*Séminaire Lotharingien de Combinatoire* **80B** (2018)
Article #61, 12 pp.

*Proceedings of the 30th Conference on Formal Power Series and Algebraic Combinatorics (Hanover)*

# Algebraic structures on integer posets

Vincent Pilaud[*1] and Viviane Pons[†2]

[1] *CNRS & LIX, École Polytechnique, Palaiseau*
[2] *LRI, Univ. Paris-Sud - CNRS - Centrale Supelec - Univ. Paris-Saclay, Orsay*

**Abstract.** We define a lattice structure and a Hopf algebra on integer posets and use them to recover relevant structures on the elements, the intervals and the faces in the permutahedron, the associahedron, the cube and more generally all permutreehedra.

**Résumé.** Nous définissons une structure de treillis et une algèbre de Hopf sur les ordre partiels sur les entiers, et nous les utilisons pour retrouver des structures importantes sur les éléments, les intervalles et les faces du permutaèdre, de l'associaèdre, du cube et en général de tous les permutarbroèdres.

Our original motivation is the fascinating interplay between the permutations, the binary trees, and the binary sequences. These combinatorial objects are deeply related, in particular through their lattice structures (weak order, Tamari lattice, and boolean lattice), their Hopf algebras (Malvenuto–Reutenauer algebra [11], Loday–Ronco algebra [10], and descent algebra [6]), and their polytopes (permutahedron, associahedron, and cube). These connections have been largely explained in [13], where permutations, binary trees and binary sequences are all seen as special instances of permutrees. Moreover, permutrees allow to interpolate between these families, leading to combinatorial objects that are structurally "half" permutations and "half" binary trees.

This abstract reports on further developments in this direction, with the objective to provide a unified understanding not only for the elements, but also for the intervals and the faces of the permutahedron, the associahedron and the cube. As explained in Section 2, the *integer posets* (*i.e.* posets on $[n]$) provide a convenient model for all these combinatorial objects. This abstract presents two algebraic structures on integer posets:

- In Section 1, we define the *weak order* on integer posets and show that it is a lattice. It contains as subposets (and even sometimes sublattices) the classical weak order, the Tamari lattice, their interval lattices and their facial lattices [9, 12, 5].
- In Section 3, we define a Hopf algebra on integer posets. The vertices, intervals and faces of the permutahedra index certain quotients of this algebra, and the vertices, intervals and faces of the associahedra index subalgebras of these quotients.

This abstract relies on three papers connected to this topic: one on permutrees [13], one on the weak order on posets [4], and one on Hopf algebras on integer posets [14]. More details and proofs can be found in the long versions.

*vincent.pilaud@lix.polytechnique.fr. Supported by the French ANR grant SC3A (15 CE40 0004 01).
†viviane.pons@lri.fr

# 1 Weak order on integer posets

## 1.1 Integer binary relations

An *integer binary relation* of size $n$ is a binary relation on $[n] := \{1, \dots, n\}$, that is, a subset $\mathrm{R}$ of $[n]^2$. As usual, we write equivalently $(a, b) \in \mathrm{R}$ or $a \,\mathrm{R}\, b$. We only consider reflexive relations and let $\mathcal{R}_n$ be the set of all reflexive binary relations on $[n]$.

The *increasing* and *decreasing subrelations* of an integer relation $\mathrm{R} \in \mathcal{R}_n$ are the relations defined by $\mathrm{R}^{\mathsf{Inc}} := \{(a, b) \in \mathrm{R} \mid a \le b\}$ and $\mathrm{R}^{\mathsf{Dec}} := \{(b, a) \in \mathrm{R} \mid a \le b\}$. In our pictures, we always represent an integer relation $\mathrm{R} \in \mathcal{R}_n$ as follows: we write the numbers $1, \dots, n$ from left to right and we draw the increasing relations of $\mathrm{R}$ above in blue and the decreasing relations of $\mathrm{R}$ below in red. Although we only consider reflexive relations, we always omit the relations $(i, i)$ in the pictures. See *e.g.* Figure 1.

We consider the *weak order* on $\mathcal{R}_n$, defined by $\mathrm{R} \preccurlyeq \mathrm{S}$ if $\mathrm{R}^{\mathsf{Inc}} \supseteq \mathrm{S}^{\mathsf{Inc}}$ and $\mathrm{R}^{\mathsf{Dec}} \subseteq \mathrm{S}^{\mathsf{Dec}}$. It is clearly a graded lattice with meet $\mathrm{R} \wedge_{\mathcal{R}} \mathrm{S} = (\mathrm{R}^{\mathsf{Inc}} \cup \mathrm{S}^{\mathsf{Inc}}) \cup (\mathrm{R}^{\mathsf{Dec}} \cap \mathrm{S}^{\mathsf{Dec}})$ and join $\mathrm{R} \vee_{\mathcal{R}} \mathrm{S} = (\mathrm{R}^{\mathsf{Inc}} \cap \mathrm{S}^{\mathsf{Inc}}) \cup (\mathrm{R}^{\mathsf{Dec}} \cup \mathrm{S}^{\mathsf{Dec}})$. The name chosen for this order is explained in Section 2, where we explore its connection to various relevant combinatorial lattices.

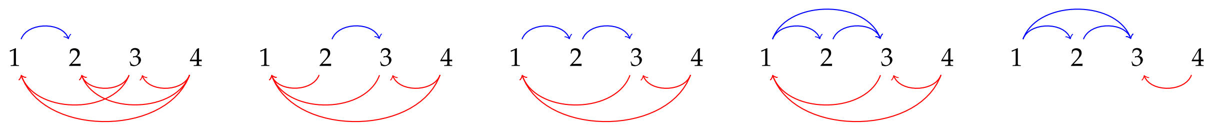
## 1.2 Integer posets

An *integer poset* is a reflexive ($a \,\mathrm{R}\, a$), antisymmetric ($a \,\mathrm{R}\, b \;\wedge\; b \,\mathrm{R}\, a \implies a = b$) and transitive ($a \,\mathrm{R}\, b \;\wedge\; b \,\mathrm{R}\, c \implies a \,\mathrm{R}\, c$) integer relation. We denote by $\mathcal{P}_n$ the set of integer posets on $[n]$. We want to show that the subposet of the weak order induced by integer posets is a lattice. Note that $\wedge_{\mathcal{R}}$ and $\vee_{\mathcal{R}}$ do not preserve transitivity, so that $\mathcal{P}_n$ does not induce a sublattice of the weak order on $\mathcal{R}_n$. For $\mathrm{R} \in \mathcal{R}_n$, define the *transitive closure* $\mathrm{R}^{\mathsf{tc}}$, the *transitive decreasing deletion* $\mathrm{R}^{\mathsf{tdd}}$ and the *transitive increasing deletion* $\mathrm{R}^{\mathsf{tid}}$ as

$$\mathrm{R}^{\mathsf{tc}} := \{(u, w) \in [n]^2 \mid \exists\, v_1, \dots, v_p \in [n] \text{ such that } u = v_1 \,\mathrm{R}\, v_2 \,\mathrm{R} \dots \mathrm{R}\, v_{p-1} \,\mathrm{R}\, v_p = w\}$$
$$\mathrm{R}^{\mathsf{tdd}} := \mathrm{R} \smallsetminus \{(b, a) \in \mathrm{R}^{\mathsf{Dec}} \mid \exists\, i \le b \text{ and } j \ge a \text{ such that } i \,\mathrm{R}\, b \,\mathrm{R}\, a \,\mathrm{R}\, j \text{ while } i \,\not{\mathrm{R}}\, j\},$$
$$\mathrm{R}^{\mathsf{tid}} := \mathrm{R} \smallsetminus \{(a, b) \in \mathrm{R}^{\mathsf{Inc}} \mid \exists\, i \ge a \text{ and } j \le b \text{ such that } i \,\mathrm{R}\, a \,\mathrm{R}\, b \,\mathrm{R}\, j \text{ while } i \,\not{\mathrm{R}}\, j\}.$$

Note that in these definitions, $i$ and $j$ may coincide with $a$ and $b$ (since we assumed that all our relations are reflexive). The next statement is illustrated below and in Figure 1.

