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A labelled variant of the matchings-Jack and hypermap-Jack conjectures

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Abstract. Introduced by Goulden and Jackson in their 1996 paper, the matchings-Jack and hypermap-Jack conjectures are two major open questions relating symmetric functions, representation theory and combinatorial maps. These conjectures state an important combinatorial interpretation of the coefficients in the power sum expansion of two related formal power series involving Jack symmetric functions. These coefficients are indexed by three partitions of a given integer n. This paper is devoted to the case when one of them is equal to (n). We exhibit some of their polynomial properties and prove a variant of the two conjectures involving labelled hypermaps and matchings in some important cases.

Résumé. Introduites par Goulden et Jackson dans leur article de 1996, les conjectures matchings-Jack et hypermap-Jack sont deux grandes questions ouvertes mettant en relation les fonctions symétriques, la théorie des représentations et les cartes combinatoires. Ces conjectures affirment une interprétation combinatoire importante des coefficients du diéveloppement dans la base des polynômes symétriques puissances de deux séries formelles impliquant des fonctions symétriques de Jack. Ces coefficients sont indexés par trois partitions d'un entier n donné. Cet article est consacré au cas où l'une d'elles est égal à (n). Nous montrons certaines de leurs propriétés polynomiales et prouvons dans certains cas importants une variante des deux conjectures impliquant des hypercartes et des couplages numérotés.

Keywords: Matchings-Jack and hypermap-Jack conjectures, Jack symmetric functions

1 Introduction

1.1 The matchings-Jack and the hypermap-Jack conjectures

For any integer n denote $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_p) \vdash n$ an integer partition of $|\lambda| = n$ with $\ell(\lambda) = p$ parts sorted in decreasing order. The set of all integer partitions is denoted

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$\mathcal{P}$. If $m_i(\lambda)$ is the number of parts of λ that are equal to i, then we may write λ as $[1^{m_1(\lambda)}\,2^{m_2(\lambda)}\cdots]$ and define $z_\lambda = \prod_i i^{m_i(\lambda)} m_i(\lambda)!$, $Aut_\lambda = \prod_i m_i(\lambda)!$. When there is no ambiguity, the one part partition of integer n, $(n) = [n^1]$ is simply denoted n. Given a parameter α, denote $p_\lambda(x)$ and $J^\alpha_\lambda(x)$ the power sum and **Jack symmetric function** indexed by λ on $x = (x_1, x_2, \cdots)$. Jack symmetric functions are orthogonal for the scalar product $\langle\cdot,\cdot\rangle_\alpha$ defined by $\langle p_\lambda, p_\mu\rangle_\alpha = \alpha^{\ell(\lambda)} z_\lambda \delta_{\lambda,\mu}$. Denote $j_\lambda(\alpha)$ the value of the scalar product $\langle J^\alpha_\lambda, J^\alpha_\mu\rangle_\alpha = j_\lambda(\alpha)\delta_{\lambda,\mu}$. This paper is devoted to the study of the following series for Jack symmetric functions introduced by Goulden and Jackson in [5].

$$\Phi(x,y,z,t,\alpha) = \sum_{\gamma\in\mathcal{P}} \frac{J^\alpha_\gamma(x)J^\alpha_\gamma(y)J^\alpha_\gamma(z)t^{|\gamma|}}{\langle J^\alpha_\gamma, J^\alpha_\gamma\rangle_\alpha}, \quad \Psi(x,y,z,t,\alpha) = \alpha t \frac{\partial}{\partial t}\log\Phi(x,y,z,t,\alpha).$$

More specifically, we focus on the coefficients $a^\lambda_{\mu,\nu}(\alpha)$ and $h^\lambda_{\mu,\nu}(\alpha)$ defined by:

$$\Phi(x,y,z,t,\alpha) = \sum_{n\geqslant 0} t^n \sum_{\lambda,\mu,\nu\vdash n} \alpha^{-\ell(\lambda)} z_\lambda^{-1} a^\lambda_{\mu,\nu}(\alpha) p_\lambda(x)p_\mu(y)p_\nu(z),$$
$$\Psi(x,y,z,t,\alpha) = \sum_{n\geqslant 0} t^n \sum_{\lambda,\mu,\nu\vdash n} h^\lambda_{\mu,\nu}(\alpha) p_\lambda(x)p_\mu(y)p_\nu(z).$$

Goulden and Jackson conjecture that $a^\lambda_{\mu,\nu}(\alpha)$ and $h^\lambda_{\mu,\nu}(\alpha)$ may have a strong combinatorial interpretation. In particular thanks to exhaustive computations of the coefficients they show that the $a^\lambda_{\mu,\nu}(\alpha)$ and $h^\lambda_{\mu,\nu}(\alpha)$ are polynomials in $\beta = \alpha - 1$ with non negative integer coefficients and of degree at most $n - \min\{\ell(\mu), \ell(\nu)\}$ for all $\lambda, \mu, \nu \vdash n \leqslant 8$. They conjecture this property for arbitrary λ, μ, ν and prove it in the limit cases $\lambda = [1^n]$ and $\lambda = [1^{n-2}2^1]$. Moreover, for λ, μ, ν partitions of a given integer n, they make the stronger suggestion that the coefficients in the powers of β in $a^\lambda_{\mu,\nu}(\alpha)$ count certain sets of **matchings** i.e. fixpoint-free involutions of the symmetric group on $2n$ elements (the **matchings-Jack conjecture**) and that the coefficients in the powers of β in $h^\lambda_{\mu,\nu}(\alpha)$ count certain sets of **locally orientable hypermaps** i.e. connected bipartite graphs embedded in a locally orientable surface (the **hypermap-Jack conjecture** or **b-conjecture**). We look at the case $\mu = (n)$ and study a variant of the conjectures involving labelled objects defined in the following section. In this specific case the two conjectures are related as

$$h^\lambda_{n,\nu}(\alpha) = \alpha n z_\lambda^{-1} \alpha^{-\ell(\lambda)} a^\lambda_{n,\nu}(\alpha). \tag{1.1}$$

However, because of the difference in the combinatorial objects involved in the two conjectures, they do not seem to be equivalent.

