*Séminaire Lotharingien de Combinatoire* **80B** (2018)
Article #68, 12 pp.

*Proceedings of the 30th Conference on Formal Power Series and Algebraic Combinatorics (Hanover)*

# Plabic graphs and zonotopal tilings (extended abstract)

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**Abstract.** We introduce the notion of *chord separation* of two sets which generalizes Leclerc and Zelevinsky's *weak separation*. We show that every maximal *by inclusion* collection of pairwise chord separated sets is also maximal *by size*. Moreover, we prove that such collections are in bijection with fine zonotopal tilings of the three-dimensional cyclic zonotope. As a result, we get that Postnikov's reduced plabic graphs are precisely the objects dual to horizontal sections of zonotopal tilings of the three-dimensional cyclic zonotope, and Postnikov's moves on plabic graphs correspond to flips of these zonotopal tilings.

**Keywords:** Zonotopal tilings, weak separation, purity phenomenon, plabic graphs, higher Bruhat order

## 1 Introduction

In 1998, Leclerc and Zelevinsky [11] defined the notion of *weak separation* while studying the $q$-deformation of the coordinate ring of the flag variety. They showed that for two subsets $S$ and $T$ of the set $[n] := \{1, 2, \ldots, n\}$, the corresponding quantum flag minors quasicommute if and only if $S$ and $T$ are *weakly separated* which is a certain natural combinatorial condition that we recall below. They raised an exciting *purity conjecture*: every maximal *by inclusion* collection of pairwise weakly separated subsets of $[n]$ is also maximal *by size*. This conjecture has been proven independently by Danilov–Karzanov–Koshevoy [6, 5] and by Oh–Postnikov–Speyer [12]. In particular, Oh, Postnikov, and Speyer showed that maximal by inclusion weakly separated collections of $k$-element sets correspond to reduced *plabic graphs* (see Figure 3 for an example). Plabic graphs have been introduced by Postnikov in [13] where he used them to construct a certain CW decomposition of the totally nonnegative Grassmannian. Since then, these objects played a major role in several seemingly unrelated contexts. For example, plabic graphs have been used as a geometric basis for computing scattering amplitudes for $\mathcal{N} = 4$ supersymmetric Yang–Mills theory [1]. Other applications include soliton solutions to the KP

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equation and cluster algebras [10, 9]. It was shown earlier by Scott [16, 15] that maximal by inclusion weakly separated collections of subsets form clusters in the cluster algebra structure on the coordinate ring of the Grassmannian.

Rather than defining *weak separation* of [11], we will work with a slightly weaker notion of *chord separation*. Namely, we say that two subsets $S, T \subset [n]$ are *chord separated* if there do not exist numbers $a < b < c < d \in [n]$ such that $a, c \in S \setminus T$ and $b, d \in T \setminus S$, or vice versa. For example, the sets $\{2,3,4,6\}$ and $\{2,5\}$ are chord separated, but the sets $\{2,3,4,6\}$ and $\{1,2,5\}$ are not. It is easy to see that for every $a < b \in \mathbb{Z}$, the *cyclic interval* $[a,b] := \{a, a+1, \dots, b-1, b\}$ (where the indices are taken modulo $n$) is chord separated from any other subset of $n$.

When $|S| = |T|$, the two sets are chord separated if and only if they are weakly separated in the sense of [11]. However, if $S$ and $T$ have different sizes then they may be chord separated but not weakly separated. We will only consider weak separation for sets of the same size, therefore we will formulate the results of [11, 12, 6, 5] in the language of chord separation instead.

We denote by $\binom{[n]}{k}$ the collection of all $k$-element subsets of $[n]$, and by $2^{[n]}$ the collection of all subsets of $[n]$. We say that a collection $\mathcal{D} \subset 2^{[n]}$ of subsets of $[n]$ is *chord separated* if any two sets $S, T \in \mathcal{D}$ are chord separated from each other. We can now state the purity phenomenon for chord separation:

**Theorem 1.1** ([12, 6, 5]). *Every maximal* by inclusion *chord separated collection $\mathcal{D} \subset \binom{[n]}{k}$ is also maximal* by size*:*

$$|\mathcal{D}| = k(n-k) + 1.$$

Oh–Postnikov–Speyer proved this theorem by showing that such collections are in a one-to-one correspondence with *reduced plabic graphs* (see Section 2.2 for the definition of the latter):