**Theorem 1.1.** *The subposet of the weak order induced by integer posets is a lattice with meet and join* $\mathrm{R} \wedge_{\mathcal{P}} \mathrm{S} = \big((\mathrm{R}^{\mathsf{Inc}} \cup \mathrm{S}^{\mathsf{Inc}})^{\mathsf{tc}} \cup (\mathrm{R}^{\mathsf{Dec}} \cap \mathrm{S}^{\mathsf{Dec}})\big)^{\mathsf{tdd}}$ *and* $\mathrm{R} \vee_{\mathcal{P}} \mathrm{S} = \big((\mathrm{R}^{\mathsf{Inc}} \cap \mathrm{S}^{\mathsf{Inc}}) \cup (\mathrm{R}^{\mathsf{Dec}} \cup \mathrm{S}^{\mathsf{Dec}})^{\mathsf{tc}}\big)^{\mathsf{tid}}$.




$\mathrm{R} \in \mathcal{P}_4$ | $\mathrm{S} \in \mathcal{P}_4$ | $(\mathrm{R}^{\mathsf{Inc}} \cup \mathrm{S}^{\mathsf{Inc}}) \cup (\mathrm{R}^{\mathsf{Dec}} \cap \mathrm{S}^{\mathsf{Dec}})$ | $(\mathrm{R}^{\mathsf{Inc}} \cup \mathrm{S}^{\mathsf{Inc}})^{\mathsf{tc}} \cup (\mathrm{R}^{\mathsf{Dec}} \cap \mathrm{S}^{\mathsf{Dec}})$ | $\mathrm{R} \wedge_{\mathcal{P}} \mathrm{S} \in \mathcal{P}_4$

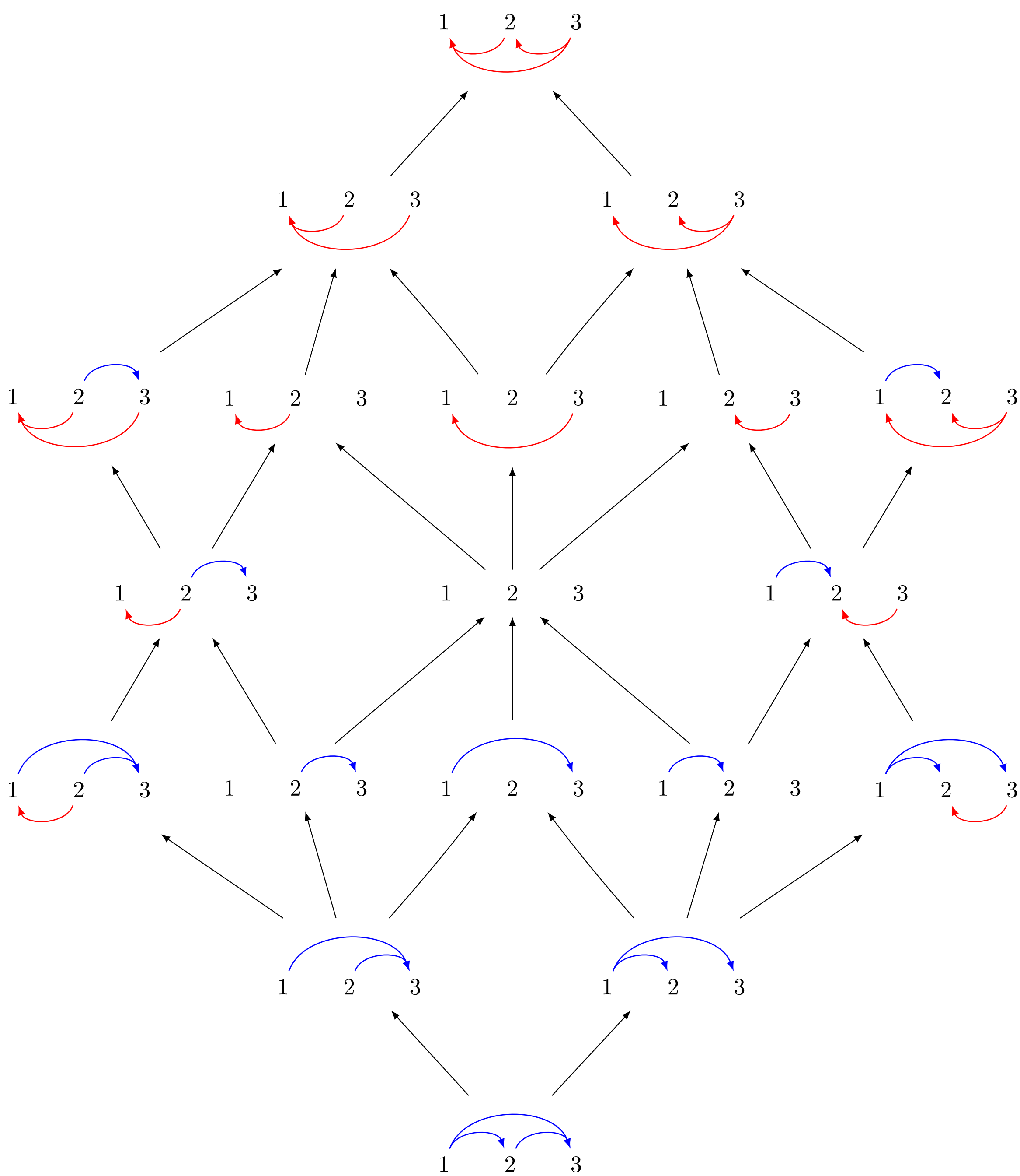



**Figure 1:** The weak order on integer posets of size 3.

# 2 Relevant families of integer posets

In this section, we observe that certain relevant combinatorial objects can be interpreted by specific integer posets and that the subposets of the weak order induced by these integer posets correspond to classical lattice structures on these combinatorial objects.

## 2.1 Permutations

For a permutation $\sigma \in \mathfrak{S}_n$, a pair $(a, b) \in [n]^2$ is a *version* if $a \le b$ and $\sigma^{-1}(a) \le \sigma^{-1}(b)$, and an *inversion* if $a \ge b$ and $\sigma^{-1}(a) \le \sigma^{-1}(b)$. The *weak order* on $\mathfrak{S}_n$ is the lattice defined by inclusion of inversions: $\sigma \preccurlyeq \tau \iff \mathrm{inv}(\sigma) \subseteq \mathrm{inv}(\tau) \iff \mathrm{ver}(\sigma) \supseteq \mathrm{ver}(\tau)$. Geometrically, its Hasse diagram can be interpreted as the graph of the *permutahedron* $\mathsf{Perm}(n) := \mathrm{conv}\,\{(\sigma(1), \dots, \sigma(n)) \mid \sigma \in \mathfrak{S}_n\}$ oriented from $[1, 2, \dots, n]$ to $[n, \dots, 2, 1]$.

Each permutation $\sigma \in \mathfrak{S}_n$ corresponds to a *weak order element poset* $\lhd_\sigma$ defined by $u \lhd_\sigma v$ if $\sigma^{-1}(u) \le \sigma^{-1}(v)$. See Figure 2 for an example with $\sigma = [2, 7, 5, 1, 3, 4, 6]$. Define $\mathsf{WOEP}_n := \{\lhd_\sigma \mid \sigma \in \mathfrak{S}_n\}$. The following statement motivates Section 1.