1.2 Combinatorial background

1.2.1 Labelled matchings

Given a non-negative integer n and a set of $2n$ vertices $V_n = \{1, \widehat{1}, \cdots, n, \widehat{n}\}$ we call a **matching** on V_n a set of n non-adjacent edges such that all the vertices are the endpoint of one edge. Given two matchings δ_1 and δ_2, the graph induced by the vertices in V_n and the $2n$ edges of $\delta_1 \cup \delta_2$ is composed of cycles of even length $2\varepsilon_1, 2\varepsilon_2, \cdots, 2\varepsilon_p$ for some $\varepsilon \vdash n$ and we denote $\Lambda(\delta_1, \delta_2) = \varepsilon$. For a partition $\lambda = (\lambda_1, \cdots, \lambda_p)$ of n, define two canonical matchings $\mathbf{g}_n$ and $\mathbf{b}_\lambda$. The matching $\mathbf{g}_n$ is obtained by drawing a gray colored edge between vertices i and $\widehat{i} = \mathbf{g}_n(i)$ for $i = 1, \cdots, n$. The matching $\mathbf{b}_\lambda$ is obtained by drawing a black colored edge between vertices $\widehat{i}$ and $\mathbf{b}_\lambda(\widehat{i})$ for $i = 1, \cdots, n$ where $\mathbf{b}_\lambda(\widehat{i}) = 1 + \sum_{k=1}^{l-1} \lambda_k$ if $i = \sum_{k=1}^{l} \lambda_k$ for some $1 \leqslant l \leqslant p$ and $\mathbf{b}_\lambda(\widehat{i}) = i + 1$ otherwise. Obviously $\Lambda(\mathbf{g}_n, \mathbf{b}_\lambda) = \lambda$. Denote by $\mathcal{G}^\lambda_{\mu,\nu}$ the set of all the matchings δ on V_n such that $\Lambda(\mathbf{g}_n, \delta) = \mu$ and $\Lambda(\mathbf{b}_\lambda, \delta) = \nu$. A matching δ in which all edges are of kind $\{i, \widehat{j}\}$ is called **bipartite**. Denote $\widetilde{b}^\lambda_{\mu,\nu} = |\mathcal{G}^\lambda_{\mu,\nu}|$ and $c^\lambda_{\mu,\nu} = |\{\delta \in \mathcal{G}^\lambda_{\mu,\nu} \mid \delta \text{ is bipartite}\}|$. One can show that the numbers $c^\lambda_{\mu,\nu}$ and $b^\lambda_{\mu,\nu} = 2^n n! \widetilde{b}^\lambda_{\mu,\nu}$ are the structure constants of the *class algebra* and the *double coset algebra* of the symmetric group (see [5], proposition 4.1). Furthermore, one has a direct connection with the coefficients of Goulden and Jackson (see e.g. [5], [10]) as

$$a^\lambda_{\mu,\nu}(1) = c^\lambda_{\mu,\nu} \quad \text{and} \quad a^\lambda_{\mu,\nu}(2) = \widetilde{b}^\lambda_{\mu,\nu}.$$

This paper is focused on the case $\mu = (n)$. For $\lambda, \nu \vdash n$ we consider the set of **labelled matchings** $\widetilde{\mathcal{G}}^\lambda_\nu$, i.e. the tuples $\delta = (\bar{\delta}, \sigma_2, \cdots)$ composed of a matching $\bar{\delta} \in \mathcal{G}^\lambda_{n,\nu}$ and a permutation σ_i on the $m_i(\nu)$ cycles of length $2i$ in $\mathbf{b}_\lambda \cup \bar{\delta}$ for all $i > 1$ (the cycles of length 2 are not labelled). Clearly, $|\widetilde{\mathcal{G}}^\lambda_\nu| = \frac{Aut_\nu}{m_1(\nu)!} |\mathcal{G}^\lambda_{n,\nu}|$. We call **squares** the cycles of length 4.

Example 1.1. *Figure 1 depicts a labelled matching from $\widetilde{\mathcal{G}}^{(4,2)}_{[2^3]}$ with three labelled squares: $\widehat{1}2\widehat{3}4$, $\widehat{2}3\widehat{6}5$ and $\widehat{4}1\widehat{5}6$.*

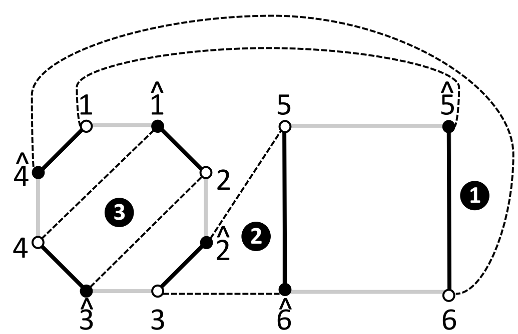


Figure 1: A labelled matching from $\widetilde{\mathcal{G}}^{(4,2)}_{[2^3]}$ with three labelled squares.

1.2.2 Labelled star hypermaps

Locally orientable hypermaps are connected bipartite graphs with black and white vertices. Each edge is composed of two half-edges both connecting the two incident vertices.

