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Enumerative Combinatorics of Prographs

Christophe Cordero*[1]

[1]*Université Paris-Est, LIGM (UMR 8049), CNRS, ENPC, ESIEE Paris, UPEM, F-77454, Marne-la-Vallée, France*

Abstract. We exhibit a bijection between reduced prographs and families of three-dimensional colored lattice paths. By a classical study of the lattice paths, we obtain recurrence relations and closed formulas for reduced prographs.

Résumé. Nous exhibons une bijection entre les prographes réduits et des familles de chemins tridimensionnels colorés. Par une étude classique des chemins, nous obtenons des relations de récurrence et des formules closes pour les prographes réduits.

Keywords: Enumerative combinatorics, lattice path, prograph

Introduction

Prographs are planar assemblies of abstract operators with multiple inputs and outputs. They can model electronic circuits, algebraic computations, or rewriting systems [9]. We also proved thanks to computer experiments and the OEIS [7] that prographs made of only one sort of operator with two inputs and three outputs can model the biological notion of tandem duplication trees [5].

We wish to count, according to their number of operators, all prographs having a fixed number of inputs and outputs and using only operators belonging to a given set. The main difficulty in enumerating prographs arises from their definition. Indeed, prographs are defined as objects obtained from base operators and two building operations [3, 10]. However, there exist relations between these operations, implying that the natural grammar for generating the prographs is ambiguous.

In order to circumvent this ambiguity, we present a canonical expression of any reduced prograph in the grammar, where reduced prographs are prographs made of operators with at least one input and one output. These canonical expressions are inspired from a reduced prograph traversal introduced by Borie in [1]. By a direct encoding of canonical expressions into paths, we obtain a bijection between reduced prographs and families of three-dimensional colored paths. The classical combinatorial study of these paths yields recurrence relations and closed formulas for reduced prographs.

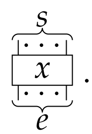
In Section 1, we recall basic definitions about prographs and set our combinatorial problem. Section 2 is devoted to the bijection between reduced prographs and families of

*Christophe.Cordero@u-pem.fr

three-dimensional colored paths. Finally, in Section 3, we compute recurrence relations and closed formulas for the paths and thus for reduced prographs.

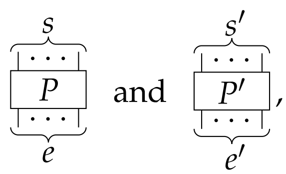
1 Prographs

A generator is an operator with a fixed number of inputs and outputs. We represent a generator x with e inputs and s outputs by

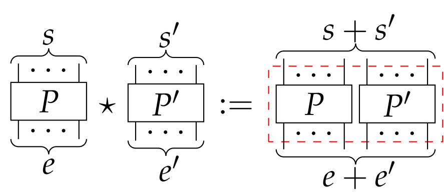


Remark 1.1. *In the cases we consider, generators always have at least one input and one output.*

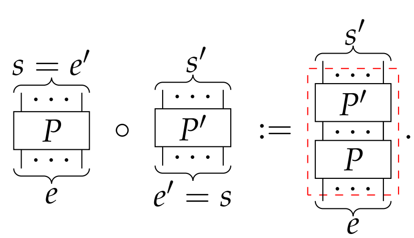
Let us see how we build prographs from a set of generators. First of all, the wire $|$ is a prograph with one input and one output and a generator with e inputs and s outputs is a prograph with e inputs and s outputs. Then, given two prographs



we have two binary operations, denoted by $\star$ and $\circ$, to build another prograph. The operation $\star$ performs a concatenation as follows:



hence building a third prograph with $e + e'$ inputs and $s + s'$ outputs. The operation $\circ$ is a partial operation, $P \circ P'$ is defined as follows if and only if $s = e'$:

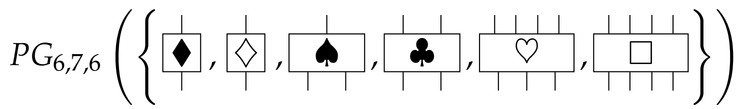


Thus we get a prograph with e inputs and s' outputs, where the first output of P is connected to the first input of P' and so on.

