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Enumerating permutations sortable by k passes through a pop-stack

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Abstract. In an exercise in the first volume of his famous series of books, Knuth considered sorting permutations by passing them through a stack. Many variations of this exercise have since been considered, including allowing multiple passes through the stack and using different data structures. We are concerned with a variation using pop-stacks that was introduced by Avis and Newborn in 1981. Let $P_k(x)$ be the generating function for the permutations sortable by k passes through a pop-stack. The generating function $P_2(x)$ was recently given by Pudwell and Smith (the case $k = 1$ being trivial). We show that $P_k(x)$ is rational for any k. Moreover, we give an algorithm to derive $P_k(x)$, and using it we determine the generating functions $P_k(x)$ for $k \leq 6$.

Keywords: permutation, pop-stack, sortable, enumeration, generating function, rational, language

1 Introduction

Knuth [9, Exercise 2.2.1.5] noted that permutations sortable by a stack are precisely those that do not contain a subsequence in the same relative order as the permutation 231. This exercise inspired a wide range of research and can be seen as the starting point of the research field we now call permutation patterns. Our interest lies in Knuth's original exercise and its variations.

In 1972 Tarjan [13] considered sorting with networks of stacks and queues, a problem that, in general, has proven itself to be beyond the reach of current methods in permutation patterns and enumeration. There has been some recent significant progress though: In 2015 Albert and Bousquet-Mélou [1] enumerated permutations sortable by two stacks in parallel. The enumeration of permutations sortable by two stacks in series is, however, an open problem. A related problem is that of sorting permutations by k passes through a stack, where the elements on the stack are required to be increasing when read from top to bottom. West [15] characterized the permutations sortable by two passes through a stack in terms of pattern avoidance and conjectured their enumeration, a conjecture

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that was subsequently proved by Zeilberger [16]. Permutations sortable by three passes have been characterized by Úlfarsson [14], but their enumeration is unknown.

In other variations of Knuth's exercise different data structures are used for sorting. One notable example is that of pop-stacks: a stack where each pop operation completely empties the stack. Avis and Newborn [5] enumerated the permutations sortable by pop-stacks in series, with the modification that each pop leads to the popped elements being immediately pushed to the following pop-stack. Atkinson and Stitt [4] considered two pop-stacks in genuine series. Permutations sortable by pop-stacks in parallel have been studied by Atkinson and Sack [3], who characterized those permutations by a finite set of forbidden patterns. They also conjectured that the generating function for their enumeration is rational, which was subsequently proved by Smith and Vatter [12], who gave an insertion encoding [2] for the sortable permutations.

Pudwell and Smith [10] recently characterized the permutations sortable by two passes through a pop-stack in terms of pattern avoidance and gave their enumeration. They also gave a bijection between certain families of polyominoes and the permutations sortable by one or two passes through a pop-stack, but noted that the bijection does not generalize to three passes. We give an algorithm to derive a rational generating function for the permutations sortable by k passes through a pop-stack, for any fixed k.

2 Sorting plans and traces

A single pass of the pop-stack sorting operator formally works as follows. Processing a permutation $\pi = a_1 a_2 \dots a_n$ of $[n] = \{1, \dots, n\}$ from left to right, if the stack is empty or its top element is smaller than the current element a_i then perform a single pop operation (a), emptying the stack and appending those elements to the output permutation; else do nothing (d). Next, push a_i onto the stack and proceed with element a_{i+1}, or if $i = n$ perform one final pop operation (a), again emptying the stack onto the output permutation, and terminate. Define $P(\pi)$ as the final output permutation and $w(\pi)$ as the word over the alphabet $\{\mathtt{a}, \mathtt{d}\}$ defined by the operations performed when processing π. For instance, with $\pi = 752491863$ we have $P(\pi) = 257419368$ and $w(\pi) = \mathtt{addaadadda}$. Note that $w(\pi)$ will always begin and end with the letter a. We will call any word of length $n+1$ with letters in $\{\mathtt{a}, \mathtt{d}\}$ that begin and end with the letter a an *operation sequence*.