**Theorem 1.2** ([12]). *There is a bijection $\mathcal{D} \mapsto \Sigma(\mathcal{D})$ between maximal chord separated collections $\mathcal{D} \subset \binom{[n]}{k}$ and* plabic tilings*, that is, planar duals of reduced bipartite plabic graphs.*

The second part of our story involves zonotopal tilings. Zonotopes are Minkowski sums of line segments, and zonotopal tilings of a given zonotope $\mathcal{Z}$ are polytopal subdivisions of $\mathcal{Z}$ into smaller zonotopes. We say that a zonotopal tiling is *fine* (also called *tight* in the literature) if all of its tiles are parallelotopes. Zonotopal tilings have been a popular subject of research for a long time, and have received much attention during the past two decades in the context of the *generalized Baues problem*, see [2], [14], and [3, Section 7.2]. Another famous conjecture, the *Extension space conjecture*, is related to zonotopal tilings via the celebrated Bohne–Dress theorem [4], which gives a correspondence between zonotopal tilings of a given zonotope and one-element liftings of the associated oriented matroid. We are going to be interested in the three-dimensional cyclic zonotope $\mathcal{Z}(n,3)$ that is associated to the *cyclic vector configuration* $\mathbf{C}(n,3)$. The endpoints of vectors in $\mathbf{C}(n,3)$ are the vertices of a convex $n$-gon lying in the affine plane $z = 1$, see Figure 1 (left). The tiles in a fine zonotopal tiling have the form $\tau_{S,\{a,b,c\}}$ for some $S \subset [n]$ and distinct $a,b,c \notin S$, see Figure 1 (middle) and Definition 2.1. Each tile $\tau_{S,\{a,b,c\}}$ intersects the horizontal plane $z = |S| + 1$ by a triangle which we color white. Similarly, it intersects the horizontal plane $z = |S| + 2$ by a triangle which we color black, see Figure 1 (right). Thus given a fine zonotopal tiling $\Delta$ of $\mathcal{Z}(n,3)$, each of its horizontal sections gives a subdivision of a convex $n$-gon $\mathcal{Z}(n,3) \cap \{z = k\}$ into black and white triangles, and the vertices of $\Delta$ are naturally labeled by subsets of $[n]$ (see Figure 2). We denote this collection of subsets by $\mathrm{Vert}(\Delta) \subset 2^{[n]}$.

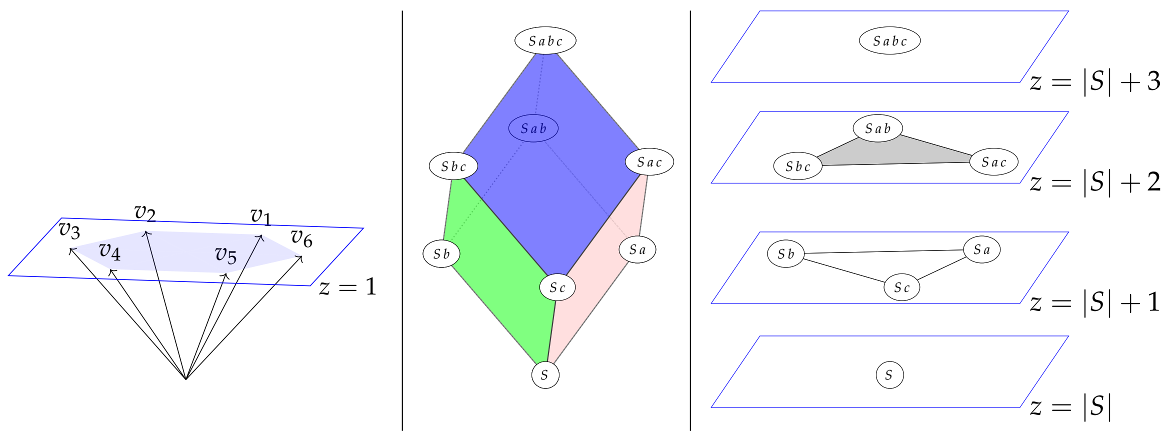



**Figure 1:** A cyclic vector configuration $\mathbf{C}(6,3)$ (left), a tile $\tau_{S,\{a,b,c\}}$ in a fine three-dimensional zonotopal tiling (middle), and its horizontal sections (right).

For a collection $\mathcal{D} \subset 2^{[n]}$ and an integer $0 \le k \le n$, let us denote $\mathcal{D}_k := \mathcal{D} \cap \binom{[n]}{k}$. We are now ready to state our main result.

