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Stability of the Heisenberg product on symmetric functions

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Abstract. The Heisenberg product is an associative product defined on symmetric functions which interpolates between the usual product and the Kronecker product. In 1938, Murnaghan discovered that the Kronecker product of two Schur functions stabilizes. We prove an analogous result for the Heisenberg product of Schur functions. We also show a rectangular symmetry for the Schur structural constants of this product.

Keywords: Heisenberg product, symmetric function, stability, Kronecker coefficient, Littlewood–Richardson coefficient

1 Introduction

Aguiar, Ferrer Santos, and Moreira introduced a new product, the Heisenberg product, on symmetric functions (also on representations of symmetric group) in [1] and [7]. Unlike the outer product and the Kronecker product, the terms appearing in the Heisenberg product of two Schur functions have different degrees. The highest degree component is the usual product. When the Schur functions have the same degree, the lowest degree component of the Heisenberg product is their Kronecker product.

In 1938, Murnaghan [8] found that the Kronecker product of two Schur functions stabilizes in the following sense. Given a partition λ of l and a sufficiently large integer n, let $\lambda[n]$ be the partition of n obtained by prepending a part of size $n-l$ to λ. Given two partitions λ and μ, the coefficients appearing in the Schur expansion of the Kronecker product $s_{\lambda[n]} * s_{\mu[n]}$ do not depend upon n when n is large enough. The aim of this paper is to show that each degree component of the Heisenberg product also has this property.

This extended abstract is organized as follows. In the second section, we give the definitions of the induction product, the Kronecker product, and the Heisenberg product, and recall some important results. In the third section, we define the Heisenberg coefficients and prove that each degree component of the Heisenberg product has a similar stabilization property as the Kronecker product. In Section 4, we define the stable

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Heisenberg coefficients, and show how to recover the usual Heisenberg coefficients from the stable ones. In the last section, we show a rectangular symmetry for the Heisenberg coefficient, which is analogous to the results in [2].

2 Preliminaries

We begin by defining the induction product on representations of symmetric groups. For an introduction to representations of symmetric groups, see [9]. For any partition α, let V_α denote the irreducible representation of $S_{|\alpha|}$ indexed by α. Let λ, μ, and ν be partitions of n, m, and $n+m$ respectively (written as $\lambda \vdash n$, $\mu \vdash m$, and $\nu \vdash m+n$). Note that the tensor product $V_\lambda \otimes V_\mu$ is a representation of $S_n \times S_m$, and $S_n \times S_m$ can be naturally embedded into S_{n+m}. The induction product of V_λ and V_μ is the induced representation of $V_\lambda \otimes V_\mu$ from $S_n \times S_m$ to S_{n+m}, written as $\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}(V_\lambda \otimes V_\mu)$. The Littlewood–Richardson coefficient $c_{\lambda,\mu}^{\nu}$ is the multiplicity of V_ν in the decomposition of $\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}(V_\lambda \otimes V_\mu)$ into irreducibles. That is,

$$\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}(V_\lambda \otimes V_\mu) = \bigoplus_{\nu \vdash n+m} c_{\lambda,\mu}^{\nu} V_\nu. \tag{2.1}$$

Let $\langle \cdot , \cdot \rangle$ denote the natural inner product on the representations of the finite groups in which the irreducible representations form an orthonormal basis. Applying the Frobenius reciprocity theorem to (2.1), we have

$$\begin{aligned} c_{\lambda,\mu}^{\nu} &= \langle \, \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}(V_\lambda \otimes V_\mu) \, , \, V_\nu \, \rangle_{S_{n+m}} \\ &= \langle \, V_\lambda \otimes V_\mu \, , \, \mathrm{Res}_{S_n \times S_m}^{S_{n+m}} V_\nu \, \rangle_{S_n \times S_m}. \end{aligned}$$

So

$$\mathrm{Res}_{S_n \times S_m}^{S_{n+m}} V_\nu = \bigoplus_{\lambda \vdash n \, , \, \mu \vdash m} c_{\lambda,\mu}^{\nu} (V_\lambda \otimes V_\mu). \tag{2.2}$$

There is a one-to-one correspondence between the irreducible representations and the Schur functions by the Frobenius map, which sends V_λ to the Schur function s_λ. So we could also express the induction product in terms of symmetric functions. Under this bijection, the induction product corresponds to the usual product (denoted by $\cdot$) on symmetric functions. i.e.

$$s_\lambda \cdot s_\mu = \sum_{\nu \vdash n+m} c_{\lambda,\mu}^{\nu} s_\nu.$$

The Littlewood–Richardson coefficient has been well-studied and it has a nice combinatorial interpretation, the Littlewood–Richardson rule, which describes this coefficient in terms of counting certain skew tableaux, see [5] (Page 143) for details about this.