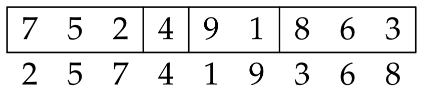
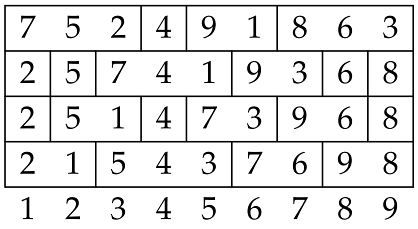


Figure 1: Single-pass sorting trace

Let us now introduce what we call sorting traces. Consider applying the pop-stack operator P to a permutation π of $[n]$. Start by interleaving $w(\pi)$ with π; for instance, with $\pi = 752491863$ (as before) we have a7d5d2a4a9d1a8d6d3a. Replacing a with a bar and d with a space, and placing $P(\pi)$ below this string, we get Figure 1.

We call the numbers between pairs of successive a's *blocks*. Above, the blocks of π are 752, 4, 91, and 863. Note that $P(\pi)$ can be obtained from π by reversing each block. An index $i \in [n-1]$ of a permutation $\pi = a_1 a_2 \dots a_n$ is an *ascent* if $a_i < a_{i+1}$. Similarly, i is a *descent* if $a_i > a_{i+1}$. With this terminology $w(\pi) = c_1 c_2 \dots c_{n+1}$ is simply the ascent/descent word of $-\infty\pi\infty$; i.e. $c_1 = c_{n+1} = \mathtt{a}$ and, for $2 \leq i \leq n$, $c_i = \mathtt{a}$ if $i-1$ is an ascent, and $c_i = \mathtt{d}$ if $i-1$ is a descent.

A figure such as Figure 1 can be extended to depict multiple passes. Applying P to the example permutation, π, until it is sorted gives Figure 2.



7	5	2	4	9	1	8	6	3
2	5	7	4	1	9	3	6	8
2	5	1	4	7	3	9	6	8
2	1	5	4	3	7	6	9	8
1	2	3	4	5	6	7	8	9

Figure 2: A sorting trace

We will call such figures sorting traces, or traces for short. The structure that remains when removing the numbers from a trace we call its sorting plan. Each row of a sorting plan corresponds to an operation sequence, and for convenience we shall number the rows 1 through k, from top to bottom. The example sorting plan can be viewed as the array of operation sequences given in Figure 3. By interpreting each column as a binary number with $a = 0$ and $d = 1$ the sorting plan can more compactly be represented, or *encoded*, with the sequence of numbers 0, 9, 10, 5, 5, 10, 5, 10, 9, 0. Formally, we define a trace and its sorting plan as follows.

Definition 2.1. *Let $A = (\alpha_1, \alpha_2, \dots, \alpha_k, \alpha_{k+1})$ be a $(k+1)$-tuple of permutations of $[n]$ in which α_{k+1} is the identity permutation. Let $M = (\mu_1, \mu_2, \dots, \mu_k)$ be a k-tuple of operation sequences, each of length $n+1$. We call $T = (A, M)$ a* trace *of* length n *and* order k, *and M its* sorting plan, *if the following conditions are satisfied for $i = 1, \dots, k$:*

1. *The word μ_i records the sequence of operations performed by the pop-stack operator when applied to α_i—in symbols, $w(\alpha_i) = \mu_i$. As to the picture of the trace, two adjacent numbers form an ascent if and only if they are separated by a bar.*

2. *The sequence of permutations in A records repeated application of the pop-stack operator to the permutation α_1—in symbols, $P(\alpha_i) = \alpha_{i+1}$. Or, assuming the first condition is satisfied,* $\operatorname{rev}(\alpha_i, \mu_i) = \alpha_{i+1}$, *where the operator* rev *is defined below.*

Definition 2.2. *Let π be a permutation of $[n]$ and $\mu = c_1 c_2 \dots c_{n+1}$ be an operation sequence. Let $i_1 < i_2 < \cdots < i_k$ be the sequence of indices i for which $c_i = \mathtt{a}$. Write $\pi = \gamma_1 \gamma_2 \dots \gamma_{k-1}$, where $|\gamma_j| = i_{j+1} - i_j$ for $j = 1, \dots, k-1$. In other words, the length of γ_j is the same as the length of the jth block of π with respect to μ. Define* $\operatorname{rev}(\pi, \mu) = \gamma_1^r \gamma_2^r \dots \gamma_{k-1}^r$, *where $(\,\cdot\,)^r$ is the reversal operator. We call this the* blockwise reversal of π according to μ.