**Proposition 2.1.**
- *A poset $\lhd$ is in $\mathsf{WOEP}_n$ if and only if $a \lhd b$ or $a \rhd b$ for all $a, b \in [n]$.*
- *For any $\sigma \in \mathfrak{S}_n$, the increasing (resp. decreasing) relations of $\lhd_\sigma$ are the versions (resp. inversions) of $\sigma$. Therefore, for any $\sigma, \sigma' \in \mathfrak{S}_n$, we have $\lhd_\sigma \preccurlyeq \lhd_{\sigma'} \iff \sigma \preccurlyeq \sigma'$.*
- *Moreover, the weak order on $\mathsf{WOEP}_n$ is a sublattice of the weak order on $\mathcal{P}_n$.*

We now present a similar approach to intervals of the weak order. For $\sigma \preccurlyeq \sigma' \in \mathfrak{S}_n$, let $[\sigma, \sigma'] := \{\tau \in \mathfrak{S}_n \mid \sigma \preccurlyeq \tau \preccurlyeq \sigma'\}$. The permutations of $[\sigma, \sigma']$ are precisely the linear extensions of the *weak order interval poset* $\lhd_{[\sigma,\sigma']} := \bigcap_{\sigma \preccurlyeq \tau \preccurlyeq \sigma'} \lhd_\tau = \lhd_\sigma \cap \lhd_{\sigma'} = \lhd_{\sigma'}^{\mathsf{Inc}} \cup \lhd_{\sigma}^{\mathsf{Dec}}$. Define $\mathsf{WOIP}_n := \{\lhd_{[\sigma,\sigma']} \mid \sigma, \sigma' \in \mathfrak{S}_n,\ \sigma \preccurlyeq \sigma'\}$.

**Proposition 2.2.**
- *A poset $\lhd$ is in $\mathsf{WOIP}_n$ if and only if it satisfies $a \lhd c \Rightarrow (a \lhd b$ or $b \lhd c)$ and $a \rhd c \Rightarrow (a \rhd b$ or $b \rhd c)$ for all $a < b < c$ [1, Theorem 6.8].*
- *For any $\sigma \preccurlyeq \sigma'$ and $\tau \preccurlyeq \tau'$ in $\mathfrak{S}_n$, we have $\lhd_{[\sigma,\sigma']} \preccurlyeq \lhd_{[\tau,\tau']} \iff \sigma \preccurlyeq \tau$ and $\sigma' \preccurlyeq \tau'$.*
- *The weak order on $\mathsf{WOIP}_n$ is a lattice with meet $\lhd_{[\sigma,\sigma']} \wedge_{\mathsf{WOIP}} \lhd_{[\tau,\tau']} = \lhd_{[\sigma \wedge_{\mathfrak{S}} \tau, \sigma' \wedge_{\mathfrak{S}} \tau']}$ and join $\lhd_{[\sigma,\sigma']} \vee_{\mathsf{WOIP}} \lhd_{[\tau,\tau']} = \lhd_{[\sigma \vee_{\mathfrak{S}} \tau, \sigma' \vee_{\mathfrak{S}} \tau']}$. However, the weak order on $\mathsf{WOIP}_n$ is not a sublattice of the weak order on $\mathcal{P}_n$.*

Finally, we consider a face of the permutahedron, that is, an ordered partition $\pi$ of $[n]$. We see $\pi$ as a *weak order face poset* $\lhd_\pi$ defined by $u \lhd_\pi v$ if $\pi^{-1}(u) < \pi^{-1}(v)$. See Figure 3 for an example with $\pi = 125|37|46$. Define $\mathsf{WOFP}_n := \{\lhd_\pi \mid \pi$ ordered partition of $[n]\}$.

**Proposition 2.3.**
- *A poset $\lhd$ is in $\mathsf{WOFP}_n$ if and only if $\lhd \in \mathsf{WOIP}_n$ and $(a \lhd b \Leftrightarrow b \rhd c)$ and $(a \rhd b \Leftrightarrow b \lhd c)$ for all $a < b < c$ with $a \not\lhd c$ and $a \not\rhd c$.*
- *For any ordered partitions $\pi, \pi'$ of $[n]$, we have $\lhd_\pi \preccurlyeq \lhd_{\pi'} \iff \pi \preccurlyeq \pi'$ in the facial weak order of [9, 12, 5].*
- *The weak order on $\mathsf{WOFP}_n$ is a lattice but not a sublattice of the weak order on $\mathcal{P}_n$, nor on $\mathsf{WOIP}_n$.*

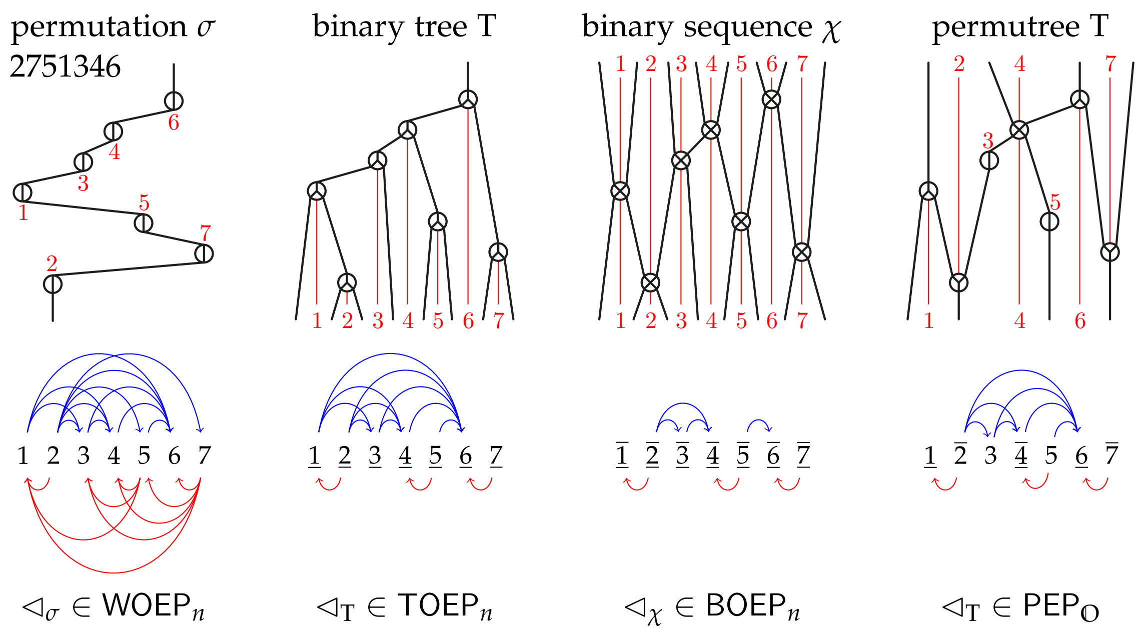



**Figure 2:** Posets for permutations, binary trees, binary sequences, and permutrees.

## 2.2 Binary trees

Let $\mathfrak{B}_n$ be the set of *rooted binary trees* with $n$ nodes (always labeled in inorder). The *Tamari lattice* on $\mathfrak{B}_n$ is the lattice $\preccurlyeq$ whose cover relations are given by *right rotations* on binary trees. It was reinterpreted in [1, 15] as a lattice quotient of the weak order as follows. Consider the surjection bt which maps a permutation $\sigma := \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$ to the binary tree $\mathrm{bt}(\sigma) \in \mathfrak{B}_n$ obtained by successive insertions of $\sigma_n, \dots, \sigma_1$ in a binary (search) tree. The fiber of a tree T is precisely the set of linear extensions of T. It is an interval of the weak order whose minimal and maximal elements respectively avoid the patterns 312 and 132. The Tamari order is given by $\mathrm{T} \preccurlyeq \mathrm{T}'$ if and only if there exist $\sigma, \sigma' \in \mathfrak{S}_n$ such that $\mathrm{bt}(\sigma) = \mathrm{T}$, $\mathrm{bt}(\sigma') = \mathrm{T}'$ and $\sigma \preccurlyeq \sigma'$. Geometrically, the Hasse diagram of the Tamari lattice is the graph of the associahedron [8] oriented from the left comb to the right comb.

We aim at reinterpreting the Tamari lattice, its interval lattice and its facial interval lattice using specific posets. Each tree T corresponds to a *Tamari order element poset* $\lhd_{\mathrm{T}}$, defined by $i \lhd_{\mathrm{T}} j$ when $i$ is a descendant of $j$ in T. In other words, the Hasse diagram of $\lhd_{\mathrm{T}}$ is the tree T oriented towards its root. See Figure 2. Define $\mathsf{TOEP}_n := \{\lhd_{\mathrm{T}} \mid \mathrm{T} \in \mathfrak{B}_n\}$.