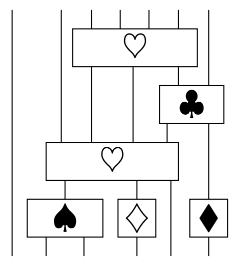
Remark 1.2. *We restrict ourselves to prographs made of generators with at least one input and one output. We might call them reduced prographs but we simply call them prographs.*

Definition 1.3. *For a set of generators $\mathbb{G}$ and a triple $(e, s, n) \in \mathbb{N}^3$, we write $PG_{e,s,n}(\mathbb{G})$ the set of prographs with e inputs, s outputs and using exactly n generators from $\mathbb{G}$.*

Example 1.4. *The set*

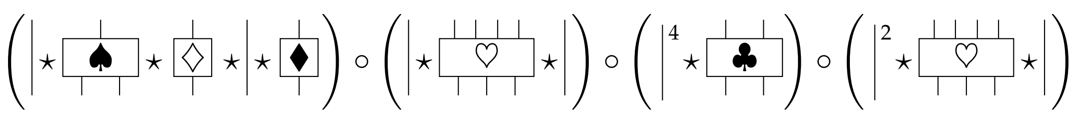


contains the prograph



(1.1)

that can be built by the expression



where for any $n \geq 0$, $|^n := \underbrace{|\star\cdots\star|}_{n}$.

Problem. *Given a set of generators $\mathbb{G}$ and a triple $(e, s, n) \in \mathbb{N}^3$, our goal is to count the prographs of $PG_{e,s,n}(\mathbb{G})$.*

The main difficulty in counting prographs is that the grammar provided by their definition is ambiguous. Indeed, the operations $\star$ and $\circ$ are associative and are related by two relations, called unitary and rectangle relations.

The unitary relation states that for a prograph P with e inputs and s outputs,

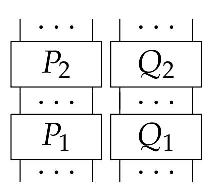
$$P \circ |^s = P = |^e \circ P.$$

This formalizes the fact that connecting a wire does not change anything.

Rectangle relation states that for prographs P_1, P_2, Q_1 and Q_2 be such that $P_1 \circ P_2$ and $Q_1 \circ Q_2$ are well-defined,

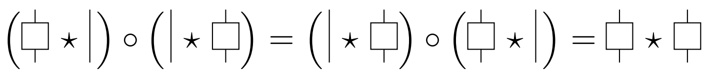
$$(P_1 \circ P_2) \star (Q_1 \circ Q_2) = (P_1 \star Q_1) \circ (P_2 \star Q_2)\,.$$

This exhibits two ways to construct the prograph



from the prographs P_1, P_2, Q_1 and Q_2.

Example 1.5. *Here is a small example of ambiguity:*



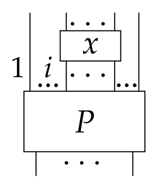
To overcome this difficulty, we propose a canonical expression of any prograph in their grammar and use it to build a bijection.

2 Bijection between prographs and some lattice paths

2.1 Canonical expressions

For a prograph P, we denote by $\downarrow(P)$ its number of inputs and by $\uparrow(P)$ its number of outputs. Let a prograph P, a generator x and $i \in \mathbb{N}_{\geq 1}$ such that $i + \downarrow(x) - 1 \leq \uparrow(P)$ then we define a new operation $\circ_i$ as follows:

$$P \circ_i x :=$$



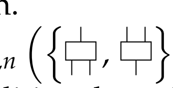
. (2.1)

A similar operation is used in [9].