Much of the information stored in a trace is redundant: As we have seen, α_1 alone determines the complete trace. Similarly, from the sorting plan M we can recover $\alpha_1, \dots, \alpha_{k+1}$. This is because the last permutation α_{k+1} is the identity, the permutation α_k is the blockwise reversal of α_{k+1} according to μ_k, the permutation α_{k-1} is the blockwise reversal of α_k according to μ_{k-1}, etc. In symbols, $\text{rev}(\alpha_i, \mu_i) = \alpha_{i+1}$ if and only if $\alpha_i = \text{rev}(\alpha_{i+1}, \mu_i)$. In this way a sorting plan uniquely determines a trace.

Our goal is to count the permutations of $[n]$ that are sortable by k passes through a pop-stack and for brevity we will sometimes refer to such permutations as *k-pop-stack-sortable*. Note that a k-pop-stack-sortable permutation is also $(k+1)$-pop-stack-sortable. Starting with a k-pop-stack-sortable permutation of $[n]$ and performing k passes of the pop-stack sorting operator results in a trace of length n and order k. Conversely, the first row of that trace is the k-pop-stack-sortable permutation we started with. Thus, k-pop-stack-sortable permutations of $[n]$ are in one-to-one correspondence with traces of length n and order k. Those are, in turn, in one-to-one correspondence with their sorting plans, and hence it suffices to count sorting plans of length n and order k.

```
addaadadda
aaaddadaaa
aadaadadaa
adaddadada
```

Figure 3

Let us call any k-tuple of operation sequences of length $n+1$ an *operation array* of length $n+1$ and order k. Note that sorting plans are operation arrays, but not all operation arrays are sorting plans. Let an operation array $(\mu_1, \mu_2, \ldots, \mu_k)$ be given. Let the permutations α_1, α_2, …, α_{k+1} be defined by requiring that the permutation α_{k+1} is the identity, the permutation α_k is the blockwise reversal of α_{k+1} according to μ_k, the permutation α_{k-1} is the blockwise reversal of α_k according to μ_{k-1}, etc. In symbols, $\text{rev}(\alpha_i, \mu_i) = \alpha_{i+1}$. Then the tuple $(\alpha_1, \ldots, \alpha_{k+1})$ is called a *semitrace*. In other words, a semitrace is defined by requiring that Property 2 of Definition 2.1 holds, but not necessarily Property 1 of the same definition.

We shall characterize those operation arrays that are sorting plans, then count the sorting plans of order k, and by extension the k-pop-stack-sortable permutations. We start by considering cases where k is small.

2.1 1-pop-stack-sortable permutations

We want to determine which operation sequences of length $n+1$ when interpreted as operation arrays are sorting plans. Consider the operation sequence `addaadada`. The semitrace corresponding to it is given in Figure 4. Is this a trace? By definition of a semitrace it satisfies Property 2 of Definition 2.1, but does it also satisfy Property 1? Yes, because the bottom-most permutation is the identity, any two adjacent numbers separated by a bar form an ascent, and two adjacent numbers within a block form a descent. Thus, each operation sequence represents a sorting plan of order 1. An operation sequence starts and ends with the letter `a`. The remaining letters can be either `a` or `d`. Thus, for $n > 0$, there are 2^{n-1} operation sequences of length $n+1$,

$$\begin{array}{|ccc|c|cc|cc|} \hline 3 & 2 & 1 & 4 & 6 & 5 & 8 & 7 \\ \hline \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{4} & \multicolumn{1}{c}{5} & \multicolumn{1}{c}{6} & \multicolumn{1}{c}{7} & \multicolumn{1}{c}{8} \end{array}$$

