sigma$ is a pair of entries $\sigma(\square) > \sigma(\square')$ such that $\square$ lies

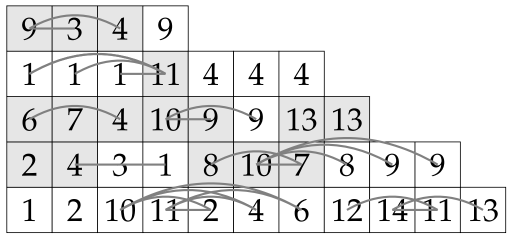


**Figure 1:** Inversion pairs in the above filling $\sigma$ are indicated by grey lines, while the set of descents is highlighted in light gray.

immediately above $\square'$. Let $\ell_\lambda(\square)$ denotes the number of boxes in $\lambda$ lying in the same column as $\square$ strictly above it. The *major index* $\mathrm{maj}(\sigma)$ of a filling $\sigma$ is defined as:

$$\mathrm{maj}(\sigma) := \sum_{\square \in \mathrm{Des}(\sigma)} (\ell_\lambda(\square) + 1), \tag{2.2}$$

where $\mathrm{Des}(\sigma) := \{\square \in \lambda : \sigma(\square) > \sigma(\square') \text{ is a descent}\}$ is a *set of descents*.

We also define the set of *inversion pairs* $(\square_1, \square_2)$, where $\square_1$ is a box lying in the same row as $\square_2$ to its left, $\square_3$ is a box lying directly below $\square_1$ and such that $\square_1, \square_2, \square_3$ are *counterclockwise increasing*. That is, one of the following conditions holds true: $\sigma(\square_1) \leq \sigma(\square_3) < \sigma(\square_2)$, or $\sigma(\square_3) < \sigma(\square_2) < \sigma(\square_1)$, or $\sigma(\square_2) < \sigma(\square_1) \leq \sigma(\square_3)$. Here, the convention is that for $\square_1, \square_2$ lying in the first row $\sigma(\square_3) < \min_{\square \in \lambda} \sigma(\square)$. The set of inversion pairs of $\sigma$ is denoted by $\mathrm{InvP}(\sigma)$. Figure 1 presents an example of above defined objects. We set

$$\mathrm{inv}(\sigma) := \#\mathrm{InvP}(\sigma). \tag{2.3}$$

Finally, the *reading order* is the linear ordering of the entries of $\lambda$ given by reading them row by row, top to bottom, and left to right within each row. With any filling $\sigma$ we associate its *reading word* $w_\sigma$ by reading its entries in the reading order.

## 2.2 $G$-parking functions and the abelian sandpile model

Let $G = (V, E)$ be a rooted, connected multigraph (multiple edges and loops are allowed) with the set of vertices $V = [r]$, where $r \geq 1$ is a positive integer and we denote the root of $G$ by $v \in [r]$. For any $i \in U \subset [r] \setminus \{v\}$ we define the *outdegree* $\mathrm{outdeg}_U(i)$ of a vertex $i$ as the number of edges in $G$ linking $i$ with some vertex $j \notin U$. We call a function $f : [r] \setminus \{v\} \to \mathbb{N}$ a *G-parking function* if for any nonempty subset $U \subset [r] \setminus \{v\}$ there exists $i \in U$ such that $f(i) < \mathrm{outdeg}_U(i)$.

When $G = K_r$ is the complete graph on $[r]$, then the set of $G$-parking functions is precisely the set of parking functions.

Postnikov and Shapiro noticed that $G$-parking functions are directly related to *recurrent configurations* in the *abelian sandpile model for* $G$, which is a model where we are trying to distribute chips among vertices of our graph. A function $u : [r] \setminus \{v\} \to \mathbb{N}$ giving the number of chips placed in vertices of $G$ different from the root is called a *configuration*. We say that a vertex $i \in [r] \setminus \{v\}$ is *unstable* if $u(i) \geq \deg(i)$ – if this is a case, this vertex can *topple* by sending chips to adjacent vertices one along each incident edge. We say that a configuration is *stable* if all the vertices $i \in [r] \setminus \{v\}$ except the root are stable. For the root we set $u(v) = -\sum_{i \in [r]\setminus\{v\}} u(i)$, and the root can always topple. Finally, we say that a configuration $u$ is *recurrent* if there exists a nontrivial configuration $u' \neq 0$ such that $u$ can be obtained from $u + u'$ by a sequence of topplings. Postnikov and Shapiro noticed that a configuration $u$ is recurrent if and only if $f : [r] \setminus \{v\} \to \mathbb{N}$ defined by $f(i) := \deg(i) - u(i) - 1$ is a $G$-parking function. We define a *weight* of a $G$-parking function $f$:

$$\mathrm{wt}(f) := \#E - r - \sum_{i \in [r]\setminus\{v\}} f(i),$$

and we associate a $q$-generating function of $G$-parking functions $\mathcal{I}_G(q) := \sum_f q^{\mathrm{wt}(f)}$.

Merino López proved [13] that $\mathcal{I}_G(q) = \mathrm{Tutte}_G(1, q)$, where $\mathrm{Tutte}_G(t, q)$ is a Tutte polynomial of $G = (V, E)$, that is

$$T_G(x, y) = \sum_{H \subset G} (x-1)^{c(H)-1} \, (y-1)^{\#E(H) - \#V + c(H)}, \tag{2.4}$$

where we sum over all (possibly disconnected) sub-multigraphs of $G$, $c(H)$ denotes the number of connected components of $H$, and $E(H)$ is the set of edges of $H$. Therefore in the special case of the complete graph $\mathcal{I}_G(q)$ is the inversion polynomial (the generating function of rooted trees with respect to the number of their inversions) and in general we call it *$G$–inversion polynomial*.

For technical reasons, we extend the definition of $\mathcal{I}_G(q)$ to the set of all finite multigraphs by setting $\mathcal{I}_G(q) = 0$ when $G$ is not connected.

## 2.3 Combinatorial formula for Macdonald cumulants

We are ready to formulate our main result.

**Theorem 2.2.** *Let $\lambda^1, \dots, \lambda^r$ be partitions. Then, the following formula holds true:*

$$\kappa(\lambda^1, \dots, \lambda^r) = \sum_{\sigma : \lambda^{[r]} \to \mathbb{N}_+} \mathcal{I}_{G^{\sigma}_{\lambda^1,\dots,\lambda^r}}(q) \, t^{\mathrm{maj}(\sigma)} \, \boldsymbol{x}^{\sigma}. \tag{2.5}$$

The summation index in (2.5) runs over all the fillings $\sigma$ of a Young diagram $\lambda^{[r]}$ by positive integers, where $\lambda^B := \bigoplus_{b \in B} \lambda^b$ and $G^{\sigma}_{\lambda^1,\dots,\lambda^r}$ is a certain multigraph and its construction is presented in the following section.

### 2.4 Consequences of the formula

Since our combinatorial formula for Macdonald cumulants is a generalization of (2.1), it has similar implications. Let us briefly summarize them, before we go into details of a description of $G^{\sigma}_{\lambda^1,\dots,\lambda^r}$.

Our main result extends Theorem 1.1 twofold. Firstly, our formula shows that the coefficients of Macdonald cumulants in the monomial expansion are positive integers which is a stronger statement. Secondly, the original proof of Theorem 1.1 relies on some abstract arguments and complicated induction – in particular it does not tell anything about the combinatorial structure of Macdonald cumulants, contrary to our formula which provides an explicit combinatorial interpretation in terms of $G$–parking functions.

Moreover, we deduce from our formula an explicit expansion of Macdonald cumulants in the fundamental quasisymmetric functions, which turns out to be $q,t$–positive.

Finally, our main result implies that Conjecture 1.2 holds true in the special case of hooks.

We describe these applications in Section 4.