This graph is embedded in a locally orientable surface such that if we cut the graph from the surface, the remaining part consists of connected components called faces or cells, each homeomorphic to an open disk. The map may be represented (not in a unique way) as a **ribbon graph** on the plane keeping the incidence order of the edges around each vertex. In such a representation, two half-edges can be parallel or cross in the middle. We say that the hypermap is **orientable** if it is embedded in an orientable surface (sphere, torus, pretzel, ...). Otherwise the hypermap is embedded in a non orientable surface (projective plane, Klein bottle, ...) and is said to be **non-orientable**. In this paper we consider only **rooted hypermaps**, i.e. hypermaps with a distinguished half-edge. The **degree** of a face, a white vertex or a black vertex is the number of edges incident to it. For any integer n and partitions λ, μ and ν of n, denote $\mathcal{L}^{\lambda}_{\mu,\nu}$ and $l^{\lambda}_{\mu,\nu}$ (resp. $\mathcal{M}^{\lambda}_{\mu,\nu}$ and $m^{\lambda}_{\mu,\nu}$) the set and the number of locally orientable (resp. orientable) hypermaps of face degree distribution λ, white vertices degree distribution μ and black vertices degree distribution ν. The following identities are proved in [5].

$$h^{\lambda}_{\mu,\nu}(1) = m^{\lambda}_{\mu,\nu} \quad \text{and} \quad h^{\lambda}_{\mu,\nu}(2) = l^{\lambda}_{\mu,\nu}.$$

In this paper, we consider **labelled star hypermaps**, i.e. hypermaps with only one white vertex and where the $\ell(\nu)$ black vertices are labelled by integers $1, \cdots, \ell(\nu)$ such that the vertex incident to the root is labelled 1. Denote d_i the degree of the black vertex indexed i, we further assume that the edges incident to the black vertex indexed i are labelled with $\sum_{1\leqslant j<i} d_j + 1, \sum_{1\leqslant j<i} d_j + 2, \cdots, \sum_{1\leqslant j\leqslant i} d_j$ with the additional condition that the root edge is labelled 1. For $\lambda, \nu \vdash n$, denote $\widetilde{\mathcal{L}}^{\lambda}_{\nu}$ the set of labelled star hypermaps with face degree distribution λ and black vertices degree distribution ν. We focus on the special case $\nu = [k^m]$. Clearly, $|\widetilde{\mathcal{L}}^{\lambda}_{[k^m]}| = (m-1)!k!^{m-1}(k-1)!l^{\lambda}_{n,[k^m]} = \frac{m!k!^m}{n} l^{\lambda}_{n,[k^m]}$.

Example 1.2. *The two ribbon graphs depicted on Figure 2 are labelled star hypermaps with $\nu = [3^m]$ (left-hand side) and $\nu = (n)$ (right-hand side).*

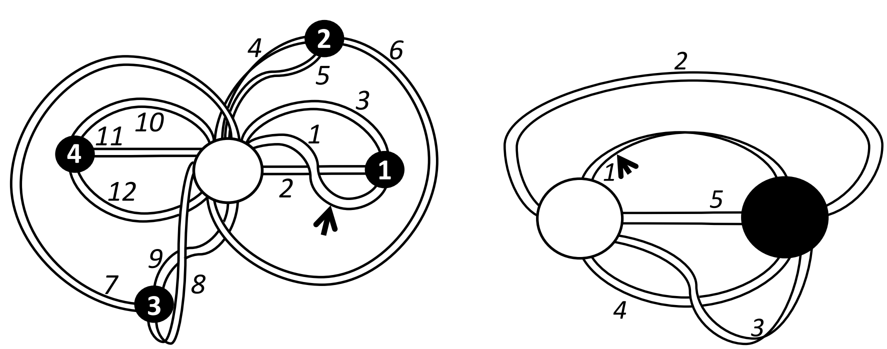


Figure 2: Examples of labelled star hypermaps.

2 Main results

We use linear operators for Jack symmetric functions to derive a new formula for the coefficients $a^{\lambda}_{n,\nu}(\alpha)$ for general λ and ν which shows their polynomial properties and, as a consequence to Equation (1.1), the polynomial properties of the coefficients $h^{\lambda}_{n,\nu}(\alpha)$. Making this formula explicit and using some bijective constructions for labelled star hypermaps and matchings, we show a variant of the matchings-Jack and the hypermap-Jack conjectures for labelled objects in some important cases.
Denote D_α, the Laplace–Beltrami operator. Namely,

$$D_\alpha = \frac{\alpha}{2}\sum_i x_i^2 \frac{\partial^2}{\partial x_i^2} + \sum_{i\neq j} \frac{x_i x_j}{x_i - x_j}\frac{\partial}{\partial x_i}$$

and let Δ and $\{\Omega_k\}_{k\geqslant 1}$ be the operators on symmetric functions defined by

$$\Delta = [D_\alpha, [D_\alpha, p_1/\alpha]], \quad \Omega_1 = [D_\alpha, p_1/\alpha], \quad \Omega_{k+1} = [\Delta, \Omega_k],$$

where $[\cdot,\cdot]$ stands for the Lie bracket. Our main result can be stated as follows

Theorem 2.1. *For any integer n and $\lambda, \nu \vdash n$, the coefficients $a^{\lambda}_{n,\nu}(\alpha)$ verify:*

$$Aut_\nu \sum_{\lambda\vdash n} z_\lambda^{-1}\alpha^{-\ell(\lambda)} a^{\lambda}_{n,\nu}(\alpha) p_\lambda = \frac{1}{\prod_{i\geqslant 1}\nu_i!}\left(\prod_{i\geqslant 2}\Omega_{\nu_i}\right)\Delta^{\nu_1 - 1}(p_1/\alpha). \tag{2.1}$$

As a consequence to Theorem 2.1, we have the following polynomial properties.

Corollary 2.2. *For $\lambda, \nu \vdash n$, $Aut_\nu |C_\lambda| a^{\lambda}_{n,\nu}(\alpha)$ and $Aut_\nu \prod_{i\geqslant 1}\nu_i! h^{\lambda}_{n,\nu}(\alpha)$ are polynomials in α with integer coefficients of respective degrees at most $n-\ell(\nu)$ and $n+1-\ell(\lambda)-\ell(\nu)$.*

Explicit computation of operators Ω_k for $k = 1,2,3$ and Δ, allows us to show:

Theorem 2.3. *For $\lambda, \nu \vdash n$, define $\widetilde{a}^{\lambda}_{n,\nu}(\alpha) = (Aut_\nu / m_1(\nu)!) a^{\lambda}_{n,\nu}(\alpha)$. If all but one part of ν are less or equal to 3, there exists a function $\mathrm{wt}\colon \widetilde{\mathcal{G}}^{\lambda}_{\nu} \to \{0,1,2,\cdots,n-\ell(\nu)\}$ such that*

$$\widetilde{a}^{\lambda}_{n,\nu}(\beta+1) = \sum_{\delta\in\widetilde{\mathcal{G}}^{\lambda}_{\nu}} \beta^{\mathrm{wt}(\delta)}$$

and $\mathrm{wt}(\delta) = 0 \iff \delta$ is bipartite.