Figure 4: Semitrace for `addaadada`

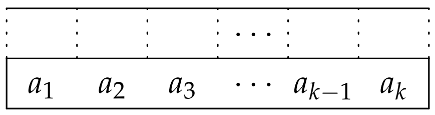
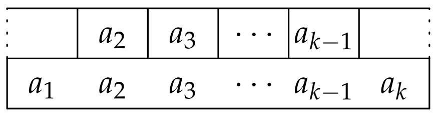


Figure 5: The neighborhood around a given block, where a dotted line can either be a bar or not


a_2 a_3 $\cdots$ a_{k-1}
a_1 a_2 a_3 $\cdots$ a_{k-1} a_k
Figure 6: The same neighborhood, as to the left, after inferring the status of most dotted lines

and hence 2^{n-1} 1-pop-stack-sortable permutations of $[n]$. It is easy to see that these are precisely the layered permutations (direct sums of decreasing permutations).

While this simple example outlines our approach to count k-pop-stack-sortable permutations, it is a little too simple, as for larger k, most operation arrays will not be sorting plans. As an example, we present the following lemma.

Lemma 2.3. *In a trace of order* 2 *or greater, each operation sequence—except for the first one—contains at most* 2 *consecutive* `d`*'s. Or, equivalently, each row of the sorting plan—except for the first one—has blocks of size at most* 3.

Proof. Given a trace of order 2 or greater, consider any block $a_1 > a_2 > \cdots > a_k$ from any row of its sorting plan, excluding the first row. For contradiction, assume that $k \geq 4$. The neighborhood around the block then looks as in Figure 5. Consider the dotted lines in the upper row, except for the two outermost dotted lines. If any of them were not solid, then at least two of $a_1, a_2, \ldots, a_k$ would be together in a block. These two elements would occur in increasing order, violating Property 1 of Definition 2.1. These dotted lines must thus be solid, as is illustrated in Figure 6. Now, however, a_2 and a_3 are two adjacent numbers in distinct blocks that do not form an ascent, again violating Property 1. □

This lemma characterizes a certain class of operation arrays that are not sorting plans, namely those that have at least one block of size 4 or larger in rows 2 to k. In the next section we will take a more general approach towards characterizing operation arrays that are not sorting plans by introducing what we call forbidden segments.

3 Forbidden segments

A semitrace fails to be a trace precisely when there is a pair of elements witnessing Property 1 of Definition 2.1 fail. With the following definition we single out such pairs.

Definition 3.1. *Let T be a semitrace of length n and let a and b be two distinct elements of $[n]$. We call (a, b) a* violating pair *of T if, in any row, a and b occur in adjacent positions such that they violate Property 1. That is, a and b form a descent and are separated by a bar, or a and b form an ascent and are not separated by a bar.*

From Definitions 2.1 and 3.1 we get the following characterization of sorting plans.

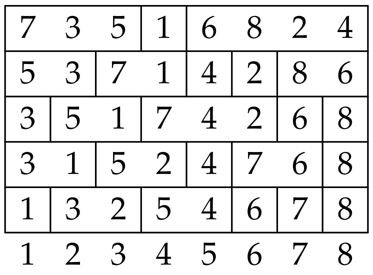


Figure 7: An example semitrace

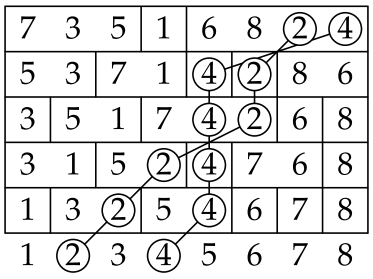


Figure 8: The progress of 2 and 4 through the semitrace to the left

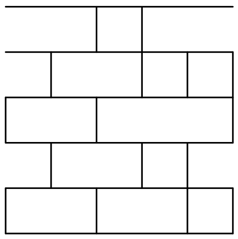


Figure 9: The segment $T_{2,4}$ of the trace above

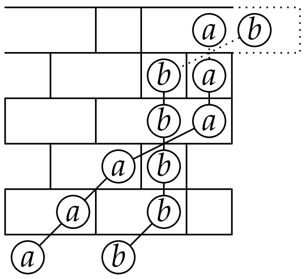


Figure 10: Following the elements a and b up the segment to the left

Lemma 3.2. *An operation array is a sorting plan if and only if its semitrace has no violating pair.*