We define the Kronecker coefficient in terms of representations of symmetric groups. Let λ, μ, and ν be partitions of n. While the tensor product $V_\lambda \otimes V_\mu$ is a representation of $S_n \times S_n$, it can also be considered as a representation of S_n (by viewing S_n as a subgroup of $S_n \times S_n$ through the diagonal map). Write it as $\mathrm{Res}_{S_n}^{S_n \times S_n}(V_\lambda \otimes V_\mu)$. The Kronecker coefficient $g_{\lambda,\mu}^{\nu}$ is the multiplicity of V_ν in the decomposition of $\mathrm{Res}_{S_n}^{S_n \times S_n}(V_\lambda \otimes V_\mu)$ into irreducibles. That is,

$$\mathrm{Res}_{S_n}^{S_n \times S_n}(V_\lambda \otimes V_\mu) = \bigoplus_{\nu \vdash n} g_{\lambda,\mu}^{\nu} V_\nu. \tag{2.3}$$

We could also express the Kronecker product (denoted by $*$) in terms of symmetric functions:

$$s_\lambda * s_\mu = \sum_{\nu \vdash n} g_{\lambda,\mu}^{\nu} s_\nu.$$

We will switch between the languages of representation theory and symmetric functions.

Given a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ and a positive integer n, let $\lambda[n]$ be the sequence $(n - |\lambda|, \lambda_1, \lambda_2, \dots)$. When $n \geq |\lambda| + \lambda_1$, $\lambda[n]$ is a partition of n. The stability of the Kronecker coefficients says that for any partitions λ, μ, and ν, the Kronecker coefficient $g_{\lambda[n],\mu[n]}^{\nu[n]}$ does not depend on n when n is large enough. This property is best shown on an example. Let $\lambda = (2)$ and $\mu = (1,1)$, we compute the Kronecker product $s_{n-2,2} * s_{n-2,1,1}$ for $n \geq 4$:

$$\begin{aligned}
s_{2,2} * s_{2,1,1} &= s_{3,1} + s_{2,1,1} \\
s_{3,2} * s_{3,1,1} &= s_{4,1} + s_{3,2} + 2s_{3,1,1} + s_{2,2,1} + s_{2,1,1,1} \\
s_{4,2} * s_{4,1,1} &= s_{5,1} + s_{4,2} + 2s_{4,1,1} + s_{3,3} + 2s_{3,2,1} + s_{3,1,1,1} + s_{2,2,1,1} \\
s_{5,2} * s_{5,1,1} &= s_{6,1} + s_{5,2} + 2s_{5,1,1} + s_{4,3} + 2s_{4,2,1} + s_{4,1,1,1} + s_{3,3,1} + s_{3,2,1,1} \\
s_{6,2} * s_{6,1,1} &= s_{7,1} + s_{6,2} + 2s_{6,1,1} + s_{5,3} + 2s_{5,2,1} + s_{5,1,1,1} + s_{4,3,1} + s_{4,2,1,1}.
\end{aligned}$$

Observe that the last two equations are only different in the first part of the indexing partitions. Indeed, for $n \geq 7$, we have

$$\begin{aligned}
s_{n-2,2} * s_{n-2,1,1} = s_{n-1,1} &+ s_{n-2,2} + 2s_{n-2,1,1} + s_{n-3,3} + 2s_{n-3,2,1} \\
&+ s_{n-3,1,1,1} + s_{n-4,3,1} + s_{n-4,2,1,1}.
\end{aligned}$$

One can also observe that the sequence of Kronecker coefficients in each column in the above example is weakly increasing from top to bottom. This was shown by Brion [4] and Manivel [6]:

Proposition 2.1. *Let λ, μ, and ν be partitions. The sequence $g_{\lambda[n],\mu[n]}^{\nu[n]}$ is weakly increasing.*

The sequence in the above proposition is eventually constant due to the stability of the Kronecker coefficients. Write $\overline{g}_{\lambda,\mu}^{\nu}$ for the stable value of this sequence and call it a reduced Kronecker coefficient. In our example, we see that $\overline{g}^{(2,1)}_{(2),(1,1)} = 2$ and $\overline{g}^{(1,1,1)}_{(2),(1,1)} = 1$. When all the Kronecker coefficients in $s_{\lambda[n]} * s_{\mu[n]}$ reach the reduced ones, we say that the Kronecker product stabilizes. In our example, the stabilization of the Kronecker product $s_{n-2,2} * s_{n-2,1,1}$ begins at $n = 7$. Moreover, Murnaghan [8] also claimed that $\overline{g}^{\nu}_{\lambda,\mu}$ vanishes unless

$$|\lambda| \leq |\mu| + |\nu|, \quad |\mu| \leq |\lambda| + |\nu|, \quad |\nu| \leq |\lambda| + |\mu|,$$

which are triangle inequalities. When $|\nu| = |\lambda| + |\mu|$, $\overline{g}^{\nu}_{\lambda,\mu}$ is equal to the Littlewood–Richardson coefficient $c^{\nu}_{\lambda,\mu}$ [8].

Briand et al. [3] determined when the Kronecker product stabilizes and provide another condition for the reduced Kronecker coefficient being nonzero.