**Proposition 2.4.**
- *A poset $\lhd$ is in $\mathsf{TOEP}_n$ if and only if (i) $(a \lhd c \Rightarrow b \lhd c)$ and $(a \rhd c \Rightarrow a \rhd b)$ for all $a < b < c$ and (ii) there exists $a < b < c$ such that $a \lhd b \rhd c$ for all $a < c$ incomparable in $\lhd$.*
- *For any binary trees $\mathrm{T}, \mathrm{T}' \in \mathfrak{B}_n$, we have $\lhd_{\mathrm{T}} \preccurlyeq \lhd_{\mathrm{T}'} \iff \mathrm{T} \preccurlyeq \mathrm{T}'$.*
- *Moreover, the Tamari lattice on $\mathsf{TOEP}_n$ is a sublattice of the weak order on $\mathcal{P}_n$.*

For $\mathrm{T} \preccurlyeq \mathrm{T}' \in \mathfrak{B}_n$, consider the Tamari interval $[\mathrm{T}, \mathrm{T}'] := \{\mathrm{S} \in \mathfrak{B}_n \mid \mathrm{T} \preccurlyeq \mathrm{S} \preccurlyeq \mathrm{T}'\}$. We see it as the *Tamari order interval poset* $\lhd_{[\mathrm{T},\mathrm{T}']} := \bigcap_{\mathrm{T} \preccurlyeq \mathrm{S} \preccurlyeq \mathrm{T}'} \lhd_{\mathrm{T}} = \lhd_{\mathrm{T}} \cap \lhd_{\mathrm{T}'} = \lhd^{\mathsf{Inc}}_{\mathrm{T}'} \cap \lhd^{\mathsf{Dec}}_{\mathrm{T}}$. Define $\mathsf{TOIP}_n := \{\lhd_{[\mathrm{T},\mathrm{T}']} \mid \mathrm{T}, \mathrm{T}' \in \mathfrak{B}_n, \mathrm{T} \preccurlyeq \mathrm{T}'\}$.

**Proposition 2.5.**
- *A poset $\lhd$ is in $\mathsf{TOIP}_n$ if and only if $(a \lhd c \Rightarrow b \lhd c)$ and $(a \rhd c \Rightarrow a \rhd b)$ for all $a < b < c$.*
- *For any $\mathrm{S} \preccurlyeq \mathrm{S}'$ and $\mathrm{T} \preccurlyeq \mathrm{T}'$ in $\mathfrak{B}_n$, we have $\lhd_{[\mathrm{S},\mathrm{S}']} \preccurlyeq \lhd_{[\mathrm{T},\mathrm{T}']} \iff \mathrm{S} \preccurlyeq \mathrm{T}$ and $\mathrm{S}' \preccurlyeq \mathrm{T}'$.*
- *Moreover, the weak order on $\mathsf{TOIP}_n$ is a sublattice of the weak order on $\mathcal{P}_n$.*

Consider now a face of the associahedron, that is, a Schröder tree S (a rooted tree where each internal node has at least two children). We label the angles between two consecutive children in inorder, meaning that each angle is labeled after the angles in its left child and before the angles in its right child. We associate to S the poset $\lhd_{\mathrm{S}}$ where $i \lhd_{\mathrm{S}} j$ if and only if the angle labeled $i$ belongs to the left or to the right child of the angle labeled $j$. See Figure 3. Note that $\lhd_{\mathrm{S}} = \lhd_{[\mathrm{T}^{\min}, \mathrm{T}^{\max}]}$, where $\mathrm{T}^{\min}$ (resp. $\mathrm{T}^{\max}$) is obtained by replacing the nodes of S by left (resp. right) combs. Define $\mathsf{TOFP}_n := \{\lhd_{\mathrm{S}} \mid \mathrm{S}$ Schröder tree on $[n]\}$.

**Proposition 2.6.**
- *A poset $\lhd$ is in $\mathsf{TOFP}_n$ if and only if $\lhd \in \mathsf{TOIP}_n$ and for all $a < c$ incomparable in $\lhd$, either there exists $a < b < c$ such that $a \not\rhd b \not\lhd c$, or for all $a < b < c$ we have $a \rhd b \lhd c$.*
- *For any Schröder trees $\mathrm{S}, \mathrm{S}'$, we have $\lhd_{\mathrm{S}} \preccurlyeq \lhd_{\mathrm{S}'} \iff \mathrm{S} \preccurlyeq \mathrm{S}'$ in the facial weak order on the associahedron $\mathsf{Asso}(n)$ studied in [12, 5]. This order is a quotient of the facial weak order on the permutahedron by the fibers of the Schröder tree insertion* st.
- *The weak order on $\mathsf{TOFP}_n$ is a lattice but not a sublattice of the weak order on $\mathcal{P}_n$, nor on $\mathsf{WOIP}_n$, nor on $\mathsf{TOIP}_n$.*

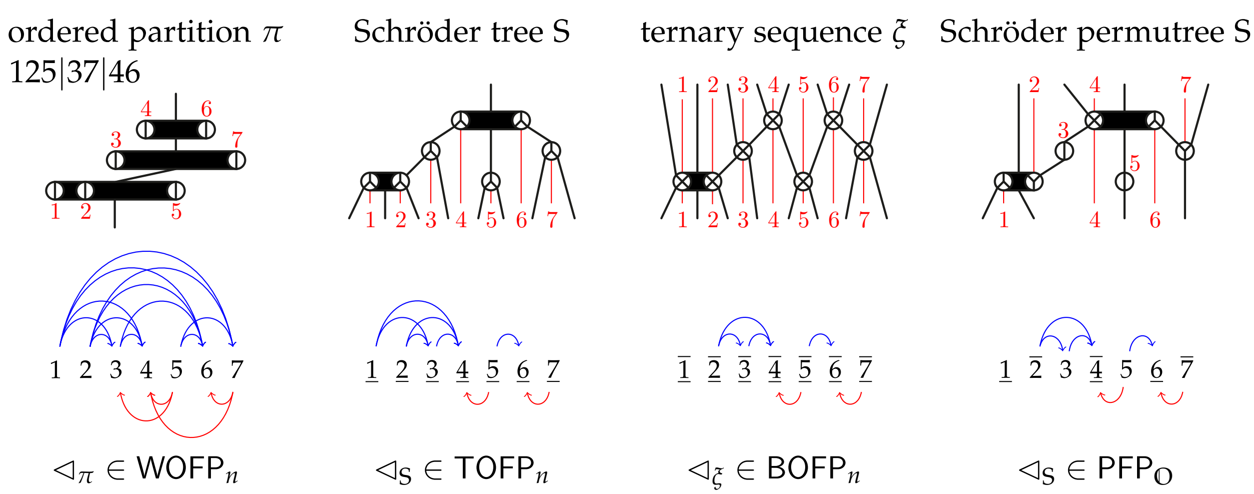



**Figure 3:** Posets for ordered partitions, Schröder trees, ternary sequences, and Schröder permutrees.

Finally, we want to show that the binary tree insertion and the Schröder tree insertion can as well be directly understood at the level of posets. For this, define the TOIP *deletion* by $\triangleleft^{\mathsf{TOIPd}} := \triangleleft \smallsetminus (\{(a,c) \mid \exists\, a < b < c,\ b \ntriangleleft c\} \cup \{(c,a) \mid \exists\, a < b < c,\ a \ntriangleright b\})$.

**Proposition 2.7.** *For any permutation $\sigma$, weak order interval $\sigma \preccurlyeq \sigma'$, and ordered partition $\pi$,*

$$(\triangleleft_\sigma)^{\mathsf{TOIPd}} = \triangleleft_{\mathsf{bt}(\sigma)}, \qquad (\triangleleft_{[\sigma,\sigma']})^{\mathsf{TOIPd}} = \triangleleft_{[\mathsf{bt}(\sigma),\, \mathsf{bt}(\sigma')]} \qquad \text{and} \qquad (\triangleleft_\pi)^{\mathsf{TOIPd}} = \triangleleft_{\mathsf{st}(\pi)}.$$

## 2.3 Permutrees

To conclude this section, we want to mention that the statements of Sections 2.1 & 2.2 extend to the permutrees, a common generalization of permutations and binary trees.