Theorem 2.4. *For $\lambda \vdash n$ and integers k and m with $n = km$, define $\widetilde{h}^{\lambda}_{n,[k^m]}(\alpha) = \frac{m!k!^m}{n} h^{\lambda}_{n,[k^m]}(\alpha)$. For all $k \in \{1,2,3,n\}$ there exists a function $\vartheta\colon \widetilde{\mathcal{L}}^{\lambda}_{[k^m]} \to \{0,1,2,\cdots,n+1-\ell(\lambda)-m\}$ such that*

$$\widetilde{h}^{\lambda}_{n,[k^m]}(\beta+1) = \sum_{M\in\widetilde{\mathcal{L}}^{\lambda}_{[k^m]}} \beta^{\vartheta(M)}$$

and $\vartheta(M) = 0 \iff M$ is orientable.

Remark 2.5. *The focus on labelled objects and this variant of the matchings-Jack and the hypermap-Jack conjectures is motivated by the coefficients Aut_ν and $\prod_{i\geqslant 1}\nu_i!$ that appear in Equation (2.1).*

3 Prior results on the matchings-Jack and the hypermap-Jack conjectures

While the matchings-Jack and the hypermap-Jack conjectures are still open in the general case, some special cases and weakened forms have been solved over the past decade. In particular, Brown and Jackson in [1] prove that for any partition $\mu \vdash 2m$, $\sum_\lambda h^\lambda_{\mu,[2^m]}(\beta+1)$ verifies a weaker form of the hypermaps-Jack conjecture. Later on, in his PhD thesis ([7]), Lacroix defines a *measure of non-orientability* ϑ for hypermaps and focuses on a stronger form of the result of Brown and Jackson. He shows that $\sum_{\ell(\lambda)=r} h^\lambda_{\mu,[2^m]}(\beta+1) = \sum_{M\in\bigcup_{\ell(\lambda)=r}\mathcal{L}^\lambda_{\mu,[2^m]}} \beta^{\vartheta(M)}$. Finally, Dolega in [2] shows that $h^n_{\mu,\nu}(\beta+1) = \sum_{M\in\mathcal{L}^n_{\mu,\nu}} \beta^{\vartheta(M)}$ holds true when either β is restricted to the values $\beta \in \{-1,0,1\}$ or β is general but $\ell(\mu)+\ell(\nu) \geqslant n-3$.

Except the limit cases $\lambda = [1^n], [1^{n-2}2^1]$ already covered by Goulden and Jackson [5], the matchings-Jack conjecture has been proved by Kanunnikov and Vassilieva [6] in the case $\mu = \nu = (n)$. The authors introduce a weight function wt for matchings in $\mathcal{G}^\lambda_{n,n}$ such that $a^\lambda_{n,n}(\beta+1) = \sum_{\delta\in\mathcal{G}^\lambda_{n,n}} \beta^{\mathrm{wt}(\delta)}$, and $\mathrm{wt}(\delta) = 0$ if and only if δ is bipartite.

In [3] and [4] Dolega and Feray focus only on the polynomiality part of the conjectures and show that the $a^\lambda_{\mu,\nu}(\alpha)$ and $h^\lambda_{\mu,\nu}(\alpha)$ are polynomials in α with **rational** coefficients for arbitrary partitions λ, μ, ν. See also [10] for a proof of the polynomiality with non-negative integer coefficients of a multi-indexed variant of $a^\lambda_{\mu,\nu}(\alpha)$ in some important special cases.

4 Sketch of the proof of Theorem 2.1 and Corollary 2.2

4.1 Theorem 2.1

For $\lambda, \mu \vdash n$, denote $\theta^\lambda_\mu(\alpha)$ the coefficient of p_μ in the power sum expansion of J^α_λ. Namely, $J^\alpha_\lambda = \sum_{\mu\vdash n} \theta^\lambda_\mu(\alpha) p_\mu$. Besides, following [8], denote E_2 and $E_2^\perp$ the conjugate operators defined by $E_2 = [D_\alpha, p_1/\alpha] = \sum_{i\geqslant 1} i p_{i+1} \frac{\partial}{\partial p_i}$ and $E_2^\perp = [p_1^\perp/\alpha, D_\alpha] = \sum_{i\geqslant 1}(i+1) p_i \frac{\partial}{\partial p_{i+1}}$. Theorem 2.1 is a consequence of the following equality.

Theorem 4.1. *For any integer $k \geqslant 1$ denote Π_k the operator defined by $\Pi_1 = \frac{1}{\alpha} p_1^\perp = \frac{\partial}{\partial p_1}$, $\Pi_{k+1} = [\Pi_k, E_2^\perp]$. Given two indeterminates x and y and integers $k, n \geqslant 1$, the following identity holds:*

$$\sum_{\rho\vdash n+k} \frac{\theta^\rho_{n+k}(\alpha) J^\alpha_\rho(x) \Pi_k J^\alpha_\rho(y)}{j_\rho(\alpha)} = \sum_{\gamma\vdash n} \frac{\theta^\gamma_n(\alpha) J^\alpha_\gamma(y) \Omega_k J^\alpha_\gamma(x)}{j_\gamma(\alpha)}. \tag{4.1}$$

Proof. (sketch) We proceed by recurrence on k. For $k = 1$, Equation (4.1) reads

$$\sum_{\rho \vdash n+1} \frac{\theta^{\rho}_{n+1}(\alpha) J^{\alpha}_{\rho}(x) p_1^{\perp} J^{\alpha}_{\rho}(y)}{j_{\rho}(\alpha)} = \alpha \sum_{\gamma \vdash n} \frac{\theta^{\gamma}_{n}(\alpha) J^{\alpha}_{\gamma}(y) E_2 J^{\alpha}_{\gamma}(x)}{j_{\gamma}(\alpha)}. \tag{4.2}$$