**Theorem 1.3.**

- *Any maximal* by inclusion *chord separated collection* $\mathcal{D} \subset 2^{[n]}$ *has* size

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}.$$

- *Given any integer* $0 \le k \le n$*, the collection* $\mathcal{D}_k$ *has size* $k(n-k)+1$*, that is, is maximal* by size *inside* $\binom{[n]}{k}$ *and thus corresponds to a plabic tiling* $\Sigma(\mathcal{D}_k)$*.*

- *The map* $\Delta \mapsto \mathrm{Vert}(\Delta)$ *gives a bijection between fine zonotopal tilings of* $\mathcal{Z}(n,3)$ *and maximal by inclusion chord separated collections* $\mathcal{D} \subset 2^{[n]}$*. For any* $0 \le k \le n$*, the intersection of* $\Delta$ *with the plane* $z = k$ *in* $\mathbb{R}^3$ *gives a triangulation of* $\Sigma(\mathcal{D}_k)$ *where* $\mathcal{D} = \mathrm{Vert}(\Delta)$ *is the corresponding chord separated collection.*

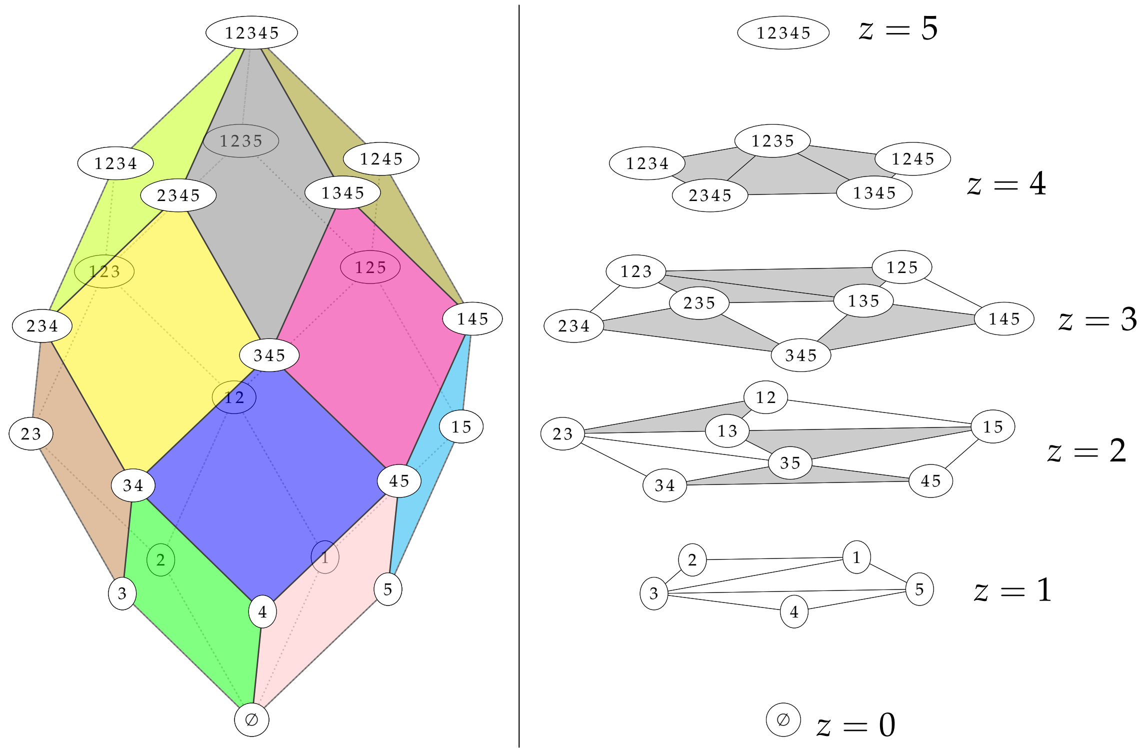



**Figure 2:** The zonotope $\mathcal{Z}(5,3)$ (left). The horizontal sections by the planes $z = 0, 1, \ldots, 5$ of one of its zonotopal tilings $\Delta$ (right). The vertex labels form a maximal by inclusion chord separated collection $\mathcal{D} = \mathrm{Vert}(\Delta)$ of subsets of $\{1,2,3,4,5\}$. Each horizontal section $\Sigma(\mathcal{D}_i)$, $i = 0, 1, \ldots, 5$, is a triangulation of a plabic tiling, dual to a trivalent reduced plabic graph. See Example 1.5.