A *permutree* is an oriented tree T with nodes labeled by $[n]$, such that

- any node has either one or two parents and either one or two children,
- if the node $j$ has two parents (resp. children), then all labels in its left ancestor (resp. descendant) subtree are smaller than $j$ while all labels in its right ancestor (resp. descendant) subtree are larger than $j$.

See *e.g.* Figure 2. The *orientation* of a permutree T is $\mathbb{O}(\mathrm{T}) = (\mathbb{O}^+, \mathbb{O}^-)$ where $\mathbb{O}^+$ is the set of labels of the nodes with two parents while $\mathbb{O}^-$ is the set of labels of the nodes with two children. We denote by $\mathfrak{P}(\mathbb{O})$ the set of permutrees with a given orientation $\mathbb{O}$. As illustrated in Figure 2, a permutree with orientation $(\varnothing, \varnothing)$ (resp. $(\varnothing, [n])$, resp. $([n],[n])$) is nothing else but a permutation (resp. a binary tree, resp. a binary sequence).

To each permutree T corresponds its *permutree element poset* $\triangleleft_\mathrm{T}$ where $i \triangleleft_\mathrm{T} j$ if there is an oriented path from $i$ to $j$ in T. Define $\mathsf{PEP}_\mathbb{O} := \{\triangleleft_\mathrm{T} \mid \mathrm{T} \in \mathfrak{P}(\mathbb{O})\}$. Many properties of permutations and binary trees extend to permutrees. In particular:

- For a permutree $\mathrm{T} \in \mathfrak{P}(\mathbb{O})$, the set of linear extensions $\mathcal{L}(\mathrm{T})$ of $\triangleleft_\mathrm{T}$ is an interval in the weak order on $\mathfrak{S}_n$ whose minimal element avoids[1] the patterns $31\text{-}\underline{2}$ and $\overline{2}\text{-}31$, and whose maximal element avoids the patterns $13\text{-}\underline{2}$ and $\overline{2}\text{-}31$.
- For any orientation $\mathbb{O}$ of $[n]$, the set $\{\mathcal{L}(\mathrm{T}) \mid \mathrm{T} \in \mathfrak{P}(\mathbb{O})\}$ forms a partition of $\mathfrak{S}_n$. This defines a surjection[2] $\Psi_\mathbb{O} : \mathfrak{S}_n \to \mathfrak{P}(\mathbb{O})$, which sends a permutation $\sigma \in \mathfrak{S}_n$ to the unique permutree $\mathrm{T} \in \mathfrak{P}(\mathbb{O})$ such that $\sigma \in \mathcal{L}(\mathrm{T})$.
- This partition defines a lattice congruence of the weak order (see [16, 15, 13] for details). This yields the *permutree lattice*[3], defined by $\mathrm{T} \preccurlyeq \mathrm{T}'$ if and only if there exist $\sigma, \sigma' \in \mathfrak{S}_n$ such that $\Psi_\mathbb{O}(\sigma) = \mathrm{T}$, $\Psi_\mathbb{O}(\sigma') = \mathrm{T}'$ and $\sigma \preccurlyeq \sigma'$. Its minimal (resp. maximal) element is the *left* (resp. *right*) *$\mathbb{O}$-comb*: it is a chain where each vertex in $\mathbb{O}^+$ has an additional empty left (resp. right) parent, while each vertex in $\mathbb{O}^-$ has an additional empty right (resp. left) child.
- Geometrically, the Hasse diagram of the permutree lattice is the graph of the *permutreehedron* $\mathsf{PT}(\mathbb{O})$ [13, Section 3] oriented from the left to the right $\mathbb{O}$-comb.

[1] A permutation $\sigma$ contains the pattern $31\text{-}\underline{2}$ if $\sigma = \sigma_1 c a \sigma_2 b \sigma_3$ with $a < b < c$ and $b \in \mathbb{O}^-$.

[2] This surjection can also be described directly as an insertion algorithm, see [13, Section 2.2].

[3] This lattice structure can equivalently be described by right rotations in permutrees [13, Section 2.6].

The next statement extends Propositions 2.1 & 2.4. An orientation $\mathbb{O} = (\mathbb{O}^+, \mathbb{O}^-)$ is said to be *covering* if $\{2, \dots, n-1\} \subseteq \mathbb{O}^+ \cup \mathbb{O}^-$.

**Proposition 2.8.**
- $\mathsf{PEP}_{\mathbb{O}}$ *is characterized by local conditions*[4] *as in Propositions 2.1 & 2.4.*
- *For any permutrees* $\mathrm{T}, \mathrm{T}' \in \mathfrak{P}(\mathbb{O})$, *we have* $\lhd_{\mathrm{T}} \preccurlyeq \lhd_{\mathrm{T}'} \iff \mathrm{T} \preccurlyeq \mathrm{T}'$.
- *The permutree lattice on* $\mathsf{PEP}_{\mathbb{O}}$ *is always a sublattice of the weak order on* $\mathsf{WOIP}_n$. *It is a sublattice of the weak order on* $\mathcal{P}_n$ *when* $\mathbb{O}$ *is covering, but not in general.*

We now consider intervals in the permutree lattice. For two permutrees $\mathrm{T}, \mathrm{T}' \in \mathfrak{P}(\mathbb{O})$ with $\mathrm{T} \preccurlyeq \mathrm{T}'$, let $[\mathrm{T}, \mathrm{T}'] := \{\mathrm{S} \in \mathfrak{P}(\mathbb{O}) \mid \mathrm{T} \preccurlyeq \mathrm{S} \preccurlyeq \mathrm{T}'\}$. It corresponds to the *permutree interval poset* $\lhd_{[\mathrm{T},\mathrm{T}']} := \lhd_{\mathrm{T}'}^{\mathsf{Inc}} \cap \lhd_{\mathrm{T}}^{\mathsf{Dec}}$. Define $\mathsf{PIP}_{\mathbb{O}} := \{\lhd_{[\mathrm{T},\mathrm{T}']} \mid \mathrm{T}, \mathrm{T}' \in \mathfrak{P}(\mathbb{O}), \mathrm{T} \preccurlyeq \mathrm{T}'\}$. The next statement extends Propositions 2.2 & 2.5.

**Proposition 2.9.**
- *A poset* $\lhd$ *is in* $\mathsf{PIP}_{\mathbb{O}}$ *if and only if* $\lhd \in \mathsf{WOIP}_n$ *and for any* $a < b < c$, *we have* $(a \lhd c \Rightarrow a \lhd b)$ *and* $(a \rhd c \Rightarrow b \rhd c)$ *when* $b \in \mathbb{O}^+$, *and* $(a \lhd c \Rightarrow b \lhd c)$ *and* $(a \rhd c \Rightarrow a \rhd b)$ *when* $b \in \mathbb{O}^-$.
- *For any* $\mathrm{S} \preccurlyeq \mathrm{S}'$ *and* $\mathrm{T} \preccurlyeq \mathrm{T}'$ *in* $\mathfrak{P}(\mathbb{O})$, *we have* $\lhd_{[\mathrm{S},\mathrm{S}']} \preccurlyeq \lhd_{[\mathrm{T},\mathrm{T}']} \iff \mathrm{S} \preccurlyeq \mathrm{T}$ *and* $\mathrm{S}' \preccurlyeq \mathrm{T}'$.
- *The weak order on* $\mathsf{PIP}_{\mathbb{O}}$ *is always a sublattice of the weak order on* $\mathsf{WOIP}_n$. *It is a sublattice of the weak order on* $\mathcal{P}_n$ *when* $\mathbb{O}$ *is covering, but not in general.*

To extend Propositions 2.3 & 2.6, we now consider facial intervals of the permutree lattice. The faces of the permutreehedron $\mathsf{PT}(\mathbb{O})$ correspond to the Schröder permutrees with orientation $\mathbb{O}$, which are obtained from the permutrees of $\mathfrak{P}(\mathbb{O})$ by edge contractions[5]. Define the *permutree face poset* $\lhd_{\mathrm{S}} = \lhd_{[\mathrm{T}^{\min}, \mathrm{T}^{\max}]}$, where $\mathrm{T}^{\min}$ (resp. $\mathrm{T}^{\max}$) is obtained by replacing the nodes of S by left (resp. right) combs (with the correct orientations). See Figure 3. Define $\mathsf{PFP}_{\mathbb{O}} := \{\lhd_{\mathrm{S}} \mid \mathrm{S}$ Schröder permutree with orientation $\mathbb{O}\}$.