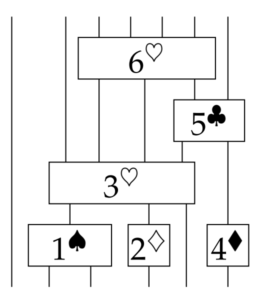
Theorem 2.1. *For $e, s, n \in \mathbb{N}$ and $P \in PG_{e,s,n}(\mathbb{G})$, there exists a unique pair of sequences $((x_1, \ldots, x_n), (i_1, \ldots, i_n)) \in \mathbb{G}^n \times \mathbb{N}^n_{\geq 1}$ such that*

$$P = \left(\cdots \left(\left(|^e \circ_{i_1} x_1 \right) \circ_{i_2} x_2 \right) \cdots \right) \circ_{i_n} x_n. \tag{2.2}$$

and for all $k \in [n-1]$, $i_k < i_{k+1} + \downarrow(x_{k+1})$. Such a decomposition is called canonical.

Sketch of proof. We just show how to compute the canonical decomposition.
In [1], Borie introduces a prograph traversal for the prographs of $PG_{1,1,n}$ , where $n \in \mathbb{N}$. It is a depth-left first numbering of generators with the additional condition that a generator can be numbered only if all the generators connected to its inputs are already numbered.

His traversal naturally runs on any prograph. For example, the generators of the prograph (1.1) are numbered like this



. (2.3)

A generator appears in the canonical decomposition at the position given by the numbering. Then the index of the operation (2.1) corresponding to the generator is uniquely determined as the number of wires on its left plus one.

In our example (1.1/2.3), the prograph (1.1) is canonically decomposed as

$$\left(\cdots\left(\left(|^6 \circ_2 \spadesuit\right) \circ_3 \diamondsuit\right) \circ_2 \heartsuit\right) \circ_6 \blacklozenge\right) \circ_5 \clubsuit\right) \circ_3 \heartsuit.$$

$\square$

Remark 2.2. *We deduce from Theorem 2.1 a simple algorithm to generate prographs of $PG_{e,s,n}(\mathbb{G})$: we just have to generate their canonical decompositions.*

The canonical decompositions are mostly sequences of inequalities and this type of constraints can be encoded by some lattice paths.

2.2 Lattice paths

Definition 2.3. *A path is a pair $(p_\mathrm{i}, w_1 \cdots w_n) \in \mathbb{Z}^3 \times \left(\mathbb{Z}^3 \times \mathbb{N}\right)^n$, where p_i is the origin of the path and $w_1 \cdots w_n$ is the sequence of three-dimensional colored steps.*

For a step $v \in \mathbb{Z}^3 \times \mathbb{N}$, we denote by γ_1, γ_2, γ_3 and γ_c respectively the abscissa, the ordinate, the applicate and the color of the step v. Formally, $v = (\gamma_1(v), \gamma_2(v), \gamma_3(v), \gamma_c(v))$. We also write for any $v \in \mathbb{Z}^3, v = (\gamma_1(v), \gamma_2(v), \gamma_3(v))$. For any set X, we denote by X^* the set $\bigsqcup_{k \geq 0} X^k$ where $\sqcup$ denotes the disjoint union.

Definition 2.4. *For $M \subset \left(\mathbb{Z}^3 \times \mathbb{N}\right)^*$ and $p_\mathrm{i}, p_\mathrm{f} \in \mathbb{Z}^3$, we denote by $\mathcal{L}(M, p_\mathrm{i}, p_\mathrm{f})$ the set of paths $(p_\mathrm{i}, w_1 \cdots w_n)$ such that*

1. *$w_1 \cdots w_n \in M$;*
2. *$\gamma_j(p_\mathrm{i}) + \sum_{\ell=1}^{n} \gamma_j(w_\ell) = \gamma_j(p_\mathrm{f})$ for any $j = 1, 2, 3$;*
3. *for $k \in [0, n], 1 \leq \gamma_2(p_\mathrm{i}) + \sum_{\ell=1}^{k} \gamma_2(w_\ell) \leq \gamma_3(p_\mathrm{i}) + \sum_{\ell=1}^{k} \gamma_3(w_\ell)$.*

In other words, $\mathcal{L}(M, p_\mathrm{i}, p_\mathrm{f})$ is the set of paths from p_i to p_f whose sequence of steps belongs to M and such that in any point of the paths the ordinate is between 1 and the applicate.