Definition 3.3. *Let ψ be the bijection mapping a sorting plan to its encoding. Let M be a sorting plan of length n and suppose that $1 \le i \le j \le n+1$. Let $\psi(M) = c_1c_2\dots c_{n+1}$ be the encoding of M. Then we call $S = \psi^{-1}(c_ic_{i+1}\dots c_j)$ a* segment *of M, and $|S| = j - i + 1$ its length.*

Definition 3.4. *Let $T = (A, M)$ be a semitrace of length n and let a and b be two distinct elements of $[n]$. Let B be the set of blocks that contain either a or b, excluding blocks in the first row. We define* the segment of T determined by a and b, *denoted $T_{a,b}$, as the smallest segment of M that fully contains all the blocks in B.*

As an example, consider the semitrace in Figure 7. It contains four violating pairs, namely $(1,5)$, $(2,4)$, $(3,5)$, and $(6,8)$. Looking at the pair $(2,4)$, Figure 8 shows how these two numbers progress through the trace. The segment $T_{2,4}$, shown in Figure 9, is the smallest segment that contains all blocks on rows 2, 3, 4, and 5 with at least one circled element.

Any sorting plan of order 5 that contains this segment, no matter where it occurs horizontally, will violate Property 1, just as the above semitrace. This is because we can follow the two elements a and b playing the roles of 2 and 4 from the bottom to the second row, as shown in Figure 10. Formally, we make the following definition.

Definition 3.5. *A segment $T_{a,b}$ is* forbidden *if, after inferring the positions of a and b in rows* 2 *to k, either the two numbers violate Property* 1 *on row i, where* $2 \leq i \leq k$*, or*

1. *a and b form a descent on row* 2 *and, if a and b are in columns i and* $i+1$ *on row* 2*, there is no bar separating columns i and* $i+1$ *on row* 1*, or*

2. *a and b are in decreasing order on row* 2 *and, if a and b are in columns i and j on row* 2*, with* $i < j$*, there is a bar immediately to the left of column i on row* 1*, a bar immediately to the right of column j on row* 1*, and exactly one bar between columns i and j on row* 1*.*

In both cases the bars that we reference belong to the segment $T_{a,b}$*. In other words, we can determine if* $T_{a,b}$ *is forbidden or not without knowing in which semitrace it is embedded.*

Lemma 3.6. *A segment* $T_{a,b}$ *is forbidden if and only if* (a, b) *is a violating pair of* T*.*

Proof. Consider a forbidden segment $T_{a,b}$. If, after inferring the positions of a and b in rows 2 to k, the two numbers violate Property 1 on row i, where $2 \leq i \leq k$, then (a, b) is a violating pair of T. Otherwise we have two cases, corresponding to the two cases of Definition 3.5, and these are illustrated in Figures 11 and 12, respectively. In the first case a and b form an ascent and are in the same block on row 1. In the second case a and b form a descent and are separated by a bar on row 1. In both cases we have found a violation of Property 1 and hence (a, b) is a violating pair of T.

Conversely, consider a violating pair (a, b) of T, as well as the segment $T_{a,b}$. If a and b violate Property 1 on row i, where $2 \leq i \leq k$, then $T_{a,b}$ is a forbidden segment. If, on the other hand, the two numbers violate Property 1 on row 1, then we have two cases.

If a and b form an ascent on row 1 and are not separated by a bar, then the two numbers are in the same block on row 1, and will form a descent on row 2. Furthermore, if the two numbers are in columns i and $i+1$ on row 2, there will not be a bar separating columns i and $i+1$ on row 1. Hence $T_{a,b}$ is a forbidden segment.

If a and b form a descent on row 1 and are separated by a bar, then the two numbers are in adjacent blocks on row 1. If i is the leftmost column that the left block intersects, and j is the rightmost column that the right block intersects, then the two numbers will be in decreasing order on row 2, with the larger number number in column i and the smaller number in column j. Furthermore, there is a bar immediately to the left of column i on row 1 and a bar immediately to the right of column j on row 1, and exactly one bar between columns i and j on row 1. Hence $T_{a,b}$ is a forbidden segment. □

From Lemmas 3.2 and 3.6 we get the following result.