## 3 Coloring of the Young diagram $\lambda^{[r]}$ and multigraphs

### 3.1 Coloring of the Young diagram $\lambda^{[r]}$

Let $\lambda^1,\dots,\lambda^r$ be partitions, and let $\pi \in \mathcal{P}([r])$ be a set partition. For each $B \in \pi$, we are going to color the columns of $\lambda^B$ by numbers $b \in B$ as follows: we observe that the Young diagram $\lambda^B$ can be constructed by sorting the columns of the diagrams $\lambda^{b_1}, \dots, \lambda^{b_t}$ in decreasing order, where $B = \{b_1,\dots,b_t\}$ and $b_1 < \cdots < b_t$. When several columns have the same length, we use the total order of $B$, that is we put first the columns of $\lambda^{b_1}$, then those of $\lambda^{b_2}$ and so on. We say that a column of $\lambda^B$ is *colored by* $b \in B$ if this column is identified with the column of $\lambda^b$ in the above construction. Similarly, we say that a box $\square \in \lambda^B$ is *colored by* $b \in B$ if it lies in the column colored by $b$; see Figure 2a (at the moment, please disregard entries). This gives a way to identify boxes of $\lambda^{[r]}$ with the boxes of $\{\lambda^B : B \in \pi\}$. To be more precise a box $\square \in \lambda^B$ which lies in the $i$-th column colored by $b$ in $\lambda^B$ (not necessarily in the $i$-th column of $\lambda^B$) and in the $j$-th row of $\lambda^B$ is identified with the box $\tilde{\square}$ of $\lambda^{[r]}$ which lies in the $i$-th column colored by $b$ in $\lambda^{[r]}$ and in the $j$-th row of $\lambda^{[r]}$.

This identification leads to a one-to-one correspondence between all the fillings $\sigma$ of $\lambda^{[r]}$ with entries from a given set $\mathcal{A}$ and between the sets of fillings $\{\sigma_B : \lambda^B \to \mathcal{A} \mid B \in \pi\}$, and for a given set of fillings $\{\sigma_b : \lambda^b \to \mathcal{A} \mid b \in B\}$ the corresponding filling $\sigma : \lambda^B \to \mathcal{A}$ is denoted by $\sigma^B$, see Figure 2a.

Let $\sigma : \lambda^{[r]} \to \mathbb{N}_+$ be a filling. We are ready to construct the multigraph $G^{\sigma}_{\lambda^1,\dots,\lambda^r} := (V, E)$. For each inversion pair in $\sigma$, we draw an edge linking its boxes, and we color its

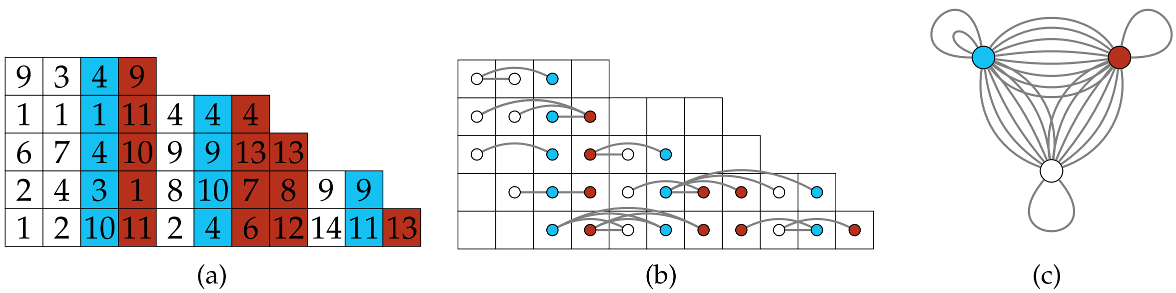


**Figure 2:** Figure 2a shows the colored filling $\sigma$ from Figure 1 with $\lambda^1 = (4^2, 3^2, 2), \lambda^2 = (3^2, 2^2, 1), \lambda^3 = (4, 3^2, 2, 1)$. Figure 2b shows the edges representing inversion pairs in $\sigma$ and colors of their endpoints. Figure 2c presents the multigraph $G^{\sigma}_{\lambda^1,\lambda^2,\lambda^3}$ for $\sigma$, and $\lambda^1, \lambda^2, \lambda^3$ from Figure 2a obtained from the edges from Figure 2b by identifying the vertices of the same color.