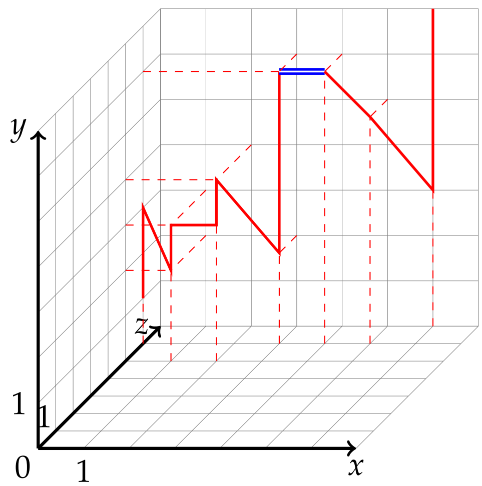
Example 2.5. *We draw the color 0 with red (simple line) and the color 1 with blue (double lines). The set*

$$\mathcal{L}\left(\left\{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\0\end{bmatrix}\right\}^*, \begin{bmatrix}0\\1\\6\end{bmatrix}, \begin{bmatrix}6\\7\\7\end{bmatrix}\right)$$

contains the path

$$\left(\begin{bmatrix}0\\1\\6\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}^2 \begin{bmatrix}1\\-1\\-1\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \begin{bmatrix}1\\-2\\1\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\\0\end{bmatrix}^4 \begin{bmatrix}1\\0\\0\\1\end{bmatrix} \begin{bmatrix}1\\-1\\0\\0\end{bmatrix} \begin{bmatrix}1\\-2\\1\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\\0\end{bmatrix}^4 \right) \tag{2.4}$$

drawn as



. (2.5)

The canonical decompositions are encoded by particular lattice paths, called propaths.

2.3 Propaths

We denote by $\mathbb{G}$ the set of generators

$$\left\{ \underbrace{\overbrace{\begin{array}{|c|}\hline \cdots \\ \hline 1 \\ \hline \cdots \\ \hline\end{array}}^{\beta_1}}_{\alpha_1}, \dots, \underbrace{\overbrace{\begin{array}{|c|}\hline \cdots \\ \hline m_1 \\ \hline \cdots \\ \hline\end{array}}^{\beta_1}}_{\alpha_1}, \dots, \underbrace{\overbrace{\begin{array}{|c|}\hline \cdots \\ \hline 1 \\ \hline \cdots \\ \hline\end{array}}^{\beta_d}}_{\alpha_d}, \dots, \underbrace{\overbrace{\begin{array}{|c|}\hline \cdots \\ \hline m_d \\ \hline \cdots \\ \hline\end{array}}^{\beta_d}}_{\alpha_d} \right\}.$$

where $\alpha_1, \dots, \alpha_d, \beta_1, \dots, \beta_d, m_1, \dots, m_d \in \mathbb{N}_{\geq 1}$ and we define ω by

$$\begin{array}{rccc} \omega : & \mathbb{G} & \longrightarrow & \mathbb{Z}^3 \times \mathbb{N} \\ & \underbrace{\overbrace{\begin{array}{|c|}\hline \cdots \\ \hline j \\ \hline \cdots \\ \hline\end{array}}^{\beta_i}}_{\alpha_i} & \mapsto & (1, 1-\alpha_i, \beta_i - \alpha_i, j) \end{array}$$

We denote by U the step $(0,1,0,0)$, by D the step $(0,-1,0,0)$ and by $\mathcal{W}$ the set of steps $\{\omega(x) \mid x \in \mathbb{G}\}$.

Definition 2.6. *We call $PP_{e,n,k,s}(\mathbb{G})$ the set $\mathcal{L}((U+\mathcal{W})^*, (0,1,e), (n,k,s))$. Its elements are called propaths.*

Propaths are refinements of prographs.