Lemma 3.7. *An operation array is a sorting plan if and only if it does not contain any forbidden segment* $T_{a,b}$*.*

Because there are potentially an infinite number of forbidden segments $T_{a,b}$ this lemma is of limited practical use as stated. This motivates the following definition.

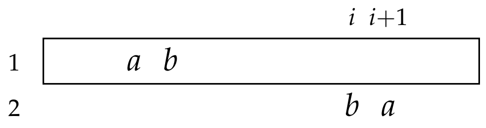


Figure 11: The first case of Definition 3.5, assuming $a < b$

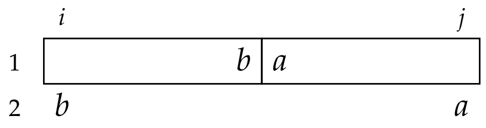


Figure 12: The second case of Definition 3.5, assuming $a < b$

Definition 3.8. *A segment of order k is* bounded *if each of its blocks in rows 2 to k has size at most 3. Equivalently, a segment is bounded if its operation array has no occurrence of three consecutive $\mathtt{d}$'s on rows 2 to k.*

Combining Lemmas 2.3 and 3.7 we arrive at the following proposition.

Proposition 3.9. *An operation array is a sorting plan if and only if it does not contain any bounded forbidden segment $T_{a,b}$ and each block on rows 2 through k is of size at most 3.*

Through a series of lemmas we shall establish that there are finitely many bounded forbidden segments $T_{a,b}$ of order k, for any fixed k.

Lemma 3.10. *Let T be a semitrace of length n and order k and let a and b be two distinct elements of $[n]$. If $T_{a,b}$ is a bounded segment, and there is a block, not on the first row, that includes both a and b, then $|T_{a,b}| \leq 4k - 5$.*

Proof. We can disregard the first row, as neither this lemma nor the definition of $T_{a,b}$ includes blocks on that row. Since $T_{a,b}$ is a bounded segment, each of the remaining blocks that either contains a or b has size at most 3. Thus, if we consider two adjacent rows and $x \in \{a, b\}$, then the horizontal distance between x in the upper row and x in the lower row is at most 2. In total, the horizontal distance between x on the second row and x on the k-th row is at most $2(k-2)$ and, consequently, the length of the segment $T_{a,b}$ is at most $4(k-2)+m$, where $m \leq 3$ is the size of the block containing a and b. □

Lemma 3.11. *Let T be a semitrace and (a, b) a violating pair of T. Then there is a block, not on the first row, that includes both a and b.*

Proof. Consider a row i where the pair (a, b) violates Property 1. If a and b form a descent and are separated by a bar, then the two elements are in distinct blocks on row i, and will still be in descending order on row $i+1$ after performing the blockwise reversals. If, on the hand, a and b form an ascent and are not separated by a bar, then the two elements are in the same block on row i, and will be in descending order on row $i+1$ after performing the blockwise reversals. In either case, a and b will be in descending order on row $i+1$. Since the two elements are in increasing order in the last permutation the two elements must be reversed on at least one of the rows between $i+1$ and k. As the relative order of elements is only reversed when they appear together in a block, there is a block on one of the rows between $i+1$ and k that includes both a and b. □

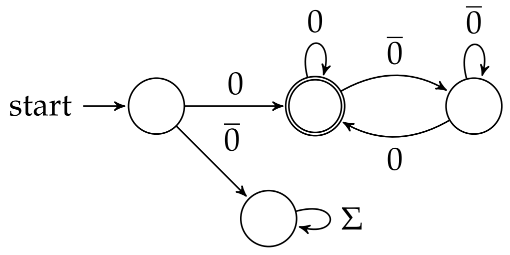


Figure 13: A DFA recognizing strings that begin and end with a solid boundary

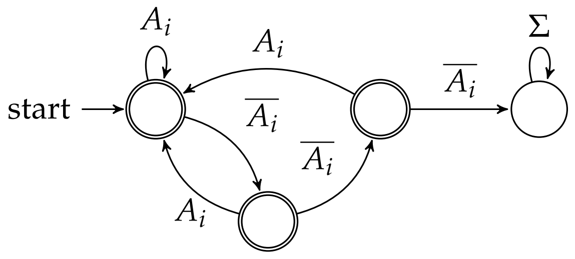


Figure 14: A DFA that when intersected with W recognizes operation arrays with blocks of size at most 3 in row i