Proposition 2.2 ([3, Theorem 1.2]). *Let λ and μ be partitions. The expansion of the Kronecker product $s_{\lambda[n]} * s_{\mu[n]}$ stabilizes at $n = |\lambda| + |\mu| + \lambda_1 + \mu_1$.*

Proposition 2.3 ([3, Theorem 3.2]). *Let λ and μ be partitions, then*

$$\max\{|\nu| + \nu_1 \,|\, \nu \text{ partition}, \overline{g}^{\nu}_{\lambda,\mu} > 0\} = |\lambda| + |\mu| + \lambda_1 + \mu_1.$$

Both the induction product and the Kronecker product are graded. Aguiar et al. [1] and Moreira [7] introduced a new (nongraded commutative) product which interpolates between these two products.

Definition 2.4 (Heisenberg product). *Let V and W be representations of S_n and S_m respectively. Fix an integer $l \in [max\{m, n\}, m + n]$, and let $a = l - m$, $b = n + m - l$, and $c = l - n$. Observe that $S_x \times S_y$ can be viewed as a subgroup of S_{x+y}: $S_x \times S_y \hookrightarrow S_{x+y}$, for any nonnegative integers x and y. Also, we can consider S_b as a subgroup of $S_b \times S_b$ through the diagonal embedding Δ_{S_b}: $S_b \hookrightarrow S_b \times S_b$. We have the diagram of inclusions:*

$$\begin{tikzcd}
S_a \times S_b \times S_b \times S_c \arrow[r, hook] & S_{a+b} \times S_{b+c} = S_n \times S_m \arrow[dl, dashed, "Res"'] \\
S_a \times S_b \times S_c \arrow[u, hook, "id_{S_a} \times \Delta_{S_b} \times id_{S_c}"] \arrow[ur] \arrow[r, hook] \arrow[r, dashed, bend left, "Ind"] & S_{a+b+c} = S_l
\end{tikzcd} \tag{2.4}$$

The Heisenberg product (denoted by #) of V and W is

$$V\#W = \bigoplus_{l=max(n,m)}^{n+m} (V\#W)_l, \tag{2.5}$$

where, following the dashed arrows in diagram (2.4),

$$(V\#W)_l = Ind^{S_l}_{S_a \times S_b \times S_c} Res^{S_n \times S_m}_{S_a \times S_b \times S_c} (V \otimes W) \tag{2.6}$$

is the degree l component.

When $l = m+n$, $(V\#W)_l = \mathrm{Ind}_{S_n\times S_m}^{S_{n+m}}(V\otimes W)$, which is the induction product of representations; when $l = n = m$, $(V\#W)_l = \mathrm{Res}_{S_l}^{S_l\times S_l}(V\otimes W)$, which is the Kronecker product of representations. The Heisenberg product connects the induction product and the Kronecker product. Remarkably, this product is associative ([1] Theorem 2.3, Theorem 2.4, Theorem 2.6). By the definition of the Heisenberg product (look at diagram (2.4)), when b is much greater than a and c, the corresponding degree component behaves like the Kronecker product.

A natural question is whether we can develop a stability result for this component. We look at an example of this.

Let us take $\lambda = (1,1)$, $\mu = (1)$. We use Sage [10] to compute the lowest degree components of $s_{(1,1)[n]}\#s_{(1)[n-1]}$:

$$
\begin{aligned}
(s_{1,1,1}\#s_{1,1})_3 &= s_3+s_{2,1},\\
(s_{2,1,1}\#s_{2,1})_4 &= s_4+3s_{3,1}+2s_{2,2}+3s_{2,1,1}+s_{1,1,1,1}.\\
(s_{3,1,1}\#s_{3,1})_5 &= s_5+3s_{4,1}+4s_{3,2}+4s_{3,1,1}+4s_{2,2,1}+3s_{2,1,1,1}+s_{1,1,1,1,1},\\
(s_{4,1,1}\#s_{4,1})_6 &= s_6+3s_{5,1}+4s_{4,2}+4s_{4,1,1}+2s_{3,3}+5s_{3,2,1}+3s_{3,1,1,1}+s_{2,2,2}+2s_{2,2,1,1}\\
&\quad +s_{2,1,1,1,1},\\
(s_{5,1,1}\#s_{5,1})_7 &= s_7+3s_{6,1}+4s_{5,2}+4s_{5,1,1}+2s_{4,3}+5s_{4,2,1}+3s_{4,1,1,1}+s_{3,3,1}+s_{3,2,2}\\
&\quad +2s_{3,2,1,1}+s_{3,1,1,1,1},\\
(s_{6,1,1}\#s_{6,1})_8 &= s_8+3s_{7,1}+4s_{6,2}+4s_{6,1,1}+2s_{5,3}+5s_{5,2,1}+3s_{5,1,1,1}+s_{4,3,1}+s_{4,2,2}\\
&\quad +2s_{4,2,1,1}+s_{4,1,1,1,1},\\
&\cdots\cdots
\end{aligned}
$$

We create a table for this:

n \ ν	n	$n-1$ 1	$n-2$ 2	$n-2$ 1 1	$n-3$ 3	$n-3$ 2 1	$n-3$ 1 1 1	$n-4$ 3 1	$n-4$ 2 2	$n-4$ 2 1 1	$n-4$ 1 1 1 1
3	1∗	1									
4	1	3∗	2	3			1				
5	1	3	4∗	4∗	2	5	3∗				1
6	1	3	4	4	2∗	5∗	3		1∗	2∗	1∗
7	1	3	4	4	2	5	3	1∗	1	2	1
8	1	3	4	4	2	5	3	1	1	2	1

Table 1: $(s_{(1,1)[n]}\#s_{(1)[n-1]})_n$ for $3\le n\le 8$.