We thus get the following relationship between plabic graphs and zonotopal tilings:

**Corollary 1.4.**

- *Given a fine zonotopal tiling of $\mathcal{Z}(n,3)$, its horizontal sections are planar duals of reduced trivalent plabic graphs.*

- *Conversely, every reduced trivalent plabic graph is a planar dual of a horizontal section of some fine zonotopal tiling of $\mathcal{Z}(n,3)$.*

We will later see in Section 3 that *moves* on plabic graphs correspond to *flips* of zonotopal tilings and *strands* in plabic graphs correspond to *pseudoplanes* in zonotopal tilings.

**Example 1.5.** Theorem 1.3 is illustrated in Figure 2. Namely, consider a collection $\mathcal{D}$ that contains all of the 22 cyclic intervals (which label the vertices of $\mathcal{Z}(5,3)$ in Figure 2 (left)) together with the sets $\{13, 35, 135, 235\}$. The size of the collection equals

$26 = 1+5+10+10$, and for each $k = 0,1,\dots,5$, the plabic tiling $\Sigma(\mathcal{D}_k)$ is shown in Figure 2 (right). Each plabic tiling is triangulated in such a way that they together form a fine zonotopal tiling $\Delta$ of $\mathcal{Z}(n,3)$: the eight vertices of any tile $\tau_{S,\{a,b,c\}}$ in $\Delta$ are labeled by $S, Sa, Sb, Sc, Sab, Sac, Sbc, Sabc \in \mathcal{D}$, see Figure 1 (middle). Here by $Sa$ we abbreviate $S \cup \{a\}$, etc. The vertices $Sa, Sb, Sc$ form a white triangle in the plabic tiling $\Sigma(\mathcal{D}_{|S|+1})$ and the vertices $Sab, Sbc, Sac$ form a black triangle in $\Sigma(\mathcal{D}_{|S|+2})$. For example, $\{1, 12, 13, 15, 123, 125, 135, 1235\}$ are the vertices of one of the 10 tiles of $\Delta$.

**Remark 1.6.** In [11, Theorem 1.6], Leclerc and Zelevinsky showed that all maximal by inclusion *strongly separated* collections have size

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2}$$

and correspond to pseudoline arrangements, or dually, to zonotopal tilings of the two-dimensional cyclic zonotope. Thus our Theorem 1.3 is a direct three-dimensional analog of their result. More generally, one can extend the notions of strong and chord separation to any oriented matroid $\mathcal{M}$. We studied the purity phenomenon in this setting in our subsequent joint work with Alex Postnikov [8].

This paper is an extended abstract to [7].


# Acknowledgments

The author is grateful to Alex Postnikov for his numerous useful suggestions and ideas.


# 2 Background

## 2.1 Zonotopal tilings

A *vector configuration* $\mathbf{V} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ is a finite subset of $\mathbb{R}^d$. For a vector configuration $\mathbf{V}$, define the corresponding *zonotope* $\mathcal{Z}_{\mathbf{V}}$ to be the Minkowski sum

$$\mathcal{Z}_{\mathbf{V}} := [0, \mathbf{v}_1] + [0, \mathbf{v}_2] + \dots + [0, \mathbf{v}_n],$$

where the *Minkowski sum* of two subsets $A, B \subset \mathbb{R}^d$ is defined by

$$A + B = \{a + b \mid a \in A, b \in B\} \subset \mathbb{R}^d.$$

**Definition 2.1.** For a pair $(S, A)$ of disjoint subsets of $[n]$, we denote by $\tau_{S,A}$ the following zonotope:

$$\tau_{S,A} := \sum_{i \in [n]} \begin{cases} v_i, & \text{if } i \in S; \\ [0, v_i], & \text{if } i \in A; \\ 0, & \text{otherwise.} \end{cases}$$

**Definition 2.2.** A collection $\Delta$ of pairs of disjoint subsets of $[n]$ is called a *zonotopal tiling* of $\mathcal{Z}_{\mathbf{V}}$ if and only if the following two conditions hold:

- $\mathcal{Z}_{\mathbf{V}} = \bigcup_{(S,A)\in\Delta} \tau_{S,A}$;
- For any two pairs $(S, A), (S', A') \in \Delta$, either the intersection $\tau_{S,A} \cap \tau_{S',A'}$ is empty or there exists $(S'', A'') \in \Delta$ such that $\tau_{S'',A''}$ is a face of $\tau_{S,A}$ and of $\tau_{S',A'}$, and
$$\tau_{S,A} \cap \tau_{S',A'} = \tau_{S'',A''}.$$