Equation (4.2) may be proved using the Pieri formulas for Jack symmetric functions showed by Lassalle in [9] and formulas relating Jack symmetric functions, the Laplace–Beltrami operator and the coefficients $\theta^{\lambda}_{[1^{n-2}2^1]}(\alpha)$ showed in [6].
Assume Equation (4.1) is true for some integer $k \geqslant 1$. The formula for $k+1$ is a consequence of the recurrence hypothesis and the following relation for Jack symmetric functions showed in [6, Theorem 5]

$$\sum_{\rho \vdash n+1} \frac{\theta^{\rho}_{n+1}(\alpha) J^{\alpha}_{\rho}(x) E_2^{\perp} J^{\alpha}_{\rho}(y)}{j_{\rho}(\alpha)} = \sum_{\gamma \vdash n} \frac{\theta^{\gamma}_{n}(\alpha) J^{\alpha}_{\gamma}(y) \Delta J^{\alpha}_{\gamma}(x)}{j_{\gamma}(\alpha)}.$$

The proof of Theorem 2.1 ends by noticing that $\Pi_k = k! \frac{\partial}{\partial p_k}$. □

4.2 Corollary 2.2

The proof of Corollary 2.2 is based on two main ingredients. On the one hand, using the expansions in the power sum basis of D_α and E_2, it is easy to show that the coefficients in the power sum expansion of the operators $\{\Omega_k\}_k$ and Δ are polynomial in α with integer coefficients. As a consequence $Aut_\nu |C_\lambda| \alpha^{-\ell(\lambda)+1} a^{\lambda}_{n,\nu}(\alpha)$ is a polynomial in α with integer coefficients. On the other hand, the upper bound on the degree is showed thanks to a symmetry property on these coefficients, which is a consequence of the application of [10, Theorem 5] to get the equality $\alpha^{\ell(\lambda)} a^{\lambda}_{n,\nu}(\alpha^{-1}) = (-\alpha)^{-n+1+\ell(\lambda)+\ell(\nu)} \alpha^{-\ell(\lambda)} a^{\lambda}_{n,\nu}(\alpha)$.

5 Sketch of the proofs of Theorem 2.3 and Theorem 2.4

While Theorem 2.1 allows us to demonstrate most of the polynomial properties of $a^{\lambda}_{n,\nu}(\alpha)$ and $h^{\lambda}_{n,\nu}(\alpha)$, it is not enough to prove Theorems 2.3 and 2.4. In particular, it is not clear from the definition of operators $\{\Omega_k\}_k$ that the coefficients of $\widetilde{a}^{\lambda}_{n,\nu}(\alpha)$ and $\widetilde{h}^{\lambda}_{n,\nu}(\alpha)$ in $\beta = \alpha - 1$ are non-negative. We overcome this issue by computing a more explicit form of operator Ω_k in the cases $k = 1, 2$ and 3 to show some recurrence formulas for $\widetilde{a}^{\lambda}_{n,\nu}(\alpha)$ (for ν with at most one part strictly greater than 3) and $\widetilde{h}^{\lambda}_{n,[k^m]}(1+\beta)$. Then we use bijective constructions for labelled matchings and hypermaps to show that the right-hand sides of the main equations of theorems 2.3 and 2.4 fulfil the same recurrence relation. This method can be extended to higher values of k but computations and bijections become cumbersome and do not shed much more light on the problem. In this extended abstract we provide the proof of one example case of each theorem (2.3 and 2.4).

5.1 Explicit recurrence relations for $a^{\lambda}_{n,[2^m]}(\alpha)$ and $h^{\lambda}_{n,n}(\alpha)$

To illustrate our approach, we focus on the case $\nu = [2^m]$ in Theorem 2.3 and the case $k = n$ in Theorem 2.4. Define when applicable for $i, j \geqslant 1$ the operations on partitions:

$$\lambda_{\downarrow(i)} = \lambda \setminus \{i\} \cup \{i-1\},\ \lambda_{\downarrow(i,j)} = \lambda \setminus \{i,j\} \cup \{i+j-1\},\ \lambda^{\uparrow(i,j)} = \lambda \setminus \{i+j+1\} \cup \{i,j\},$$
$$\lambda_{\Downarrow(i)} = \lambda \setminus \{i\} \cup \{i-2\},\ \ \lambda_{\Downarrow(i,j)} = \lambda \setminus \{i,j\} \cup \{i+j-2\},\ \lambda^{\Uparrow(i,j)} = \lambda \setminus \{i+j+2\} \cup \{i,j\}.$$

We show the following recurrence relations for the coefficients $\widetilde{a}^{\lambda}_{n,[2^m]}(\alpha)$ and $\widetilde{h}^{\lambda}_{n,n}(\alpha)$. We omit the reference to variable α in the coefficients for clarity.

Lemma 5.1. *For integer partition λ of $n = 2m$, the coefficients $\widetilde{a}^{\lambda}_{n,[2^m]}$ verify*

$$\widetilde{a}^{\lambda}_{n,[2^m]} = (\alpha-1)\sum_{i:\ \lambda_i>2} \frac{\lambda_i(\lambda_i-1)}{2}\widetilde{a}^{\lambda_{\Downarrow\lambda_i}}_{n-2,[2^{m-1}]} + \frac{1}{2}\sum_i \lambda_i \sum_{d=1}^{\lambda_i-3} \widetilde{a}^{\lambda^{\Uparrow(\lambda_i-2-d,d)}}_{n-2,[2^{m-1}]} + \alpha \sum_{i<j,\lambda_i+\lambda_j>2} \lambda_i\lambda_j \widetilde{a}^{\lambda_{\Downarrow(\lambda_i,\lambda_j)}}_{n-2,[2^{m-1}]}.$$