The first column gives the values of n. The first row lists all the terms which may appear in the component, and we use the indexing partitions (written in columns) to denote the corresponding Schur functions. We can see that the last two rows have the same Heisenberg coefficients in the Schur expansion, and the only difference is the first part of the indexing partitions. The stabilization of the lowest degree component begins at $n = 7$. When $n \geq 7$, we have

$$\begin{aligned}(s_{n-2,1,1}\#s_{n-2,1})_n = {} & s_n + 3s_{n-1,1} + 4s_{n-2,2} + 4s_{n-2,1,1} + 2s_{n-3,3} \\ & + 5s_{n-3,2,1} + 3s_{n-3,1,1,1} + s_{n-4,3,1} + s_{n-4,2,2} + 2s_{n-4,2,1,1} \\ & + s_{n-4,1,1,1,1}.\end{aligned} \tag{2.7}$$

The main result of this paper is the following:

Theorem 2.5. *Given nonnegative integers r and t and two partitions λ and μ, the expansion of $(V_{\lambda[n]}\#V_{\mu[n-r]})_{n+t}$ stabilizes when $n \geq |\lambda| + |\mu| + \lambda_1 + \mu_1 + 3t + 2r$. Moreover, this is where the stabilization begins.*

From Table 1, we can also see that different columns stabilize at different steps. In Section 4, we give an upper bound for when each column stabilizes in (Corollary 4.2), and we add $*$'s to the cells in Table 1 corresponding to the upper bounds.

3 Proof of Theorem 2.5

Let λ be a partition. Define λ^+ to be the partition obtained from λ by adding 1 to the first part $\lambda^+ = (\lambda_1 + 1, \lambda_2, \lambda_3, \dots)$; similarly, set $\lambda^- = (\lambda_1 - 1, \lambda_2, \lambda_3, \dots)$. Let $\overline{\lambda} = (\lambda_2, \lambda_3, \dots)$ be the partition obtained from λ by removing the first part. For partitions λ and μ, we set $\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots)$, and $\lambda - \mu = (\lambda_1 - \mu_1, \lambda_2 - \mu_2, \dots)$ (when μ is contained in λ). Using the Littlewood–Richardson rule, we can show the following lemma:

Lemma 3.1. *Let λ, μ and ν be partitions with $|\nu| = |\lambda| + |\mu|$*

(1) If $\nu_1 - \nu_2 \geq |\lambda|$, then $c^{\nu}_{\lambda,\mu} = c^{\nu^+}_{\lambda,\mu^+}$.

(2) If $\mu_1 - \mu_2 \geq |\lambda|$, then $c^{\nu}_{\lambda,\mu} = c^{\nu^+}_{\lambda,\mu^+}$.

Remark 3.2. *When λ, μ, and ν do not satisfy the conditions in Lemma 3.1, we can show that $c^{\nu}_{\lambda,\mu} \leq c^{\nu^+}_{\lambda,\mu^+}$. In other words, the sequence $c^{\overline{\nu}[n+|\lambda|]}_{\lambda,\overline{\mu}[n]}$ is weakly increasing and is constant when n is large.*

The Heisenberg coefficient $h^{\nu}_{\lambda,\mu}$ is the multiplicity of V_ν in $V_\lambda\#V_\mu$, i.e.

$$V_\lambda\#V_\mu = \bigoplus_{l=\max(n,m)}^{n+m} \bigoplus_{\nu\vdash l} h^{\nu}_{\lambda,\mu} V_\nu.$$

and we set $h^{\nu}_{\lambda,\mu} = 0$ if λ, μ, or ν is not a partition. The first part of Theorem 2.5 states that when $n \geq |\lambda| + |\mu| + \lambda_1 + \mu_1 + 3t + 2r$,

$$h^{\nu^-}_{\lambda[n],\mu[n-r]} = h^{\nu}_{\lambda[n+1],\mu[n-r+1]} \tag{3.1}$$

for all $\nu \vdash n+t+1$.