A zonotopal tiling $\Delta$ is called *fine* if for every $(S, A) \in \Delta$ we have $|A| \leq d$, that is, if all the top-dimensional tiles are parallelotopes. For a fine zonotopal tiling $\Delta$, its *set of vertices* is defined as
$$\mathrm{Vert}(\Delta) := \{S \mid (S, \emptyset) \in \Delta\} \subset 2^{[n]}.$$

For $1 \leq d \leq n$, the *cyclic vector configuration* $\mathbf{C}(n, d) = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subset \mathbb{R}^d$ is given by
$$\mathbf{v}_i = (x_i^{d-1}, \dots, x_i, 1),$$
where $0 < x_1 < x_2 < \cdots < x_n \in \mathbb{R}$ are any increasing positive real numbers.

## 2.2 Plabic graphs

A *planar bicolored graph* (*plabic graph* for short) $G$ is a planar graph embedded in a disc so that every non-boundary vertex of $G$ is colored either black or white. Two adjacent vertices are not required to have opposite colors. Plabic graphs were introduced by Postnikov in [13] in his study of the totally nonnegative Grassmannian.

Given a plabic graph $G$, a *strand* in $G$ is a path that "makes a sharp right turn" at every black vertex and "makes a sharp left turn" at every white vertex. Let $b_1, \dots, b_n$ be the boundary vertices of $G$ in the counterclockwise order. Then the *strand permutation* $\pi_G$ of $[n]$ is defined by $\pi_G(i) = j$ whenever the strand that starts at $b_i$ ends at $b_j$. We denote such a strand by $i \to j$. See Figure 3 for an example.

We say that two strands have an *essential intersection* if there is an edge that they traverse in opposite directions. We say that two strands have a *bad double crossing* if they have essential intersections at two edges $e_1$ and $e_2$ and each of the strands first passes through $e_1$ and then through $e_2$.

**Definition 2.3.** A plabic graph $G$ is called *reduced* (see [13, Theorem 13.2]) if

- there are no closed strands in $G$;
- no strand in $G$ has an essential self-intersection;

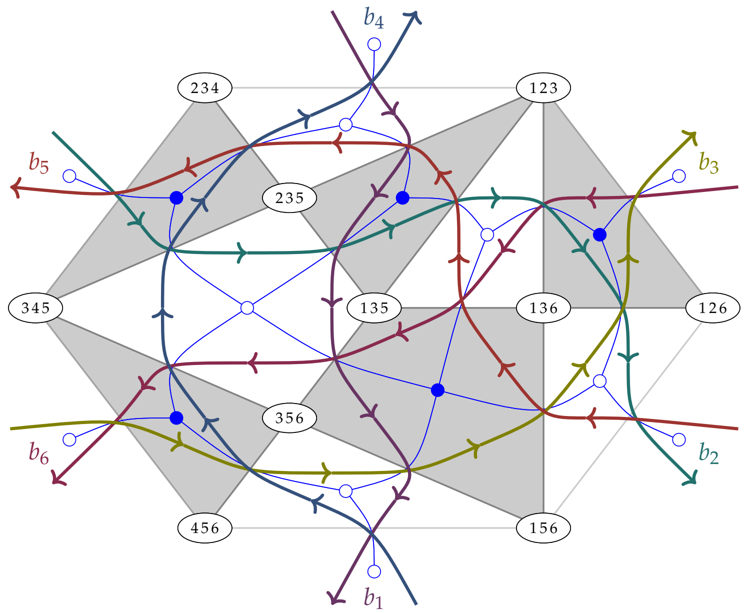



**Figure 3:** A plabic graph $G$ (blue). For each $j \in [n]$, the faces whose labels contain $j$ are to the left of the strand $i \to j$. The collection of face labels is a maximal by inclusion chord separated collection in $\binom{[n]}{k}$. The planar dual of $G$ is a plabic tiling which is a polygonal subdivision of an $n$-gon into black and white polygons.