endpoints by the colors of these boxes from the colored diagram $\lambda^{[r]}$; then we identify all the endpoints of the same color – see Figures 2b and 2c for a construction of $G^{\sigma}_{\lambda^1,\dots,\lambda^r}$ for $r = 3$ and $\sigma, \lambda^1, \lambda^2, \lambda^3$ as in Figure 2a. More formally, $G^{\sigma}_{\lambda^1,\dots,\lambda^r} := (V, E)$ is defined by the following data:

1. the set of vertices $V$ is equal to $[r]$;

2. the number $e_{i,j}(G)$ of edges connecting vertices $i$ and $j$ ($i$ and $j$ are not necessarily distinct) is equal to the number of inversion pairs in $\sigma$ colored by $\{i, j\}$.

## 3.2 Idea of the proof of Theorem 2.2

The first important step towards the proof is the following lemma.

**Lemma 3.1.** *Let $G = (V, E)$ be a multigraph. Then*

$$\mathcal{I}_G(q) = (q-1)^{1-\#V} \sum_{\pi \in \mathcal{P}(V)} (-1)^{\#\pi - 1} (\#\pi - 1)! \prod_{B \in \pi} q^{\#E|_B}, \tag{3.1}$$

*where $E|_B \subset E$ consists of the edges with both endpoints lying in the set $B \subset V$.*

*Proof.* Let $G$ be a multigraph (possibly disconnected), and we define the generating functions $nc_G(q) = \sum_{H \subset G} q^{\#E(H)}$, where we sum over all (possibly disconnected) submultigraphs of $G$ and $c_G(q) = \sum_{H \subset G} q^{\#E(H)}$, where we sum over all connected submultigraphs of $G$. Then, clearly

$$nc_G(q) = \sum_{\pi \in \mathcal{P}(V)} \prod_{B \in \pi} c_{G|_B}(q),$$

where $G|_B$ is a sub-multigraph of $G$ supported on $B$. Thus the Möbius inversion formula on the set–partition lattice implies that

$$c_G(q) = \sum_{\pi \in \mathcal{P}(V)} (-1)^{\#\pi - 1}(\#\pi - 1)! \prod_{B \in \pi} nc_{G|_B}(q).$$

Using (2.4) we relate $c_G(q)$ with a Tutte polynomial:

$$c_G(q) = (q)^{\#V-1}\,\mathrm{Tutte}_G(1, q+1). \tag{3.2}$$

Plugging $nc_{G|_B}(q) = (1+q)^{\#E|_B}$ and (3.2) into the above equality and substituting $\mathcal{I}_G(q)$ $= \mathrm{Tutte}(1, q)$ yield the desired result. □

We are ready to sketch the proof of Theorem 2.2.

*Sketch of the proof of Theorem 2.2.* We use definitions (1.3) and (1.2) of Macdonald cumulants and Haglund's formula (2.1) to rewrite the left hand side of (2.5) as follows:

$$\begin{aligned}(q-1)^{1-r} &\sum_{\pi \in \mathcal{P}([r])} (-1)^{\#\pi - 1}(\#\pi - 1)! \prod_{B \in \pi} \tilde{H}_{\lambda^B}(\boldsymbol{x}; q, t) \\ &= \sum_{\sigma : \lambda^{[r]} \to \mathbb{N}_+} t^{\mathrm{maj}(\sigma)} \left( (q-1)^{1-r} \sum_{\pi \in \mathcal{P}([r])} (-1)^{\#\pi - 1}(\#\pi - 1)! \prod_{B \in \pi} q^{\mathrm{inv}(\sigma^B)} \right) \boldsymbol{x}^{\sigma}. \end{aligned} \tag{3.3}$$

This is a consequence of the one-to-one correspondence between fillings of a given diagram and the sets of fillings of its subdiagrams described in Section 3.1 and the following identity:

$$\mathrm{maj}(\sigma) = \sum_{\square \in \mathrm{Des}(\sigma)} (\ell_{\lambda^{[r]}}(\square) + 1) = \sum_{B \in \pi} \mathrm{maj}(\sigma^B).$$