**Proposition 2.10.**
- $\mathsf{PFP}_{\mathbb{O}}$ *is characterized by local conditions*[2] *as in Propositions 2.3 & 2.6.*
- *For any Schröder permutrees* $\mathsf{S}, \mathsf{S}'$, *we have* $\lhd_{\mathsf{S}} \preccurlyeq \lhd_{\mathsf{S}'} \iff \mathsf{S} \preccurlyeq \mathsf{S}'$ *in the* facial weak order *on the permutreehedron* $\mathsf{PT}(\mathbb{O})$ *studied in [13, 5].*
- *The weak order on* $\mathsf{PFP}_{\mathbb{O}}$ *is a lattice but not a sublattice of the weak order on* $\mathcal{P}_n$, *nor on* $\mathsf{WOIP}_n$, *nor on* $\mathsf{PIP}_{\mathbb{O}}$.

Finally, for $\lhd \in \mathcal{P}_n$, we define its $\mathsf{PIP}_{\mathbb{O}}$ *deletion* $\lhd^{\mathsf{PIP}_{\mathbb{O}}\mathsf{d}}$ to be the poset obtained by removing simultaneously from $\lhd$ all increasing relations $(a, c)$ such that:
- either $\exists\, a < b_1 < \dots < b_k < c$ with $a \not\lhd b_1 \not\lhd \dots \not\lhd b_k \not\lhd c$ (preventing $\lhd \in \mathsf{WOIP}$),
- or $\exists\, a \le n < p \le c$ with $n \in \{a\} \cup \mathbb{O}^-$, $p \in \{c\} \cup \mathbb{O}^+$, and $n \not\lhd p$.

and their analogue decreasing relations. We now give an analogue of Proposition 2.7.

**Proposition 2.11.** *For any permutation* $\sigma$, *weak order interval* $\sigma \preccurlyeq \sigma'$, *and ordered partition* $\pi$,

$$(\lhd_\sigma)^{\mathsf{PIP}_{\mathbb{O}}\mathsf{d}} = \lhd_{\Psi_{\mathbb{O}}(\sigma)}, \quad (\lhd_{[\sigma,\sigma']})^{\mathsf{PIP}_{\mathbb{O}}\mathsf{d}} = \lhd_{[\Psi_{\mathbb{O}}(\sigma),\, \Psi_{\mathbb{O}}(\sigma')]} \quad \textit{and} \quad (\lhd_\pi)^{\mathsf{PIP}_{\mathbb{O}}\mathsf{d}} = \lhd_{\Psi_{\mathbb{O}}(\pi)},$$

*where* $\Psi_{\mathbb{O}}(\sigma)$ *(resp.* $\Psi_{\mathbb{O}}(\pi)$*) is the permutree (resp. Schröder permutree) associated to* $\sigma$ *(resp.* $\pi$*).*

[4] The detailed conditions are too intricate for this abstract but can be found in [4, Propositions 60 & 63].

[5] A more direct but more technical definition of Schröder permutrees can also be found in [13, Section 5].

# 3 Hopf algebra on integer posets

In this section, we construct a Hopf algebra on integer posets. We then show that the permutations, the weak order intervals, and the ordered partitions index certain quotients of the integer poset algebra, and that the binary trees, the Tamari intervals, and the Schröder trees index subalgebras of these quotients. Define $\mathcal{R}_{\mathbb{N}} := \bigsqcup_{n \ge 0} \mathcal{R}_n$ and similarly for the other families of relations considered in this paper.

## 3.1 Hopf algebras on binary relations and integer posets

As for the weak order, we first define our Hopf algebra at the level of binary relations. We consider the vector space $\mathbb{K}\mathcal{R}_{\mathbb{N}} := \bigoplus_{n \ge 0} \mathbb{K}\mathcal{R}_n$ indexed by all integer binary relations of arbitrary size. We denote by $\mathrm{R}_X := \{(i,j) \in [k]^2 \mid x_i \,\mathrm{R}\, x_k\}$ the *restriction* of an integer relation $\mathrm{R} \in \mathcal{R}_n$ to a subset $X = \{x_1, \dots, x_k\} \subseteq [n]$. We define:

- the *product* of two integer relations $\mathrm{R} \in \mathcal{R}_m$ and $\mathrm{S} \in \mathcal{R}_n$ by $\mathrm{R} \cdot \mathrm{S} := \sum \mathrm{T}$ where $\mathrm{T}$ ranges over all integer relations $\mathrm{T} \in \mathcal{R}_{m+n}$ with $\mathrm{T}_{[m]} = \mathrm{R}$ and $\mathrm{T}_{[n+m] \smallsetminus [m]} = \mathrm{S}$.
- the *coproduct* of an integer relation $\mathrm{T} \in \mathcal{R}(p)$ by $\triangle(\mathrm{T}) := \sum \mathrm{T}_X \otimes \mathrm{T}_Y$ where the sum ranges over all partitions $X \sqcup Y \subseteq [p]$ such that $x \,\mathrm{T}\, y$ and $y \,\not\mathrel{\mathrm{T}}\, x$ for all $(x,y) \in X \times Y$.

*E.g.* $$1\ 2 \cdot 1 = 1\ 2\ 3 + 1\ 2\ 3 + 1\ 2\ 3 + \cdots + 1\ 2\ 3 + \cdots + 1\ 2\ 3\,,$$
and $$\triangle(1\ 2\ 3) = 1\ 2\ 3 \otimes \varnothing + 1 \otimes 1\ 2 + 1\ 2 \otimes 1 + \varnothing \otimes 1\ 2\ 3\,,$$

where the terms in the coproduct arise from $X$ ranging in $\{1,2,3\}$, $\{1\}$, $\{1,3\}$, and $\varnothing$. Note that the product $\mathrm{R} \cdot \mathrm{S}$ is the sum over the interval between $\mathrm{R} \backslash \mathrm{S} := \mathrm{R} \cup \overline{\mathrm{S}} \cup ([m] \times \overline{[n]})$ and $\mathrm{R} / \mathrm{S} := \mathrm{R} \cup \overline{\mathrm{S}} \cup (\overline{[n]} \times [m])$ in the weak order on $\mathcal{R}_{m+n}$ (where we denote the shifts $\overline{\mathrm{S}} := \{(m+i, m+j) \mid (i,j) \in \mathrm{S}\}$ and $\overline{[n]} := [m+n] \smallsetminus [m]$).

**Proposition 3.1.** *The product $\cdot$ and coproduct $\triangle$ endow $\mathbb{K}\mathcal{R}_{\mathbb{N}}$ with a Hopf algebra structure, i.e. $\triangle(\mathrm{R} \cdot \mathrm{S}) = \triangle(\mathrm{R}) \cdot \triangle(\mathrm{S})$ where the last product is $(a \otimes b) \cdot (c \otimes d) = (a \cdot c) \otimes (b \cdot d)$.*

We now use this Hopf algebra on binary relations to get a Hopf algebra on posets.