To prove (3.1), we first express the Heisenberg coefficient in terms of the Littlewood–Richardson coefficients and the Kronecker coefficients. Using (2.1), (2.2), (2.3), and (2.6) we get the following lemma:

Lemma 3.3. *For each $\nu \vdash l$,*

$$h^{\nu}_{\lambda,\mu} = \sum_{\substack{\alpha \vdash a,\, \rho \vdash c,\, \tau \vdash n \\ \beta,\eta,\delta \vdash b}} c^{\lambda}_{\alpha,\beta}\, c^{\mu}_{\eta,\rho}\, g^{\delta}_{\beta,\eta}\, c^{\tau}_{\alpha,\delta}\, c^{\nu}_{\tau,\rho} \tag{3.2}$$

where $\max(n,m) \leq l \leq n+m$, $a = l-m$, $b = m+n-l$, *and* $c = l-n$.

We set $c^{\nu}_{\lambda,\mu} = 0$ when λ, μ, or ν is not a partition. Then (3.2) holds for all compositions ν of l. Combining (3.1) and (3.2), shows that to prove the first part of Theorem 2.5, it is enough to show that, when $n \geq |\lambda| + |\mu| + \lambda_1 + \mu_1 + 3t + 2r$,

$$\begin{aligned} &\sum_{(\alpha,\beta,\eta,\rho,\delta,\tau)\in T} c^{\lambda[n]}_{\alpha,\beta}\, c^{\mu[n-r]}_{\eta,\rho}\, g^{\delta}_{\beta,\eta}\, c^{\tau}_{\alpha,\delta}\, c^{\nu^-}_{\tau,\rho} = \\ &\qquad \sum_{(\alpha^*,\beta^*,\eta^*,\rho^*,\delta^*,\tau^*)\in T^*} c^{\lambda[n+1]}_{\alpha^*,\beta^*}\, c^{\mu[n+1-r]}_{\eta^*,\rho^*}\, g^{\delta^*}_{\beta^*,\eta^*}\, c^{\tau^*}_{\alpha^*,\delta^*}\, c^{\nu}_{\tau^*,\rho^*} \end{aligned} \tag{3.3}$$

for all $\nu \vdash n+t+1$, where

$$\begin{aligned} T &= \{(\alpha,\beta,\eta,\rho,\delta,\tau) \mid \alpha \vdash r+t, \rho \vdash t, \tau \vdash n, \beta,\eta,\delta \vdash n-r-t\}; \\ T^* &= \{(\alpha^*,\beta^*,\eta^*,\rho^*,\delta^*,\tau^*) \mid \alpha^* \vdash r+t, \rho^* \vdash t, \tau^* \vdash n+1, \\ &\qquad\qquad \beta^*,\eta^*,\delta^* \vdash n-r-t+1\}. \end{aligned}$$

Define $f : T \longmapsto \mathbb{Z}_{\geq 0}$ and $f^* : T^* \longmapsto \mathbb{Z}_{\geq 0}$ as follows:

$$f(\alpha,\beta,\eta,\rho,\delta,\tau) = c^{\lambda[n]}_{\alpha,\beta}\, c^{\mu[n-r]}_{\eta,\rho}\, g^{\delta}_{\beta,\eta}\, c^{\tau}_{\alpha,\delta}\, c^{\nu^-}_{\tau,\rho},$$

$$f^*(\alpha^*,\beta^*,\eta^*,\rho^*,\delta^*,\tau^*) = c^{\lambda[n+1]}_{\alpha^*,\beta^*}\, c^{\mu[n+1-r]}_{\eta^*,\rho^*}\, g^{\delta^*}_{\beta^*,\eta^*}\, c^{\tau^*}_{\alpha^*,\delta^*}\, c^{\nu}_{\tau^*,\rho^*}.$$

Then equation (3.3) becomes:

$$\sum_{(\alpha,\beta,\eta,\rho,\delta,\tau)\in T} f(\alpha,\beta,\eta,\rho,\delta,\tau) = \sum_{(\alpha^*,\beta^*,\eta^*,\rho^*,\delta^*,\tau^*)\in T^*} f^*(\alpha^*,\beta^*,\eta^*,\rho^*,\delta^*,\tau^*). \tag{3.4}$$

Some terms in the sums of (3.4) vanish. Let us consider only the nonvanishing terms.

Let $T_0 = T \setminus f^{-1}(0)$ and $T_0^* = T^* \setminus f^{*-1}(0)$, so that T_0 and T_0^* index the nonzero terms. Then (3.4) becomes

$$\sum_{(\alpha,\beta,\eta,\rho,\delta,\tau)\in T_0} f(\alpha,\beta,\eta,\rho,\delta,\tau) = \sum_{(\alpha^*,\beta^*,\eta^*,\rho^*,\delta^*,\tau^*)\in T_0^*} f^*(\alpha^*,\beta^*,\eta^*,\rho^*,\delta^*,\tau^*). \tag{3.5}$$

Lemma 3.4. *The natural embedding φ from T to T^*:*

$$\varphi(\alpha,\beta,\eta,\rho,\delta,\tau) = (\alpha,\beta^+,\eta^+,\rho,\delta^+,\tau^+)$$

induces a map $\varphi|_{T_0}$ from T_0 to T_0^*. Moreover, $f|_{T_0} = f^* \circ \varphi|_{T_0}$.

$$\begin{tikzcd}
T \arrow[r, hook, "\varphi"] & T^* \\
T_0 \arrow[u, hook] \arrow[ur, hook, "\varphi|_{T_0}"] \arrow[r, "\varphi|_{T_0}"'] & \varphi(T_0) \arrow[u, hook]
\end{tikzcd}$$

Proof. For all $(\alpha,\beta,\eta,\rho,\delta,\tau) \in T_0$, we show that β, η, δ, and τ have large enough first parts so that we can apply Proposition 2.2 and Lemma 3.1 to the Kronecker coefficients and the Littlewood–Richardson coefficients appearing in the definition of f.