Theorem 2.7. *For $(e, s, n) \in \mathbb{N}^3$, we have $|PP_{e,n,s,s}(\mathbb{G})| = |PG_{e,s,n}(\mathbb{G})|$.*

Sketch of proof. Let $P \in PG_{e,s,n}(\mathbb{G})$, according to Theorem 2.1 P as a unique canonical decomposition of the form:

$$P = \left(\cdots\left(\left(|^e \circ_{i_1} x_1\right) \circ_{i_2} x_2\right)\cdots\right) \circ_{i_n} x_n.$$

Then our bijection maps this decomposition to the propath

$$\left((0,1,e),\, U^{i_1+\downarrow(x_1)-1-i_0}\omega(x_1)\cdots U^{i_n+\downarrow(x_n)-1-i_{n-1}}\omega(x_n)\, U^{s-i_n}\right) \in PP_{e,n,s,s}(\mathbb{G})$$

where $i_0 = 1$. Intuitively, the propath abscissa encodes the number of generators, its ordinate encodes the indexes of the operations (2.1) in the decomposition, its applicate encodes the number of outputs and its colors allow to distinguish generators with the same number of inputs and outputs. □

For example, our bijection maps the (decomposition of the) prograph (1.1) to the propath

$$\left((0,1,6),\, U^2\,\omega(\spadesuit)\,U\,\omega(\diamondsuit)\,U\,\omega(\heartsuit)\,U^4\,\omega(\blacklozenge)\,\omega(\clubsuit)\,\omega(\heartsuit)\,U^4\right)$$

which is equal to the path (2.4).

Lattice paths are classical objects in combinatorics (see [8] for a summary). Let us now study propaths in order to obtain, thanks to Theorem 2.7, recurrences and closed formulas for prographs.

3 Recurrences and closed formulas

3.1 Recurrences formulas

In order to simplify notation, we denote by $p(n,k,s)$ the number $|PP_{e,n,k,s}(\mathbb{G})|$, where $e \in \mathbb{N}_{\geq 1}$ is fixed.

Proposition 3.1. *The number of propaths satisfies the following recurrence relation:*

$$p(n,k,s) = \begin{cases} 1 \ \text{if } n = 0,\ k = 1 \text{ and } s = e; \\ p(n,k-1,s) + \sum_{i=1}^{d} m_i\, p(n-1, k-1+\alpha_i, s-\beta_i+\alpha_i) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } n \geq 0 \text{ and } 1 \leq k \leq s; \\ 0 \ \text{otherwise.} \end{cases}$$

Proof. The recurrence relation is a direct translation of the following unambiguous grammar that generates the propaths: a propath belonging to $PP_{e,n,k,s}(\mathbb{G})$ is

- either the path reduced to the point $(0, 1, e)$,

- or a path belonging to $PP_{e,n,k-1,s}(\mathbb{G})$ concatenated to a step U,
- or a path belonging to $PP_{e,n-1,k-1+\alpha_i,s-\beta_i+\alpha_i}(\mathbb{G})$ concatenated to a step $(1, 1-\alpha_i, \beta_i-\alpha_i, c)$ where c is one of the m_i colors.

□

According to Theorem 2.7, it is enough to specialize k to s in order to obtain a recurrence relation satisfied by prographs. The following theorem gets rid of the refinement parameter k, so it provides a recurrence relation directly on the prographs.