**Proposition 3.2.** *If $\mathrm{R} \in \mathcal{R}_{\mathbb{N}}$ is not a poset, then none of the summands in $\mathrm{R} \cdot \mathrm{S}$ (resp. $\triangle(\mathrm{R})$) is a poset (resp. the tensor product of two posets). In other words, the vector subspace of $\mathbb{K}\mathcal{R}_{\mathbb{N}}$ generated by integer relations which are not posets is a Hopf ideal of $(\mathbb{K}\mathcal{R}_{\mathbb{N}}, \cdot, \triangle)$. The quotient of $(\mathbb{K}\mathcal{R}_{\mathbb{N}}, \cdot, \triangle)$ by this ideal is thus a Hopf algebra $(\mathbb{K}\mathcal{P}_{\mathbb{N}}, \cdot, \triangle)$ on integer posets.*

*E.g.* $$1\ 2 \cdot 1 = 1\ 2\ 3 + 1\ 2\ 3 + 1\ 2\ 3 + 1\ 2\ 3 + 1\ 2\ 3 + 1\ 2\ 3 \text{ in } \mathbb{K}\mathcal{P}_{\mathbb{N}}.$$

**Proposition 3.3.** *For $\lhd \in \mathcal{P}_m$ and $\blacktriangleleft \in \mathcal{P}_n$, the product $\lhd \cdot \blacktriangleleft$ is the interval between $\lhd \backslash \blacktriangleleft$ and $\lhd / \blacktriangleleft$ in the weak order on $\mathcal{P}_{m+n}$.*

## 3.2 Hopf algebras on permutations, intervals, and ordered partitions

We now want to construct Hopf algebras on $\mathsf{WOEP}_{\mathbb{N}}$, $\mathsf{WOIP}_{\mathbb{N}}$ and $\mathsf{WOFP}_{\mathbb{N}}$ as quotients of the integer poset Hopf algebra $(\mathcal{P}_{\mathbb{N}}, \cdot, \triangle)$. The important point is that all these families of posets are defined by local conditions on their relations, and that a contradiction to these conditions cannot be destroyed by the product or the coproduct.

We start with weak order element posets $\mathsf{WOEP}_{\mathbb{N}}$. For two permutations $\sigma \in \mathfrak{S}_m$ and $\tau \in \mathfrak{S}_n$, define the *shifted shuffle* $\sigma \,\bar{\sqcup\!\sqcup}\, \tau$ (resp. the *convolution* $\sigma \star \tau$) as the permutation of $\mathfrak{S}_{m+n}$ whose first $m$ values (resp. positions) are in the same relative order as $\sigma$ and whose last $n$ values (resp. positions) are in the same relative order as $\tau$. For example,

$$12 \,\bar{\sqcup\!\sqcup}\, 231 = \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\},$$
$$\text{and} \quad 12 \star 231 = \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}.$$

Recall that the Malvenuto–Reutenauer Hopf algebra [11] is the Hopf algebra on permutations with product $\sigma \cdot \tau := \sum_{\rho \in \sigma \bar{\sqcup\!\sqcup} \tau} \rho$ and coproduct $\triangle(\rho) := \sum_{\rho \in \sigma \star \tau} \sigma \otimes \tau$. This Hopf algebra can be interpreted as a quotient of the integer poset Hopf algebra $(\mathbb{K}\mathcal{P}_{\mathbb{N}}, \cdot, \triangle)$.

**Proposition 3.4.**
- *The vector subspace of $\mathbb{K}\mathcal{P}_{\mathbb{N}}$ generated by non-total integer posets is a Hopf ideal of $(\mathbb{K}\mathcal{P}_{\mathbb{N}}, \cdot, \triangle)$. The quotient of the poset Hopf algebra $(\mathbb{K}\mathcal{P}_{\mathbb{N}}, \cdot, \triangle)$ by this ideal is thus a Hopf algebra $(\mathbb{K}\mathsf{WOEP}_{\mathbb{N}}, \cdot, \triangle)$ on total orders.*
- *The map $\sigma \mapsto \lhd_\sigma$ defines a Hopf algebra isomorphism from the Malvenuto–Reutenauer Hopf algebra on permutations [11] to $(\mathbb{K}\mathsf{WOEP}_{\mathbb{N}}, \cdot, \triangle)$.*
- *For any permutations $\sigma \in \mathfrak{S}_m$ and $\tau \in \mathfrak{S}_n$, the product $\lhd_\sigma \cdot \lhd_\tau$ is the interval between $\lhd_\sigma \backslash \lhd_\tau$ and $\lhd_\sigma / \lhd_\tau$ in the weak order on $\mathsf{WOEP}_{m+n}$.*

*E.g.* $1\ 2 \cdot 1 = 1\ 2\ 3 + 1\ 2\ 3 + 1\ 2\ 3$ in $\mathbb{K}\mathsf{WOEP}_{\mathbb{N}}$.

We now consider weak order interval posets $\mathsf{WOIP}_{\mathbb{N}}$, characterized in Proposition 2.2.

**Proposition 3.5.**
- *The vector subspace of $\mathbb{K}\mathcal{P}_{\mathbb{N}}$ generated by $\mathcal{P}_{\mathbb{N}} \smallsetminus \mathsf{WOIP}_{\mathbb{N}}$ is a Hopf ideal of $(\mathbb{K}\mathcal{P}_{\mathbb{N}}, \cdot, \triangle)$. The quotient of the integer poset algebra $(\mathbb{K}\mathcal{P}_{\mathbb{N}}, \cdot, \triangle)$ by this ideal is thus a Hopf algebra $(\mathbb{K}\mathsf{WOIP}_{\mathbb{N}}, \cdot, \triangle)$ on weak order intervals.*
- *For any $\sigma \preccurlyeq \sigma' \in \mathfrak{S}_m$ and $\tau \preccurlyeq \tau' \in \mathfrak{S}_n$, the product $\lhd_{[\sigma,\sigma']} \cdot \lhd_{[\tau,\tau']}$ is the interval between $\lhd_{[\sigma,\sigma']} \backslash \lhd_{[\tau,\tau']}$ and $\lhd_{[\sigma,\sigma']} / \lhd_{[\tau,\tau']}$ in the weak order on $\mathsf{WOIP}_{m+n}$.*

*E.g.* $1\ 2 \cdot 1 = 1\ 2\ 3 + 1\ 2\ 3 + 1\ 2\ 3 + 1\ 2\ 3 + 1\ 2\ 3$ in $\mathbb{K}\mathsf{WOIP}_{\mathbb{N}}$.

Finally, a similar statement holds for weak order face posets $\mathsf{WOFP}_{\mathbb{N}}$, characterized in Proposition 2.3. It turns out that the resulting algebra was studied in [2].

**Proposition 3.6.** • *The vector subspace of $\mathbb{K}\mathcal{P}_{\mathbb{N}}$ generated by $\mathcal{P}_{\mathbb{N}} \smallsetminus \mathsf{WOIP}_{\mathbb{N}}$ is a Hopf ideal of $(\mathbb{K}\mathcal{P}_{\mathbb{N}}, \cdot, \triangle)$. The quotient of the poset Hopf algebra $(\mathbb{K}\mathcal{P}_{\mathbb{N}}, \cdot, \triangle)$ by this ideal is thus a Hopf algebra $(\mathbb{K}\mathsf{WOFP}_{\mathbb{N}}, \cdot, \triangle)$ on faces of the permutahedron.*

- *The map $\pi \mapsto \lhd_\pi$ defines a Hopf algebra isomorphism from the Chapoton Hopf algebra on ordered partitions [2] to $(\mathbb{K}\mathsf{WOFP}_{\mathbb{N}}, \cdot, \triangle)$.*
- *For any ordered partitions $\pi$ of $[m]$ and $\omega$ of $[n]$, the product $\lhd_\pi \cdot \lhd_\omega$ is the interval between $\lhd_\pi \backslash \lhd_\omega$ and $\lhd_\pi / \lhd_\omega$ in the weak order on $\mathsf{WOFP}_{m+n}$.*

*E.g.* $1\ \ 2 \cdot 1 = 1\ \ 2\ \ 3 + 1\ \ 2\ \ 3 + 1\ \ 2\ \ 3$ in $\mathbb{K}\mathsf{WOFP}_{\mathbb{N}}$.

### 3.3 Hopf algebras on binary trees, Tamari intervals, and Schröder trees

To conclude, we use the TOIP deletion defined in the end of Section 2.2 to construct Hopf subalgebras of $\mathbb{K}\mathsf{WOEP}_{\mathbb{N}}$, $\mathbb{K}\mathsf{WOIP}_{\mathbb{N}}$ and $\mathbb{K}\mathsf{WOFP}_{\mathbb{N}}$ respectively indexed by $\mathsf{TOEP}_{\mathbb{N}}$, $\mathsf{TOIP}_{\mathbb{N}}$ and $\mathsf{TOFP}_{\mathbb{N}}$. This idea mimics the construction of the Loday–Ronco algebra on binary trees [10, 7], that can be defined as a Hopf subalgebra of the Malvenuto–Reutenauer algebra on permutations [11].