Since $n \geq |\lambda| + |\mu| + \lambda_1 + \mu_1 + 3t + 2r$, we can easily see

$$\lambda[n]_1 - \lambda[n]_2 \geq |\alpha| \qquad \text{and} \qquad \mu[n-r]_1 - \mu[n-r]_2 \geq |\rho|.$$

Using Lemma 3.1 (1), we get

$$c^{\lambda[n]}_{\alpha,\beta} = c^{\lambda[n+1]}_{\alpha,\beta^+} \quad \text{and} \quad c^{\mu[n-r]}_{\eta,\rho} = c^{\mu[n+1-r]}_{\eta^+,\rho}.$$

As $\beta \subset \lambda[n]$, $|\overline{\beta}| \leq |\lambda| < n-r-t$ and $(\overline{\beta})_1 \leq \lambda_1$. Similarly, we have $|\overline{\eta}| \leq |\mu| < n-r-t$ and $(\overline{\eta})_1 \leq \mu_1$. Since β and η are both partitions of $n-r-t$, they can be written as $\beta = \overline{\beta}[n-r-t]$ and $\eta = \overline{\eta}[n-r-t]$ respectively. They both have large first parts. More specifically, we have

$$n-r-t \geq |\lambda| + |\mu| + \lambda_1 + \mu_1 + 2t + r \geq |\overline{\beta}| + |\overline{\eta}| + (\overline{\beta})_1 + (\overline{\eta})_1.$$

By Proposition 2.2, we have

$$g^{\delta}_{\beta,\eta} = g^{\delta^+}_{\beta^+,\eta^+} = \overline{g}^{\overline{\delta}}_{\overline{\beta},\overline{\eta}}.$$

From Proposition 2.3,

$$|\overline{\delta}| + (\overline{\delta})_1 \leq |\overline{\beta}| + |\overline{\eta}| + (\overline{\beta})_1 + (\overline{\eta})_1,$$

otherwise $g^{\delta}_{\beta,\eta} = 0$, and so is f. Hence,

$$|\delta| - \delta_1 + \delta_2 \leq |\lambda| + |\mu| + \lambda_1 + \mu_1$$

which gives us

$$\delta_1 - \delta_2 \geq n - r - t - |\lambda| - |\mu| - \lambda_1 - \mu_1 \geq 2t + r \geq |\alpha|$$

Applying Lemma 3.1 (2), we get

$$c^{\tau}_{\alpha,\delta} = c^{\tau^+}_{\alpha,\delta^+}.$$

Since $c^{\tau}_{\alpha,\delta} \neq 0$, by the Little–Richardson rule, we have

$$\tau_2 \leq \delta_2 + |\alpha| \quad \text{and} \quad \tau_1 \geq \delta_1.$$

So $\tau_1 - \tau_2 \geq \delta_1 - (\delta_2 + |\alpha|) \geq 2t + r - (r + t) = t = |\rho|$.
Hence, by Lemma 3.1 (2), we get

$$c^{\nu^-}_{\tau,\rho} = c^{\nu}_{\tau^+,\rho}.$$

So

$$f(\alpha, \beta, \eta, \rho, \delta, \tau) = f^*(\varphi(\alpha, \beta, \eta, \rho, \delta, \tau))(\neq 0), \tag{3.6}$$

which means $\varphi(T_0) \subset T_0^*$ and $f|_{T_0} = f^* \circ \varphi|_{T_0}$. □

Proof of Theorem 2.5. The map of Lemma 3.4 is reversible as the map $(\alpha, \beta, \eta, \rho, \delta, \tau) \longrightarrow (\alpha, \beta^-, \eta^-, \rho, \delta^-, \tau^-)$ gives a well-defined injection from T_0^* to T_0. So $\varphi|_{T_0}$ is a bijection from T_0 to T_0^*. With this and (3.6), we prove (3.5), and hence the first part of Theorem 2.5.