Theorem 3.2. *Let $a_{n,s} := |PP_{e,n,s,s}(\mathbb{G})| = |PG_{e,s,n}(\mathbb{G})|$. It satisfies the recurrence relation:*

$$a_{n,s} = \begin{cases} 1 \text{ if } n = 0 \text{ and } s = e; \\ \sum_{\ell=1}^{n} (-1)^{\ell+1} \sum_{c_1+\cdots+c_d=\ell} \binom{\ell}{c_1,\ldots,c_d} \binom{s+\ell-\sum_{i=1}^{d} c_i\beta_i}{\ell} m_1^{c_1}\ldots m_d^{c_d} a_{n-\ell,\, s-\sum_{i=1}^{d} c_i(\beta_i-\alpha_i)} & \text{if } n,s \geq 1; \\ 0 \text{ otherwise.} \end{cases}$$

Proof. Let $n, s \geq 1$. For $\ell \in [0; n]$, we denote by A_ℓ the set

$$\mathcal{L}\left((U^*\mathcal{W})^{n-\ell}(DD^*\mathcal{W})^{\ell}U^*, (0,1,e), (n,s,s)\right).$$

We have $|A_n| = 0$, $a_{n,s} = |A_0|$ and $|A_\ell| = |A_\ell \sqcup A_{\ell+1}| - |A_{\ell+1}|$ for all $0 \leq \ell \leq n-1$. Thus by iteration, we obtain

$$a_{n,s} = \sum_{\ell=1}^{n} (-1)^{\ell+1} |A_{\ell-1} \sqcup A_\ell|.$$

Moreover, for $\ell \in [n]$, $A_{\ell-1} \sqcup A_\ell$ is equal to

$$\mathcal{L}\left((U^*\mathcal{W})^{n-\ell}(U^* + DD^*)\mathcal{W}(DD^*\mathcal{W})^{\ell-1}U^*, (0,1,e), (n,s,s)\right).$$

So $|A_{\ell-1} \sqcup A_\ell|$ decomposes into

$$\sum_{j\geq 0} a_{n-\ell,j} \left|\mathcal{L}\left(D^*\mathcal{W}(DD^*\mathcal{W})^{\ell-1}D^*, (n-\ell,j,j), (n,1,s)\right)\right|.$$

Let us compute

$$\left|\mathcal{L}\left(D^*\mathcal{W}(DD^*\mathcal{W})^{\ell-1}D^*, (n-\ell,j,j), (n,1,s)\right)\right|. \tag{3.1}$$

To build the paths belonging to (3.1), we choose at first ℓ steps in $\mathcal{W}$. For a composition $c_1 + \cdots + c_d = \ell$ we take c_i steps $(1, 1-\alpha_i, \beta_i-\alpha_i, c)$ where c is one of the m_i colors. So there are

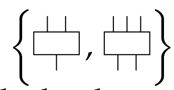
$$\binom{\ell}{c_1,\ldots,c_d} m_1^{c_1}\ldots m_d^{c_d}$$

ways to choose the steps from $\mathcal{W}$. We now have to add the steps D. Let us denote by b the number of steps D that we have to add to obtain a path of (3.1). Necessarily,

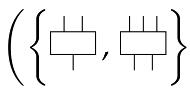
$j = s - \sum_{i=1}^{d} c_i(\beta_i - \alpha_i)$ and $b = \ell - 1 + s - \sum_{i=1}^{d} c_i\beta_i$. The $\ell - 1$ first steps from $\mathcal{W}$ in (3.1) are followed by a step D, so we just have to place $s - \sum_{i=1}^{d} c_i\beta_i$ steps D in $\ell + 1$ zones; so there are $\binom{s+\ell-\sum_{i=1}^{d} c_i\beta_i}{\ell}$ possibilities. Thus $|A_{\ell-1} \sqcup A_\ell|$ is equal to

$$\sum_{c_1+\cdots+c_d=\ell} \binom{\ell}{c_1,\ldots,c_d} \binom{s+\ell-\sum_{i=1}^{d} c_i\beta_i}{\ell} m_1^{c_1} \ldots m_d^{c_d}\, a_{n-\ell,s-\sum_{i=1}^{d} c_i(\beta_i-\alpha_i)},$$

hence the formula. $\square$

Application. *The number of outputs of the prographs made of generators*  *are determined by their number of inputs and their number of generators. Indeed, when we add a generator, we increase the number of outputs by one. Thus, the enumeration of these prographs with a single input amounts to studying the number*

$$a_n := \left| PG_{1,n+1,n} \left(\{\ldots\} \right) \right|,$$



where $n \geq 0$.