Notice now that setting $G^{\sigma}_{\lambda^1,\dots,\lambda^r} := (V, E)$, the expression in parentheses in (3.3) is given by the following formula:

$$(q-1)^{1-\#V} \sum_{\pi \in \mathcal{P}(V)} (-1)^{\#\pi - 1}(\#\pi - 1)! \prod_{B \in \pi} q^{\#E|_B},$$

which is equal to $\mathcal{I}_{G^{\sigma}_{\lambda^1,\dots,\lambda^r}}(1, q)$ by Lemma 3.1. Indeed, strictly from the construction of $G^{\sigma}_{\lambda^1,\dots,\lambda^r}$ one has $V = [r]$, and it is easy to check that $\#E|_B = \mathrm{inv}(\sigma_B)$. This concludes the proof. □

# 4 Consequences of the formula

In this section we present some important consequences of Theorem 2.2

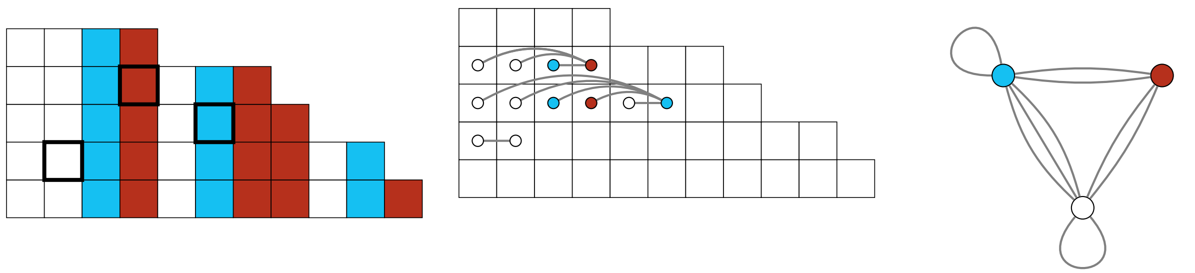

**Figure 3:** The graph $G^{\square_1,\square_2,\square_3}_{\lambda^1,\lambda^2,\lambda^3}$ for $\lambda^1, \lambda^2, \lambda^3$ from Figure 2.

## 4.1 Expansion in fundamental quasisymmetric functions

*Definition* 4.1 ([7]). For any nonnegative integer $n$ and a subset $D \subset [n-1]$ a *fundamental quasisymmetric function* $F_{n,D}(\boldsymbol{x})$ of degree $n$ in variables $\boldsymbol{x} = x_1, x_2, \dots$ is defined by the formula

$$F_{n,D}(\boldsymbol{x}) := \sum_{\substack{i_1 \leq \cdots \leq i_n \\ j \in D \Rightarrow i_j < i_{j+1}}} x_{i_1} \cdots x_{i_n}. \tag{4.1}$$

Let $\lambda^1, \dots, \lambda^r$ be partitions and let $n = |\lambda^1| + \cdots + |\lambda^r|$. Then, using formula (2.5) and a verbatim argumentation as in [9, Section 4] (see [2] for more details) we have the expansion:

$$\kappa(\lambda^1, \dots, \lambda^r)(\boldsymbol{x}) = \sum_{\sigma \in \mathfrak{S}_n} \mathcal{I}_{G^{\sigma}_{\lambda^1,\dots,\lambda^r}}(q)\, t^{\mathrm{maj}_{\lambda^{[r]}}(\sigma)}\, F_{n,\mathrm{iDes}(\sigma)}(\boldsymbol{x}), \tag{4.2}$$

where we abuse notation by denoting both a permutation by $\sigma$, and the associated standard filling of $\lambda^{[r]}$ with the reading word given by $\sigma$ (see Section 2.1).