To prove that the lower bound is where the stabilization begins, we just need show that there is some $\nu \vdash n + t$ with $\nu_1 = \nu_2$ such that $h^{\nu}_{\lambda[n],\mu[n-r]} \neq 0$ when $n = |\lambda| + |\mu| + \lambda_1 + \mu_1 + 3t + 2r$. We use the formula (3.2) for $h^{\nu}_{\lambda[n],\mu[n-r]} \neq 0$ (replace λ and ν by $\lambda[n]$ and $\mu[n-r]$ respectively, and set $l = n + t$), and take

$$\alpha = (a) = (r + t),\ \rho = (c) = (t),\ \beta = \lambda[n] - \alpha,\ \eta = \mu[n-r] - \rho,$$

$$\delta = (\overline{\beta} + \overline{\eta})[n - r - t],\ \tau = (\delta_1, \delta_2 + |\alpha|, \delta_3, \dots),\ \nu = (\tau_1, \tau_2 + |\rho|, \tau_3 \dots).$$

By the Pieri rule, $1 = c^{\lambda[n]}_{\alpha,\beta} = c^{\mu[n-d]}_{\eta,\rho} = c^{\tau}_{\alpha,\delta} = c^{\nu}_{\tau,\rho}$, as α and ρ have only one part each. Since $|\overline{\delta}| = |\overline{\beta}| + |\overline{\eta}|$, we have $g^{\delta}_{\beta,\eta} = \overline{g}^{\overline{\delta}}_{\overline{\beta},\overline{\eta}} = c^{\overline{\delta}}_{\overline{\beta},\overline{\eta}}$ (note that $\overline{\delta} = \overline{\beta} + \overline{\eta}$) which is also nonzero due to the Littlewood–Richardson rule.

So $h^{\nu}_{\lambda[n],\mu[n-r]} \neq 0$ and $\nu_1 = \nu_2 = \lambda_1 + \lambda_2 + 2t + r$, this proves that $n = |\lambda| + |\mu| + \lambda_1 + \mu_1 + 3t + 2r$ is where the stabilization begins. □

When $n < |\lambda| + |\mu| + \lambda_1 + \mu_1 + 3t + 2r$, Lemma 3.4 is not true for some ν. However, using Remark 3.2 and Proposition 2.1, we know that the map φ in Lemma 3.4 induces an injection from T_0 to T_0^* with $f|_{T_0} \leq f^* \circ \varphi|_{T_0}$. This gives us the following corollary:

Corollary 3.5. *Given three partitions λ, ν, and μ and two nonnegative integers r and t, the sequence $h^{\nu[n+t]}_{\lambda[n],\mu[n-r]}$ is weakly increasing.*

4 Stable Heisenberg Coefficients

By the Jacobi–Trudi determinant formula:

$$s_\lambda = \det(h_{\lambda_j+i-j})_{i,j}$$

where h_k is the complete homogeneous symmetric function, and we set $h_k = 0$ when k is negative and $h_0 = 1$. We no longer require λ to be a partition; λ can be any finite integer sequence. Then the Jacobi–Trudi determinant will give us 0 or ± 1 times some Schur function.

Murnaghan [8] pointed out that the reduced Kronecker coefficients determine the Kronecker product, and Briand et al. gave an exact formula for this in [3]. We show an analogous result for the Heisenberg product.

Given partitions λ, μ, and ν, Theorem 2.5 says that the sequence $\left\{h^{\overline{\nu}[n+|\nu|]}_{\overline{\lambda}[n+|\lambda|],\overline{\mu}[n+|\mu|]}\right\}_{n=0}^{\infty}$ is eventually constant. We write $\overline{h}^{\nu}_{\lambda,\mu}$ for that constant value, and call it a stable Heisenberg coefficient. By the way we define the stable Heisenberg coefficient, we have

$$\overline{h}^{\nu}_{\lambda,\mu} = \overline{h}^{\overline{\nu}[n+|\nu|]}_{\overline{\lambda}[n+|\lambda|],\overline{\mu}[n+|\mu|]}, \quad \text{for all nonnegative integers } n.$$

The reason we restrict n to nonnegative integers is that $\overline{\lambda}[n+|\lambda|]$, $\overline{\mu}[n+|\mu|]$, and $\overline{\nu}[n+|\nu|]$ need to be partitions. But we can remove this restriction if we extend the definition of $\overline{h}^{\nu}_{\lambda,\mu}$ to the case where λ, μ, and ν, starting from the second position, are finite weakly decreasing sequences of positive integers, i.e. $\lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \cdots \geq 0$, $\mu_2 \geq \mu_3 \geq \mu_4 \geq \cdots \geq 0$, and $\nu_2 \geq \nu_3 \geq \nu_4 \geq \cdots \geq 0$. Then we have

$$\overline{h}^{\nu}_{\lambda,\mu} = \overline{h}^{\overline{\nu}[n+|\nu|]}_{\overline{\lambda}[n+|\lambda|],\overline{\mu}[n+|\mu|]}, \quad \text{for all integers } n.$$

The stable Heisenberg coefficients determine the Heisenberg product, even for small values of n. Let us look at an example to see how this works. Consider the lowest degree component of $s_{2,1,1}\#s_{2,1}$. Let $n = 4$, then (2.7) gives us

$$\begin{aligned}(s_{2,1,1}\#s_{2,1})_4 = s_4 + 3s_{3,1} + 4s_{2,2} + 4s_{2,1,1} + 2s_{1,3} + 5s_{1,2,1} \\ + 3s_{1,1,1,1} + s_{0,3,1} + s_{0,2,2} + 2s_{0,2,1,1} + s_{0,1,1,1,1}.\end{aligned} \tag{4.1}$$