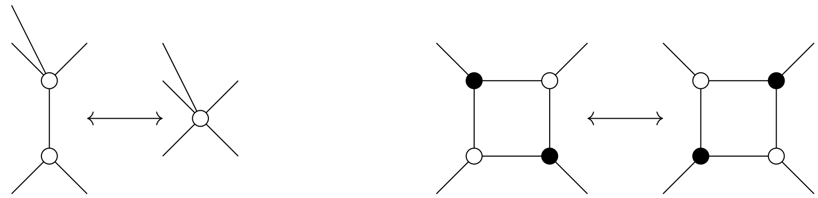
- no two strands in $G$ have a bad double crossing;
- if $\pi_G(i) = i$ then $G$ has a boundary leaf attached to $b_i$.

It follows from this definition that a plabic graph cannot have loops or parallel edges. We will additionally make a minor assumption that it has no vertices of degree two.

We will from now on restrict our attention to reduced plabic graphs whose strand permutation sends $i$ to $i+k$ modulo $n$, for all $i \in [n]$. We denote this permutation by $\sigma^{(k,n)}$. In the stratification of the totally nonnegative Grassmannian from [13], $\sigma^{(k,n)}$ corresponds to the top-dimensional cell of $\mathrm{Gr}_{k,n}^{\mathrm{tnn}}$.

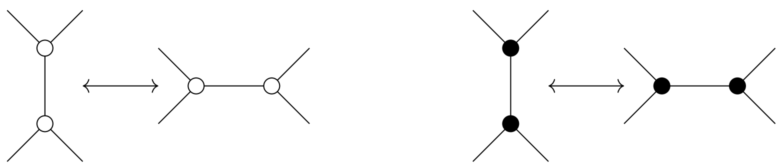
**Remark 2.4.** According to our Theorem 1.3 combined with the results of [12], reduced plabic graphs are precisely the objects dual to horizontal sections of fine zonotopal tilings of $\mathcal{Z}(n,3)$. It is quite surprising that all the conditions from Definition 2.3 are somehow incorporated into the concept of a zonotopal tiling.

As it was shown in [13], all reduced plabic graphs with the same strand permutation are connected by certain moves which we now recall. There are two kinds of moves, *unicolored contraction/uncontraction moves* and *square moves*, see Figure 4. Note that in order to perform a square move, all four vertices are required to have degree 3.



A contraction/uncontraction move A square move (M2)

**Figure 4:** Two types of moves on plabic graphs



A white trivalent move (M1) A black trivalent move (M3)

**Figure 5:** Trivalent contraction-uncontraction moves

Using contraction/uncontraction moves, one can always transform a plabic graph into a *trivalent plabic graph* for which all non-boundary vertices have degree 3. Any two trivalent plabic graphs are connected by trivalent versions of the moves from Figure 4, namely, by square moves and contraction-uncontraction moves from Figure 5. We denote the square move by (M2) and the white (resp., black) trivalent contraction-uncontraction move by (M1) (resp., (M3)).

**Example 2.5.** For $k = 1$ and any $n$, there is only one *bipartite* reduced plabic graph with strand permutation $\sigma^{(1,n)}$. Its planar dual is just a plabic tiling which consists of one white $n$-gon with vertices labeled by sets $\{1\}, \{2\}, \dots, \{n\}$. However, there is Catalan many *trivalent* reduced plabic graphs with this strand permutation, and their planar duals are triangulations of this white $n$-gon into white triangles. We note that the horizontal section of any zonotopal tiling of $\mathcal{Z}(n, 3)$ by the plane $z = 1$ yields a triangulation of an $n$-gon with vertices labeled by sets $\{1\}, \{2\}, \dots, \{n\}$ into white triangles. This is the first non-trivial case of the correspondence described in Theorem 1.3.

**Theorem 2.6** ([13])**.** *All reduced plabic graphs with the same strand permutation are connected by moves from Figure 4. All trivalent reduced plabic graphs with the same strand permutation are connected by moves (M1)–(M3).*

Given a reduced plabic graph $G$, one can associate to it a certain collection $\mathcal{F}(G) \subset 2^{[n]}$ of *face labels* of $G$. Namely, for each face $F$ of $G$, its *label* $\lambda(F) \subset [n]$ contains all indices $j \in [n]$ such that $F$ is to the left of the strand $i \to j$. We set

$$\mathcal{F}(G) = \{\lambda(F) \mid F \text{ is a face of } G\}.$$

Clearly, if faces $F_1$ and $F_2$ share an edge then $\lambda(F_1)$ and $\lambda(F_2)$ have the same size, and thus $\mathcal{F}(G)$ only contains sets of the same size.