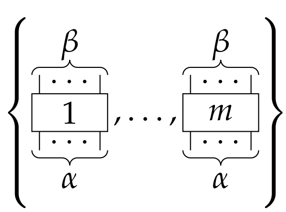
The sequence $(a_n)_{n\geq 0}$ *is equal to Heinz's sequence, entry A224776 of [7]. Indeed, they satisfy the same recurrence relation and the same initial case (see Proposition 3.1 and A224776 of [7]). According to Theorem 3.2, we can add that Heinz's sequences satisfies*

$$a_n = \begin{cases} 1 & \text{if } n = 0; \\ \sum_{k=1}^{n} (-1)^{k+1} q(n,k) a_{n-k} & \text{if } n \geq 1; \\ 0 & \text{otherwise.} \end{cases}$$

where $q(n,k) = \sum_{\ell=1}^{k} \binom{n+1+\ell-2k}{k}\binom{k}{\ell}$. *The sequence* $(q(n+2k-1,k))_{k\geq 0}$ *is known as the crystal ball sequence for the n-dimensional cubic lattice [4].*

3.2 Closed formulas

In this section, we restrict ourselves to the case $d = 1$. Which means that the set of generators $\mathbb{G}$ is

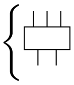


In this case the number of outputs of a prograph is determined by the other parameters. Indeed, the prographs with e inputs and n generators from the set $\mathbb{G}$ necessarily have $e + (\beta - \alpha)n$ outputs, because a generator from $\mathbb{G}$ removes α outputs and creates β outputs. So in this section we study the terms

$$a_n := \left|PP_{e,n,e+(\beta-\alpha)n,e+(\beta-\alpha)n}(\mathbb{G})\right| = \left|PG_{e,e+(\beta-\alpha)n,n}(\mathbb{G})\right|.$$

Example 3.3. *We discovered thanks to the OEIS [7] that for $m = 1$, $\alpha = 2$, $\beta = 3$ and $e = 2$ the prographs are in bijection with tandem duplication trees [5]. Precisely, for all $n \geq 0$,*

$$\left|PG_{2,2+n,n}\left(\left\{\quad\right\}\right)\right|$$



is equal to the number of rooted tandem duplication trees on $n+1$ gene segments. Indeed, they satisfy the same recurrence relation and the same initial case (see Proposition 3.1 and [6]).

Theorem 3.4. $a_n = m^n det\left(\binom{e-i(\alpha-1)+(j-1)(\beta-1)}{i-j+1}_{(i,j)\in[n]^2}\right)$.

Proof. In the particular case $d = 1$, Theorem 3.2 can be reformulated in terms of matrices as follows: $A[a_0,\ldots,a_n]^t = [1,0,\ldots,0]^t$, where $A_{i,j} = (-1)^{i-j}m\binom{e+(i+1)(\alpha-1)+j(\beta-1)}{i-j}$. According to Cramer's rule, $a_n = \frac{A_{n+1}}{det(A)}$, where A_{n+1} is the matrix A whose last column is replaced by the vector $[1,0,\ldots,0]$. Knowing that $det(A) = 1$, by developing according to the last column we get $a_n = (-1)^n B$, where B is the determinant of $\left((-1)^{i-j}m\binom{e-i(\alpha-1)+(j-1)(\beta-1)}{i-j+1}_{(i,j)\in[n]^2}\right)$. Multiplying all odd index lines and all even index columns of the determinant B by -1 gives the formula. $\square$

Theorem 3.5. $a_n = m^n \sum\limits_{k\geq 1}(-1)^{n+k} \sum\limits_{\substack{t_1+\cdots+t_k=n \\ t_i\neq 0}} \prod\limits_{j=1}^{k}\binom{(\beta-\alpha)\left(t_1+\cdots+t_{j-1}\right)+e-(\alpha-1)t_j}{t_j}$.