Lemma 3.12. *For a fixed k, there are finitely many bounded forbidden segments $T_{a,b}$ of order k, and they can be listed.*

Proof. If $T_{a,b}$ is a bounded forbidden segment then (a, b) is a violating pair by Lemma 3.6. Using Lemma 3.11 we find a block, not on the first row, that includes a and b; and Lemma 3.10 gives $|T_{a,b}| \leq 4k - 5$. Thus, there are at most as many forbidden segments as there are words in $\{\mathtt{a}, \mathtt{d}\}^{(4k-5)k}$, and they can be listed by checking against Definition 3.5. □

4 Regular language

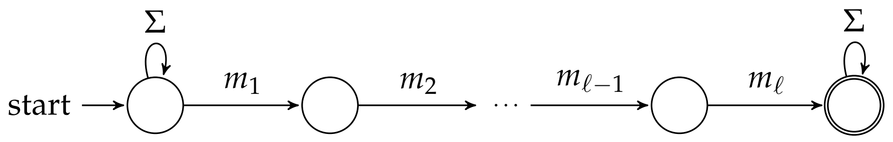
We will use the characterization of sorting plans in terms of forbidden segments to count sorting plans of order k, and hence the k-pop-stack-sortable permutations. To that end, we will employ the theory of formal languages. We will assume that the reader is familiar with basic constructions such as taking the complement of a deterministic finite automaton (DFA) or the intersection of two DFAs [8].

Recall that we can encode an operation array of length n and order k as a sequence of n integers, each in the range $[0, 2^k - 1]$. In this way we can consider operation arrays as strings of a formal language over the alphabet $\Sigma = \{0, 1, \ldots, 2^k - 1\}$. Conversely, strings over this alphabet can be considered as operation arrays, under one condition: that they both begin and end with a solid boundary. Noting that a solid boundary corresponds to the integer 0 from Σ, and letting $\overline{0} = \Sigma \setminus \{0\}$, the DFA W in Figure 13 recognizes the strings over Σ that begin and end with a solid boundary, i.e. the strings that correspond to an operation array.

For ease of notation we will use the name of a DFA to also denote the language that it recognizes. Here, W denotes the DFA recognizing strings that begin and end with a solid boundary as well as the language consisting of such strings. We want to find the subset of W corresponding to sorting plans. Recall from Proposition 3.9 that an operation array is a sorting plan if and only if it does not contain any bounded forbidden segments and each block on rows 2 through k is of size at most 3. We shall start with the latter condition.

If A_i is the set of symbols from Σ that represent a column from the operation array that has a bar in the ith row, and $\overline{A_i} = \Sigma \setminus A_i$, then the intersection of W with the automaton, R_i, in Figure 14 recognizes the operation arrays that have blocks of size at most 3 in row i. Therefore, the set of operation arrays that have blocks of size at most 3 in all but the first row is recognized by the automaton $W \cap R_2 \cap \cdots \cap R_k$.

The other condition that sorting plans satisfy is that they do not contain any bounded forbidden segments. Consider a segment M and let us encode it in the same manner as we encode operation arrays, resulting in the sequence $m_1, \ldots, m_\ell$. Note that an operation array A contains M if and only if the encoding of A contains $m_1 \cdots m_\ell$ as a factor. Furthermore, the following nondeterministic finite automaton (NFA), Q_M, recognizes the set of strings over Σ that contain the encoding of M as a factor:



Taking the complement of Q_M we get an automaton $\overline{Q_M}$ that recognizes the set of strings over Σ that do not contain the factor M. In particular, if F is a forbidden segment, then $W \cap \overline{Q_F}$ recognizes the set of operation arrays that do not contain F. Let $\mathcal{F}$ be the set of bounded forbidden segments, which is finite by Lemma 3.12. Then the automaton

$$S = W \cap \bigcap_{i=2}^{k} R_i \cap \bigcap_{F \in \mathcal{F}} \overline{Q_F}$$

recognizes the set of operation arrays that have blocks of size at most 3 in rows 2 through k, and do not contain any bounded forbidden segments. Hence, by Proposition 3.9, S recognizes exactly the set of sorting plans, giving us the following proposition:

Proposition 4.1. *The language $S = \{ w \in \Sigma^* \mid w \text{ is a sorting plan} \}$ is regular.*

We can now present our main theorem.

Theorem 4.2. *For a fixed k, the generating function $P(x) = \sum_{n=0}^{\infty} p_n x^n$, where p_n is the number of k-pop-stack-sortable permutations of length n, is rational.*

Proof. We have a bijection between the k-pop-stack-sortable permutations of length n and the sorting plans of order k and length n, so the two sets are equinumerous. The sorting plans of length n are in bijection with words of length $n+1$ recognized by the automaton S, which is regular by Proposition 4.1. It is well known that regular languages have rational generating functions [11], and they can be derived from the corresponding DFA by setting up a system of linear equations. If $S(x)$ is the rational generating function for S, it is clear that $P(x) = S(x)/x$ and that this generating function is rational. □

k	Generating function
1	$(x-1)/(2x-1)$
2	$(x^3+x^2+x-1)/(2x^3+x^2+2x-1)$
3	$(2x^{10}+4x^9+2x^8+5x^7+11x^6+8x^5+6x^4+6x^3+2x^2+x-1)/(4x^{10}+8x^9+4x^8+10x^7+22x^6+16x^5+8x^4+6x^3+2x^2+2x-1)$
4	$(64x^{25}+448x^{24}+1184x^{23}+1784x^{22}+2028x^{21}+1948x^{20}+1080x^{19}+104x^{18}-180x^{17}+540x^{16}+1156x^{15}+696x^{14}+252x^{13}+238x^{12}+188x^{11}+502x^{10}+806x^9+544x^8+263x^7+185x^6+99x^5+33x^4+13x^3+3x^2+x-1)/(128x^{25}+896x^{24}+2368x^{23}+3568x^{22}+3928x^{21}+3064x^{20}+176x^{19}-2304x^{18}-2664x^{17}-1580x^{16}-352x^{15}-576x^{14}-1104x^{13}-760x^{12}-138x^{11}+686x^{10}+1238x^9+869x^8+382x^7+210x^6+102x^5+27x^4+12x^3+3x^2+2x-1)$

Table 1: The generating functions for the k-pop-stack-sortable permutations, $k \leq 4$

Note that all of the above results are constructive, meaning that the generating function can be computed for any fixed k. We did so for $k = 1, \ldots, 6$ and in Table 1 we list the resulting generating functions, except for $k = 5$ and $k = 6$ whose expressions are too large to display. For each of them the degree of the polynomial in the numerator is the same as the degree of the polynomial in the denominator. Those degrees, the growth rates of coefficients of the generating functions, and the corresponding sequences for the number of vertices and edges in the final DFAs can be found in the table below.

k	1	2	3	4	5	6
degree	1	3	10	25	71	213
growth rate	2.0000	2.6590	3.4465	4.2706	5.1166	5.9669
vertices	4	5	12	32	99	339
edges	8	11	34	120	477	2010

All the generating functions, source code, and text files defining the DFAs can be found on GitHub [6]. The growth rate of the coefficients of a rational power series $p(x)/q(x)$ is given by $\max\{1/|\zeta| : q(\zeta) = 0\}$ and Sage [7] code for calculating the approximate growth rates, in the table above, can be found on the same GitHub page.

It would be interesting to find a closed formula for the generating functions, possibly leading to the distribution of the number of passes needed to sort a permutation using a pop-stack. It is not clear whether our approach can be used as a basis for such a formula. Finally, as we only considered their enumeration, finding a useful permutation pattern characterization of the k-pop-stack-sortable permutations, for $k \geq 3$, remains open.

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