**Theorem 2.7** (see [12])**.** *For any reduced plabic graph $G$ with $\pi_G = \sigma^{(k,n)}$, the collection $\mathcal{F}(G)$ is a maximal by inclusion chord separated collection inside $\binom{[n]}{k}$, and conversely, $\mathcal{D} \subset \binom{[n]}{k}$ is a maximal by inclusion chord separated collection if and only if there is a reduced plabic graph $G$ with $\pi_G = \sigma^{(k,n)}$ and $\mathcal{F}(G) = \mathcal{D}$.*

Given that all reduced plabic graphs $G$ with $\pi_G = \sigma^{(k,n)}$ are connected by moves from Figure 4 which do not change the cardinality of $\mathcal{F}(G)$, we get that all maximal *by inclusion* chord separated collections in $\binom{[n]}{k}$ have the same size, so the purity phenomenon (Theorem 1.1) follows from Theorems 2.7 and 2.6.

Figure 3 shows a plabic graph $G$ with $\pi_G = \sigma^{(3,6)}$ together with its face labels and its planar dual (which is a plabic tiling).

# 3 Plabic graphs as sections of zonotopal tilings

In this section, we discuss informally how to view notions related to plabic graphs as sections of three-dimensional objects related to zonotopal tilings.

Let $\Delta$ be a fine zonotopal tiling of $\mathcal{Z}(n,3)$. For $0 \le k \le n$, let $\Delta_k$ denote the horizontal sections of $\Delta$ by hyperplanes $H_k := \{(x,y,z) \mid z = k\} \subset \mathbb{R}^3$. First, as we have already mentioned in Theorem 1.3, the sections $\Delta_k$ are triangulations of plabic tilings corresponding to maximal *by inclusion* chord separated collections inside $\binom{[n]}{k}$. Conversely, every plabic tiling is a horizontal section of some fine zonotopal tiling of $\mathcal{Z}(n,3)$: indeed, the collection of its vertices is contained in some maximal *by inclusion* chord separated collection $\mathcal{D} \subset 2^{[n]}$, and by Theorem 1.3, we have $\mathcal{D} = \mathrm{Vert}(\Delta)$ for some zonotopal tiling $\Delta$ of $\mathcal{Z}(n,3)$. This proves Corollary 1.4.

Recall that the dual object to a triangulation of a plabic tiling is a trivalent reduced plabic graph $G$ such that $\pi_G = \sigma^{(k,n)}$. Thus we get *plabic graphs satisfying $\pi_G = \sigma^{(k,n)}$ as duals of horizontal sections of fine zonotopal tilings $\Delta$ of $\mathcal{Z}(n,3)$.*

Next, the objects dual to fine zonotopal tilings of $\mathcal{Z}(n,3)$ are *three-dimensional pseudoplane arrangements*, which are analogous to the well-studied *pseudoline arrangements* in two dimensions. Loosely speaking, given $\Delta$ and any $j \in [n]$, we get a *pseudoplane $P_j$* which is a smooth embedding of $\mathbb{R}^2$ (together with an orientation) into $\mathbb{R}^3$ that only intersects the tiles $\tau_{S,A}$ of $\Delta$ satisfying $j \in A$, i.e. $P_j$ intersects all segments of $\Delta$ that are parallel to $\mathbf{v}_j$. In other words, the collection of vertex labels of $\Delta$ lying on the *negative* side of $P_j$ consists precisely of those elements of $\mathrm{Vert}(\Delta)$ that do not contain $j$.

The formal definition of a pseudoplane arrangement is somewhat more complicated and involves other topological conditions like the intersection of $P_i$ and $P_j$ being homeomorphic to a line for $i \neq j \in [n]$. We will impose an extra condition that the normal to $P_j$ at any point cannot be vertical for all $j \in [n]$.