Proof. For $i \in [n-1]$ we denote by A_i the set

$$\mathcal{L}\left(\left(\left(D^*+U^*\right)\mathcal{W}\right)^i DD^*\mathcal{W}\left(\left(D^*+U^*\right)\mathcal{W}\right)^{n-i-1}U^*,(0,1,e),(n,s,s)\right)$$

and by E the set

$$\mathcal{L}\left(\left(\left(D^*+U^*\right)\mathcal{W}\right)^n U^*,(0,1,e),(n,s,s)\right).$$

We have $a_n = \left|\bigcap\limits_{i=1}^{n-1}A_i\right| = |E| - \left|\bigcup\limits_{i=1}^{n-1}A_i\right|$ and according to the inclusion–exclusion principle

$$\left|\bigcup_{k=1}^{n-1}A_k\right| = \sum_{k\geq 1}^{n-1}(-1)^{k-1}\Omega_k,$$

where Ω_k is the cardinality of the k-tuple-wise intersection of sets $A_1,\ldots,A_{n-1}$.

If for $k \in [n-1]$ we denote by $A_{t_1,\ldots,t_k}$ the set

$$\mathcal{L}\left(U^*\left[\mathcal{W}, DD^*\right]^{t_1}\left(DD^* + U^*\right)\left[\mathcal{W}, DD^*\right]^{t_2}\cdots\left[\mathcal{W}, DD^*\right]^{t_k} U^*, (0,1,e), (n,s,s)\right)$$

where for $v \in \mathbb{N}_{\geq 1}$, $[\mathcal{W}, DD^*]^v$ is $(\mathcal{W}DD^*)^{v-1}\mathcal{W}$, then Ω_k can be rewritten as

$$\sum_{\substack{t_1+\cdots+t_{n-k}=n \\ t_i \neq 0}} |A_{t_1,\ldots,t_{n-k}}|$$

and E as $A_{\underbrace{1,\ldots,1}_{n}}$. Moreover, by the change of variable $k \leftarrow n-k$, we obtain

$$a_n = \sum_{k=1}^{n} (-1)^{n+k} \sum_{\substack{t_1+\cdots+t_k=n \\ t_i \neq 0}} |A_{t_1,\ldots,t_k}|.$$

But $|A_{t_1,\ldots,t_k}|$ is equal to

$$\prod_{j=1}^{k} \left|\mathcal{L}\left(D^*[W, DD^*]^{t_j} D^*, p(j), p(j+1)\right)\right|$$

where $p(j) = \left(\sum\limits_{\ell=1}^{j-1} t_\ell, e + \sum\limits_{\ell=1}^{j-1} t_\ell(\beta-\alpha), e + \sum\limits_{\ell=1}^{j-1} t_\ell(\beta-\alpha)\right)$ so with a computation similar to (3.1) we obtain that $|A_{t_1,\ldots,t_k}|$ equals

$$\prod_{j=1}^{k} \binom{e + (\beta-\alpha)\left(t_1+\cdots+t_{j-1}\right) - (\alpha-1)t_j}{t_j}.$$

□

Conclusion and perspectives

We believe that prographs can encode or model structures that are not captured by the classical combinatorics on trees. To measure the expressive power of prographs, we propose to study their generating series. Formally, given a set of generators $\mathbb{G}$, we wonder in which class (rational, algebraic, D-finite, D-algebraic,... [11]) the series

$$\varphi(\mathbb{G}) := \sum_{e,n,s\geq 0} |PG_{e,s,n}(\mathbb{G})|\, x^n y^e z^s = \sum_{e,n,s\geq 0} |PP_{e,n,s,s}(\mathbb{G})|\, x^n y^e z^s \tag{3.2}$$

belongs to. We noticed that the families of paths in bijection with prographs are part of the paths studied by [2] and we computed a functional equation satisfied by (3.2). A study of the singularities of (3.2) might conclude that it is not D-finite in most cases.

Another generalization that we have not explored is the extension of the combinatorial study to prographs made of generators without inputs or outputs.

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