**Proposition 3.7.** • *The vector subspace $\mathbb{K}\mathsf{TOEP}_{\mathbb{N}}$ of $\mathbb{K}\mathsf{WOEP}_{\mathbb{N}}$ generated by the sums of the fibers $\{\lhd \in \mathsf{WOEP}_{\mathbb{N}} \mid \lhd^{\mathsf{TOIPd}} = \blacktriangleleft\}$ for all $\blacktriangleleft \in \mathsf{TOEP}_{\mathbb{N}}$ is stable by $\cdot$ and $\triangle$.*

- *For any binary trees $\mathrm{S} \in \mathfrak{B}_m$ and $\mathrm{T} \in \mathfrak{B}_n$, the product $\lhd_{\mathrm{S}} \cdot \lhd_{\mathrm{T}}$ is the interval between $\lhd_{\mathrm{S}} \backslash \lhd_{\mathrm{T}}$ and $\lhd_{\mathrm{S}} / \lhd_{\mathrm{T}}$ in the weak order on $\mathsf{TOEP}_{m+n}$.*
- *The map $\mathrm{T} \mapsto \lhd_{\mathrm{T}}$ is a Hopf algebra isomorphism from the Loday–Ronco algebra on binary trees [10, 7] to $(\mathbb{K}\mathsf{TOEP}_{\mathbb{N}}, \cdot, \triangle)$.*

We now proceed to the same construction for Tamari intervals. To the best of our knowledge, our approach provides the first Hopf algebra on Tamari intervals.

**Proposition 3.8.** • *The vector subspace $\mathbb{K}\mathsf{TOIP}_{\mathbb{N}}$ of $\mathbb{K}\mathsf{WOIP}_{\mathbb{N}}$ generated by the sums of the fibers $\{\lhd \in \mathsf{WOIP}_{\mathbb{N}} \mid \lhd^{\mathsf{TOIPd}} = \blacktriangleleft\}$ for all $\blacktriangleleft \in \mathsf{TOIP}_{\mathbb{N}}$ is stable by $\cdot$ and $\triangle$.*

- *For any Tamari intervals $\mathrm{S} \preccurlyeq \mathrm{S}' \in \mathfrak{B}_m$ and $\mathrm{T} \preccurlyeq \mathrm{T}' \in \mathfrak{B}_n$, the product $\lhd_{[\mathrm{S},\mathrm{S}']} \cdot \lhd_{[\mathrm{T},\mathrm{T}']}$ is the interval between $\lhd_{[\mathrm{S},\mathrm{S}']} \backslash \lhd_{[\mathrm{T},\mathrm{T}']}$ and $\lhd_{[\mathrm{S},\mathrm{S}']} / \lhd_{[\mathrm{T},\mathrm{T}']}$ in the weak order on $\mathsf{WOIP}_{m+n}$.*

The same construction for faces recovers Chapoton's Schröder trees Hopf Algebra [2].

**Proposition 3.9.** • *The vector subspace $\mathbb{K}\mathsf{TOFP}_{\mathbb{N}}$ of $\mathbb{K}\mathsf{WOFP}_{\mathbb{N}}$ generated by the sums of the fibers $\{\lhd \in \mathsf{WOFP}_{\mathbb{N}} \mid \lhd^{\mathsf{TOIPd}} = \blacktriangleleft\}$ for all $\blacktriangleleft \in \mathsf{TOFP}_{\mathbb{N}}$ is stable by $\cdot$ and $\triangle$.*

- *For any Schröder trees $\sigma[S]$ on $[m]$ and $\mathrm{T}$ on $[n]$, the product $\lhd_{\mathrm{S}} \cdot \lhd_{\mathrm{T}}$ is the interval between $\lhd_{\mathrm{S}} \backslash \lhd_{\mathrm{T}}$ and $\lhd_{\mathrm{S}} / \lhd_{\mathrm{T}}$ in the weak order on $\mathsf{WOFP}_{m+n}$.*
- *The map $\mathrm{S} \mapsto \lhd_{\mathrm{S}}$ is a Hopf algebra isomorphism from the Chapoton algebra on Schröder trees [2] to $(\mathbb{K}\mathsf{TOFP}_{\mathbb{N}}, \cdot, \triangle)$.*

Finally, let us mention that similar ideas can be used to uniformly construct Hopf algebra structures on permutrees, permutree intervals, and Schröder permutrees. Following [3, 13], one first defines some decorated versions of the Hopf algebras $\mathbb{K}\mathsf{WOEP}_{\mathbb{N}}$, $\mathbb{K}\mathsf{WOIP}_{\mathbb{N}}$ and $\mathbb{K}\mathsf{WOFP}_{\mathbb{N}}$, where each poset on $[n]$ appears $4^n$ times with all possible different orientations. One then constructs Hopf algebras on $\mathbb{K}\mathsf{PEP}$, $\mathbb{K}\mathsf{PIP}$ and $\mathbb{K}\mathsf{PFP}$ using the fibers of the surjective map $(\lhd, \mathbb{O}) \mapsto \lhd^{\mathsf{PIP}_{\mathbb{O}}\mathsf{d}}$. See [13, 14] for details.

# References


[1] A. Björner and M.L. Wachs. “Permutation statistics and linear extensions of posets”. *J. Combin. Theory Ser. A* **58**.1 (1991), pp. 85–114. DOI: 10.1016/0097-3165(91)90075-R.

[2] F. Chapoton. “Algèbres de Hopf des permutahèdres, associahèdres et hypercubes”. *Adv. Math.* **150**.2 (2000), pp. 264–275. DOI: 10.1006/aima.1999.1868.

[3] G. Châtel and V. Pilaud. “Cambrian Hopf Algebras”. *Adv. Math.* **311** (2017), pp. 598–633.

[4] G. Châtel, V. Pilaud, and V. Pons. “The weak order on integer posets”. 2017. arXiv: 1701.07995.

[5] A. Dermenjian, C. Hohlweg, and V. Pilaud. “The facial weak order and its lattice quotients”. *Trans. Amer. Math. Soc.* **370**.2 (2018), pp. 1469–1507. DOI: 10.1090/tran/7307.

[6] I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retakh, and J.-Y. Thibon. “Noncommutative symmetric functions”. *Adv. Math.* **112**.2 (1995), pp. 218–348. DOI: 10.1006/aima.1995.1032.

[7] F. Hivert, J.-C. Novelli, and J.-Y. Thibon. “The algebra of binary search trees”. *Theoret. Comput. Sci.* **339**.1 (2005), pp. 129–165. DOI: 10.1016/j.tcs.2005.01.012.

[8] Loday J.-L. “Realization of the Stasheff polytope”. *Arch. Math. (Basel)* **83**.3 (2004), pp. 267–278. DOI: 10.1007/s00013-004-1026-y.

[9] D. Krob, M. Latapy, J.-C. Novelli, Phan H.-D., and Schwer S. “Pseudo-Permutations I: First Combinatorial and Lattice Properties”. 13th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2001). 2001.

[10] J.-L. Loday and M.O. Ronco. “Hopf algebra of the planar binary trees”. *Adv. Math.* **139**.2 (1998), pp. 293–309. DOI: 10.1006/aima.1998.1759.

[11] C. Malvenuto and C. Reutenauer. “Duality between quasi-symmetric functions and the Solomon descent algebra”. *J. Algebra* **177**.3 (1995), pp. 967–982. DOI: 10.1006/jabr.1995.1336.

[12] P. Palacios and M.O. Ronco. “Weak Bruhat order on the set of faces of the permutohedron and the associahedron”. *J. Algebra* **299**.2 (2006), pp. 648–678. URL.

[13] V. Pilaud and V. Pons. “Permutrees”. *Algebraic Combin.* **1**.2 (2018), pp. 173–224. URL.

[14] V. Pilaud and V. Pons. “The Hopf algebra of integer binary relations”. 2018. arXiv: 1807.03277.

[15] N. Reading. “Cambrian lattices”. *Adv. Math.* **205**.2 (2006), pp. 313–353. URL.

[16] N. Reading. “Lattice congruences of the weak order”. *Order* **21**.4 (2004), pp. 315–344. URL.