Proof. The proof is obtained by using Theorem 2.1 in the case $\nu = [2^m]$ and computing explicitly operator Ω_2. □

For a partition ε denote $m_{i,j}(\varepsilon) = m_i(\varepsilon)(m_j(\varepsilon) - \delta_{i,j})$. We have

Lemma 5.2. *For integer partition λ of $n \geqslant 2$, the coefficients $\widetilde{h}^{\lambda}_{n,n}$ verify*

$$\widetilde{h}^{\lambda}_{n,n} = \sum_{i\in\lambda}\Bigg[(\alpha-1)(i-1)^2 m_{i-1}(\lambda_{\downarrow(i)})\widetilde{h}^{\lambda_{\downarrow(i)}}_{n-1,n-1} + \sum_{i,j\in\lambda}(i+j-1)m_{i+j-1}(\lambda_{\downarrow(i,j)})\widetilde{h}^{\lambda_{\downarrow(i,j)}}_{n-1,n-1}$$
$$+\ \alpha\sum_{i\in\lambda,d\geqslant 1}(i-1-d)d\,m_{i-1-d,d}(\lambda^{\uparrow(i-1-d,d)})\widetilde{h}^{\lambda^{\uparrow(i-1-d,d)}}_{n-1,n-1}\Bigg].$$

Proof. The proof is obtained by specialising Theorem 2.1 to the case $\nu = (n)$ extracting the coefficient $a^{\lambda}_{n,n}(\alpha)$, applying Equation (1.1) to get $h^{\lambda}_{n,n}(\alpha)$ and multiplying by $(n-2)!$. □

5.2 Bijective constructions for labelled matchings

We define recursively a weight function wt for labelled matchings. The value of wt on the empty matching is zero. For any integer $m > 1$, and $\lambda \vdash 2m$, we associate to a labelled matching $\delta \in \widetilde{\mathcal{G}}^{\lambda}_{[2^m]}$, a labelled matching $\delta' \in \widetilde{\mathcal{G}}^{\lambda'}_{[2^{m-1}]}$ for some $\lambda' \vdash 2m-2$ obtained by removing the square labelled m (denoted $\square_m$) in the labelled graph $\Gamma_{\lambda}(\delta)$ induced by $\mathbf{g}_{2m}$, $\mathbf{b}_{\lambda}$ and δ according to the following (non-bijective) **square removal procedure**. $\square_m$ is necessarily a graph on 4 vertices with labels $\{\widehat{i}, \mathbf{b}_{\lambda}(\widehat{i}), \widehat{j}, \mathbf{b}_{\lambda}(\widehat{j})\}$ for some integers $i < j$

such that $\{\delta(\widehat{i}), \delta(\mathbf{b}_\lambda(\widehat{i}))\} = \{\widehat{j}, \mathbf{b}_\lambda(\widehat{j})\}$. New (labelled) matchings $\mathbf{g}_{2m-2}$, $\mathbf{b}_{\lambda'}$ and δ' are obtained by (i) deleting these four vertices and all their incident edges, (ii) drawing a new gray edge between vertices i and $\mathbf{g}_{2m} \circ \delta(\widehat{i})$ and between $\mathbf{g}_{2m} \circ \mathbf{b}_\lambda(\widehat{i})$ and $\mathbf{g}_{2m} \circ \delta \circ \mathbf{b}_\lambda(\widehat{i})$, (iii) relabelling the remaining vertices in some canonical way (note that the squares' labels are not changed). Thanks to item (ii) $\Lambda(\mathbf{g}_{2m-2}, \delta') = (2m-2)$. Note that item (iii) may imply that some vertices with a non-hat index in $\Gamma_\lambda(\delta)$ are relabelled with a hat one in $\Gamma_{\lambda'}(\delta')$ and vice-versa. The two edges of $\square_m \cap \delta$ are either both bipartite or both non-bipartite. We say that $\square_m$ is bipartite in the former case. Define:

$$\mathrm{wt}(\delta) := \begin{cases} \mathrm{wt}(\delta') & \text{if } \square_m \text{ is bipartite,} \\ \mathrm{wt}(\delta') + 1 & \text{otherwise.} \end{cases}$$

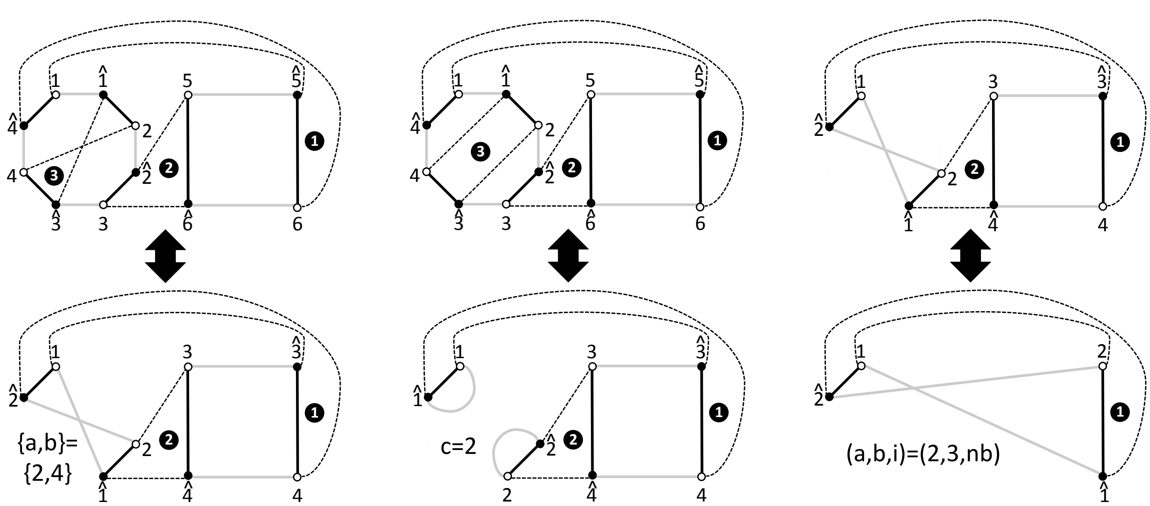
The next step is to show that $\mathfrak{S}^\lambda_{[2^m]}(\beta) = \sum_{\delta \in \widetilde{\mathcal{G}}^\lambda_{[2^m]}} \beta^{\mathrm{wt}(\delta)}$ and $\widetilde{a}^\lambda_{2m,[2^m]}(1+\beta)$ verify the same recurrence relation. First, notice that $\mathfrak{S}^2_2(\beta) = \beta = \widetilde{a}^2_{2,2}(1+\beta)$ and $\mathfrak{S}^{(1,1)}_2(\beta) = 1 + \beta = \widetilde{a}^{(1,1)}_{2,2}(1+\beta)$. Assume now $m > 1$. We use three bijective constructions for labelled matchings δ in $\widetilde{\mathcal{G}}^\lambda_{[2^m]}$ depending on the properties of $\square_m$.