## 4.2 Expansion in Schur symmetric functions

Let $\lambda^1, \dots, \lambda^r$ be partitions, and let $1 \leq s \leq |\lambda^1| + \cdots + |\lambda^r|$ be a positive integer. For any subset $\{\square_1, \dots, \square_s\} \subset \lambda^{[r]}$ of boxes we construct a graph $G^{\square_1,\dots,\square_s}_{\lambda^1,\dots,\lambda^r} := (V, E)$ as follows: we draw an edge between each $\square_i$ and each box to its left lying in the same row, and we color its endpoints by the colors of the corresponding boxes in $\lambda^{[r]}$; then we identify all the endpoints of the same color – see Figure 3 for a construction of $G^{\square_1,\square_2,\square_3}_{\lambda^1,\dots,\lambda^r}$ for $r = 3$ and $\lambda^1, \lambda^2, \lambda^3$ as in Figure 2. In other words

- the set of vertices $V$ is equal to $[r]$,

- the number of edges linking vertices $i, j \in V$ is equal to the number of pairs $(\Box_k, \Box')$ such that $\Box'$ is lying in the same row as $\Box_k$ to its left, and the pair $(\Box_k, \Box')$ is colored by $\{i, j\}$, where $1 \leq k \leq s$.

We are ready to prove Conjecture 1.2 in the case of hooks.

**Theorem 4.2.** *Let $\lambda^1, \dots, \lambda^r$ be partitions with $|\lambda^{[r]}| = n$. Then, for any nonnegative integer $s$, the coefficient of $(-u)^s$ in $\kappa(\lambda^1, \dots, \lambda^r)[1-u]$ is equal to*

$$\kappa(\lambda^1, \dots, \lambda^r)[1-u]|_{(-u)^s} = \sum_{\{\Box_1, \dots, \Box_s\} \subset \lambda^{[r]}} \mathcal{I}_{G^{\Box_1, \dots, \Box_s}_{\lambda^1, \dots, \lambda^r}}(q)\ t^{\sum_{1 \leq i \leq s} \ell'_{\lambda^{[r]}}(\Box_i)}. \tag{4.3}$$

*Equivalently, the multivariate $q,t$-Kostka number $\widetilde{K}_{(n-s,1^s);\lambda^1,\dots,\lambda^r}(q,t) := [s_{(n-s,1^s)}]\kappa(\lambda^1, \dots, \lambda^r)$ is a polynomial in $q,t$ with nonnegative integer coefficients given by the following formula:*

$$\widetilde{K}_{(n-s,1^s);\lambda^1,\dots,\lambda^r}(q,t) = \sum_{\{\Box_1, \dots, \Box_s\} \subset \lambda^{[r]} \setminus (1,1)} \mathcal{I}_{G^{\Box_1, \dots, \Box_s}_{\lambda^1, \dots, \lambda^r}}(q)\ t^{\sum_{1 \leq i \leq s} \ell'_{\lambda^{[r]}}(\Box_i)}. \tag{4.4}$$

Let us describe the main ingredients of the proof of Theorem 4.2. For the detailed proof, we refer to [2].

*Sketch of the proof of Theorem 4.2.* Firstly, we sketch an argument of Macdonald showing the equivalence of formulas (4.3) and (4.4). Macdonald proved [12, Section VI.8, Example 2] that

$$s_\lambda[1-u] = \begin{cases} 0 & \text{if } \lambda \text{ is not a hook,} \\ (1-u)(-u)^s & \text{if } \lambda = (n-s, 1^s). \end{cases}$$

Applying this to any homogeneous symmetric function

$$f := \sum_{\lambda \vdash n} c_\lambda\, s_\lambda,$$

we obtain the following relation:

$$f[1-u]|_{(-u)^s} = c_{(n-s,1^s)} + c_{(n-s+1,1^{s-1})}. \tag{4.5}$$

Thus, we need to prove that the relation (4.5) is satisfied by (4.3) and (4.4). This follows easily from the construction of $G^{\Box_1, \dots, \Box_s}_{\lambda^1, \dots, \lambda^r}$.

Haglund, Haiman and Loehr [9] explained how to compute $\tilde{H}_\mu(\boldsymbol{x}; q, t)[1-u]|_{(-u)^s}$ from their formula (2.1). Using small modifications of their argumentation and simple properties of graphs $G^{\Box_1, \dots, \Box_s}_{\lambda^1, \dots, \lambda^r}$, we are able to compute $\kappa(\lambda^1, \dots, \lambda^r)[1-u]|_{(-u)^s}$ relying on (2.5) in a similar manner. This yields the desired (4.3). $\Box$

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