Using Jacobi–Trudi determinant, we have

$$s_{1,3} = -s_{2,2}, \quad s_{0,3,1} = -s_{2,1,1}, \quad s_{0,2,1,1} = -s_{1,1,1,1}, \quad \text{and}$$
$$s_{1,2,1} = s_{0,2,2} = s_{0,1,1,1,1} = 0.$$

So (4.1) gives us

$$(s_{2,1,1}\#s_{2,1})_4 = s_4 + 3s_{3,1} + 2s_{2,2} + 3s_{2,1,1} + s_{1,1,1,1},$$

which coincides with the result we had in Section 2. Using the process in the above example, we can recover the Heisenberg coefficients from the stable ones.

Theorem 4.1. *Let λ, μ, and ν be partitions with $|\nu| \geq |\lambda| \geq |\mu|$, then*

$$h^{\nu}_{\lambda,\mu} = \sum_{i=1}^{4|\nu|-|\lambda|-|\mu|} (-1)^{i-1} \overline{h}^{\nu^{\dagger i}}_{\lambda,\mu}, \tag{4.2}$$

where $\nu^{\dagger i} = (\nu_i - i + 1, \nu_1 + 1, \nu_2 + 1, \ldots, \nu_{i-1} + 1, \nu_{i+1}, \nu_{i+2}, \ldots)$.

Using Theorem 2.5 and Theorem 4.1, we can estimate when $h^{\nu[n+t]}_{\lambda[n],\mu[n-r]}$ stabilizes for given partitions λ, μ, and ν and nonnegative integers r and t.

Corollary 4.2. *The Heisenberg coefficient $h^{\nu[n+t]}_{\lambda[n],\mu[n-r]}$ stabilizes when $n \geq \frac{1}{2}(|\lambda| + |\mu| + |\nu| + \lambda_1 + \mu_1 + \nu_1 - 1) + r + t$.*

5 Rectangular Symmetry for Heisenberg Coefficients

Briand et al. [2] showed that four families of coefficients (Kronecker coefficients, plethysm coefficients, Littlewood–Richardson coefficients, and the Kostka–Foulkes polynomials) share symmetries related to the operations of taking complements with respect to rectangles. We follow the notations that are used in [2], and prove an analogous result for the Heisenberg coefficients.

In this section, we use "bialternants" to define Schur functions. Let $X = \{x_1, x_2, \ldots\}$ be a countable set of independent variables. For $n \geq 0$, we set $X_n = \{x_1, x_2, \ldots, x_n\}$, and $X_n{}^{\vee} = \{x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\}$ be the set of the inverses of variables in X_n. The Schur function $s_\lambda[X_n]$ is

$$s_\lambda[X_n] = \frac{\det(x_i^{\lambda_i+j-1})_{1\leq i,j\leq n}}{\det(x_i^{j-1})_{1\leq i,j\leq n}},$$

where the number of nonzero parts of λ, $\ell(\lambda) \leq n$. If $\ell(\lambda) > n$, we set $s_\lambda[X_n] = 0$.

Let (k^n) denote the partitions with n parts all equal to k. Given a partition λ, and integers $k \geq \lambda_1$, $n \geq \ell(\lambda)$, let $\square_{k,n}(\lambda) = (k - \lambda_n,\, k - \lambda_{n-1},\, \ldots,\, k - \lambda_1)$ be the complement of λ in the $n \times k$ rectangle (k^n).

From [1] (Theorem 12.1), we have

$$s_\nu(XY + X + Y) = \sum_{\lambda,\nu} h^{\nu}_{\lambda,\nu} s_\lambda(X) s_\mu(Y). \tag{5.1}$$

Restricting variables to X_m and Y_n and taking the inverses of variables, get

$$s_\nu(X_m{}^{\vee} Y_n{}^{\vee} + X_m{}^{\vee} + Y_n{}^{\vee}) = \sum_{\lambda,\nu} h^{\nu}_{\lambda,\nu} s_\lambda(X_m{}^{\vee}) s_\mu(Y_n{}^{\vee}).$$

Multiplying both sides by $(\prod_{i,j} x_i y_j)^k (\prod_i x_i)^k (\prod_j y_j)^k = (\prod_i x_i)^{kn+k} (\prod_j y_j)^{km+k}$, for k sufficiently large, and using (5.1), we get a rectangle symmetry for the Heisenberg coefficient:

Theorem 5.1. *Let m, n, and k be nonnegative integers and λ, μ, and ν be three partitions such that $\lambda \subset ((kn+k)^m)$, $\mu \subset ((km+k)^n)$, and $\nu \subset ((k)^{mn+m+n})$, then*

$$h^{\nu}_{\lambda,\mu} = h^{\square_{k,\,mn+m+n}(\nu)}_{\square_{kn+k,\,m}(\lambda),\,\square_{km+k,\,n}(\mu)} \tag{5.2}$$

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