- $\square_m$ is not bipartite and its black edges lie inside the i-th cycle of length $2\lambda_i$ in $\mathbf{g}_{2m} \cup \mathbf{b}_\lambda$. Using the square removal procedure, one can show that $\lambda_i > 2$, $\lambda' = \lambda_{\Downarrow\lambda_i}$ and there is a bijection between such labelled matchings in $\widetilde{\mathcal{G}}^\lambda_{[2^m]}$ and triples (δ', a, b) where $\delta' \in \widetilde{\mathcal{G}}^{\lambda_{\Downarrow\lambda_i}}_{[2^{m-1}]}$ and a and b are the non-hat labels in $\square_m$ (we have $\sum_j^{i-1} \lambda_j \leqslant a < b \leqslant \sum_j^{i} \lambda_j$). In this case $\mathrm{wt}(\delta) = 1 + \mathrm{wt}(\delta')$.
- $\square_m$ is bipartite and its black edges lie inside the i-th cycle. We have $\lambda_i > 3$, $\lambda' = \lambda^{\Uparrow(\lambda_i - d - 2, d)}$ for some d and there is a bijection between such labelled matchings in $\widetilde{\mathcal{G}}^\lambda_{[2^m]}$ and pairs (δ', c) where $\delta' \in \bigcup_d \widetilde{\mathcal{G}}^{\lambda^{\Uparrow(\lambda_i - d - 2, d)}}_{[2^{m-1}]}$ and, if $a < b$ are the non-hat labels in $\square_m$, $c = a$ if $b - a \leqslant \lambda_i/2$ and $c = b$ otherwise. In this case $\mathrm{wt}(\delta) = \mathrm{wt}(\delta')$.
- The black edges of $\square_m$ lie in two different cycles, namely the i-th and the j-th one. In this case, $\lambda_i + \lambda_j > 2$, $\lambda' = \lambda_{\Downarrow(\lambda_i, \lambda_j)}$ and there is a bijection between such labelled matchings in $\widetilde{\mathcal{G}}^\lambda_{[2^m]}$ and quadruples (δ', a, b, i) where $\delta' \in \widetilde{\mathcal{G}}^{\lambda_{\Downarrow(\lambda_i,\lambda_j)}}_{[2^{m-1}]}$ a and b are the non-hat labels in $\square_m$ and $i \in \{b, nb\}$ indicates whether the square is bipartite or not (the two cases are always possible). In this case $\mathrm{wt}(\delta) = \mathrm{wt}(\delta')$ if $i = b$ and $\mathrm{wt}(\delta) = 1 + \mathrm{wt}(\delta')$ otherwise.

Using these three bijections (see Figure 3 for an illustration), one gets:

$$\mathfrak{S}^\lambda_{[2^m]}(\beta) = \sum_{\substack{i \geqslant 1 \\ \lambda_i > 2}} \binom{\lambda_i}{2} \sum_{\delta' \in \widetilde{\mathcal{G}}^{\lambda_{\Downarrow(\lambda_i)}}_{[2^{m-1}]}} \beta^{1+\mathrm{wt}(\delta')} + \frac{1}{2} \sum_{i,d \geqslant 1} \lambda_i \sum_{\delta' \in \widetilde{\mathcal{G}}^{\lambda^{\Uparrow(\lambda_i-d-2,d)}}_{[2^{m-1}]}} \beta^{\mathrm{wt}(\delta')} + \sum_{\substack{i<j \\ \lambda_i+\lambda_j>2}} \lambda_i \lambda_j \sum_{\delta' \in \widetilde{\mathcal{G}}^{\lambda_{\Downarrow(\lambda_i,\lambda_j)}}_{[2^{m-1}]}} \left(\beta^{\mathrm{wt}(\delta')} + \beta^{1+\mathrm{wt}(\delta')}\right)$$

As a result, $\mathfrak{S}^{\lambda}_{[2^m]}(\beta) = \widetilde{a}^{\lambda}_{2m,[2^m]}(1+\beta)$ for all $m \geqslant 1$ and any $\lambda \vdash 2m$.



(a) First bijection. (b) Second bijection. (c) Third bijection.

Figure 3: Examples of application of the three bijections for labelled matchings.

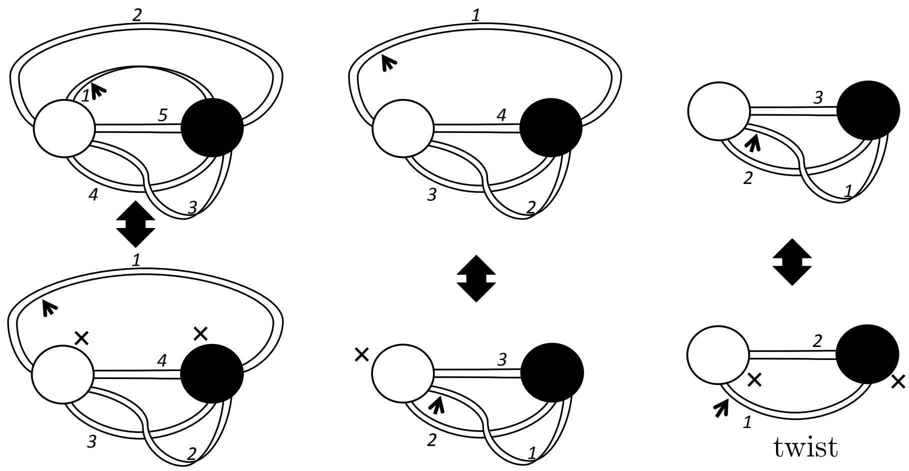
5.3 Bijective constructions for labelled star hypermaps

We adapt the definition of the measure of non-orientability introduced by Lacroix in [7, Definition 4.1] to the case of labelled star hypermaps. In what follows, we name **a leaf** an edge connecting a black vertex of degree 1 and the white vertex.

Definition 5.3 (Measure of non-orientability for labelled star hypermaps)**.** *To any labelled star hypermap M of face distribution $\lambda \neq \emptyset$, we associate a labelled star hypermap M' of face distribution λ' obtained by (i) deleting the root edge, (ii) defining the new root as the edge labelled with 2 in M and (iii) relabelling all the remaining edges by its label in M minus 1. Define recursively the function ϑ on labelled star hypermaps such that if M is the empty hypermap of face distribution $\lambda = \emptyset$, define $\vartheta(M) = 0$ otherwise M has at least one (root) edge and (i) if the root of M is a leaf then $\vartheta(M) = \vartheta(M')$ (ii) otherwise we have $|\ell(\lambda') - \ell(\lambda)| \leqslant 1$ and:*

- *if $\ell(\lambda') = \ell(\lambda)$ the root of M is a* **cross-border** *and $\vartheta(M) = 1 + \vartheta(M')$,*
- *if $\ell(\lambda') = \ell(\lambda) - 1$ the root of M is a* **border** *and $\vartheta(M) = \vartheta(M')$,*
- *if $\ell(\lambda') = \ell(\lambda) + 1$ the root of M is a* **handle**. *In this case, there is a second hypermap $\tau(M)$ obtained from M by twisting the ribbon associated with its root. The root of $\tau(M)$ is also a handle and deleting it from $\tau(M)$ also produces M'. Define $\{\vartheta(M), \vartheta(\tau(M))\} = \{\vartheta(M'), 1 + \vartheta(M')\}$. At most one of M and $\tau(M)$ is orientable, and any canonical choice such that if M is orientable, then $\vartheta(M) = 0$ and $\vartheta(\tau(M)) = 1$ is acceptable.*

Now we look at the recurrence relation verified by $\Sigma_n^\lambda(\beta) = \sum_{M \in \widetilde{\mathcal{L}}_n^\lambda} \beta^{\vartheta(M)}$. In the case $n = 1$, $\Sigma_1^\lambda(\beta) = 1 = \widetilde{h}_{1,1}^1(\beta+1)$. If $n > 1$, star hypermaps in $\widetilde{\mathcal{L}}_n^\lambda$ do not contain any leaf and we may split these maps in three sets according to the type (cross border, border or handle) of their root edge that we delete to get the following bijective constructions.



(a) First bijection. (b) Second bijection. (c) Third bijection.

Figure 4: Examples of application of the three bijections for labelled star hypermaps.

- (Cross border) The set of labelled star hypermaps with one black vertex, face distribution λ and a cross border root incident to a face of degree i is in bijection with the set of labelled star hypermaps with (i) one black vertex, (ii) face distribution $\lambda_{\downarrow(i)}$, (iii) one marked position around the white vertex incident to a face of degree $i-1$, (iv) one marked position around the black vertex incident to the same face.

- (Border) The set of labelled star hypermaps with one black vertex, face distribution λ and a border root incident to both a face of degree i and a face of degree j is in bijection with the set of labelled star hypermaps with (i) one black vertex, (ii) face distribution $\lambda_{\downarrow(i,j)}$, (iii) one marked position around the white vertex incident to a face of degree $i+j-1$ (once the position around the white vertex is chosen there is only one position around the black vertex such that connecting these two positions with a border cuts the face of degree $i+j-1$ into two faces of degree i and j).

- (Handle) The set of labelled star hypermaps with one black vertex, face distribution λ and a handle root incident to a face of degree i such that removing the root yields a face of degree d and one of degree $i-1-d$ is in bijection with the set of labelled star hypermaps with (i) one black vertex, (ii) face distribution $\lambda^{\uparrow(i-1-d,d)}$, (iii) one marked position around the white vertex incident to a face of degree $i-1-d$, (iv) one marked position around the black vertex incident to a face of degree d and (v) a type for the removed root: twist or untwist (as noted above, twisting the ribbon of a handle root yields another hypermap).

Example 5.4. *Figure 4 illustrates the three bijections described above.*

As a consequence, one gets

$$\Sigma_n^\lambda(\beta) = \sum_{i\in\lambda}(i-1)^2 m_{i-1}(\lambda_{\downarrow(i)}) \sum_{M'\in\widetilde{\mathcal{L}}_{n-1}^{\lambda_{\downarrow(i)}}} \beta^{1+\vartheta(M')} + \sum_{i,j\in\lambda}(i+j-1)m_{i+j-1}(\lambda_{\downarrow(i,j)}) \sum_{M'\in\widetilde{\mathcal{L}}_{n-1}^{\lambda_{\downarrow(i,j)}}} \beta^{\vartheta(M')}$$
$$+ \sum_{i\in\lambda, d\geqslant 1}(i-d-1)dm_{i-d-1,d}(\lambda^{\uparrow(i-1-d,d)}) \sum_{M'\in\widetilde{\mathcal{L}}_{n-1}^{\lambda^{\uparrow(i-1-d,d)}}} \left(\beta^{\vartheta(M')} + \beta^{1+\vartheta(M')}\right)$$

As a conclusion, for any integer $n \geqslant 1$, $\Sigma_n^\lambda(\beta) = \widetilde{h}_{n,n}^\lambda(\beta+1)$.

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