The reason we are considering pseudoplanes is that their horizontal sections are strands in plabic graphs. Let $G$ be the plabic graph dual to $\Delta_k$, and let $i = \pi_G^{-1}(j) \in [n]$ so that $G$ has a strand labeled $i \to j$. Then *this strand $i \to j$ can be viewed as the intersection of $P_j$ with $H_k$*. Let us explain this more rigorously. Let $\mathcal{R}, \mathcal{L} \subset \binom{[n]}{k}$ be the collections of sets in $\operatorname{Vert}(\Delta_k)$ lying on the right, resp., left side of the strand $i \to j$. So

$$\mathcal{R} = \{S \in \operatorname{Vert}(\Delta_k) \mid j \notin S\}; \quad \mathcal{L} = \{T \in \operatorname{Vert}(\Delta_k) \mid j \in T\}.$$

From this one easily observes that $\mathcal{R}$, resp., $\mathcal{L}$ are the collections of sets in $\operatorname{Vert}(\Delta_k)$ that are on the negative, resp., positive sides of $P_j$. Finally, by our assumption on the normal of $P_j$ not being vertical, it follows that $P_j \cap \mathcal{Z}(n,3) \cap H_k$ is a simple curve that divides the polygon $\mathcal{Z}(n,3) \cap H_k$ into two parts in essentially the same way as the strand $i \to j$.

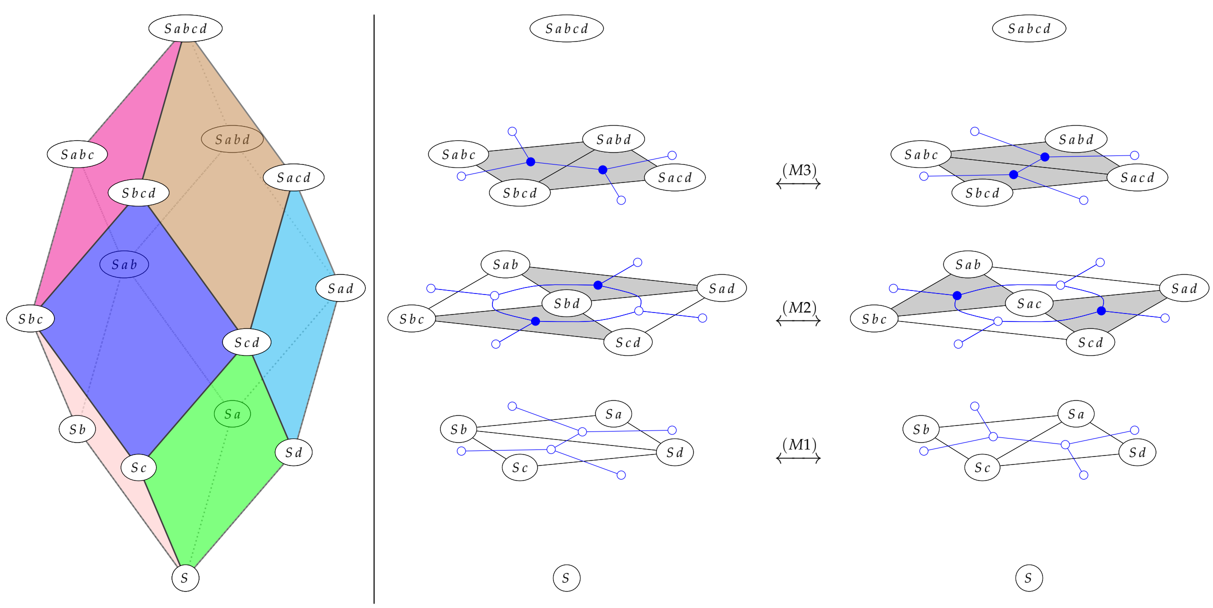



**Figure 6:** The zonotope $\mathcal{Z}(4,3)$ (left) has two fine zonotopal tilings $\Delta, \Delta'$ (right). A flip $\Delta \leftrightarrow \Delta'$ corresponds to performing three simultaneous moves on plabic graphs: (M1) on level $|S|+1$, (M2) on level $|S|+2$, and (M3) on level $|S|+3$.

Finally we consider the well-studied notion of *flips* (also called *mutations*) of zonotopal tilings. It is easy to see that there are only two fine zonotopal tilings of $\mathcal{Z}(4,3)$. Given any two distinct fine zonotopal tilings $\Delta$ and $\Delta'$ of $\mathcal{Z}(n,3)$, we say that they *differ by a flip* if there is a pair $(S, A)$ of disjoint subsets of $[n]$ with $A = \{a, b, c, d\}$ such that for any tile $\tau_{S',A'}$ we have

$$(S', A') \in \Delta \Longleftrightarrow (S', A') \in \Delta'$$

unless $S \subset S' \subset S \cup A$ and $A' \subset A$. This can be reformulated as follows. We see that $\tau_{S,A}$ is a zonotope that is combinatorially equivalent to $\mathcal{Z}(4,3)$, see Figure 6 (left). Hence there are only two fine zonotopal tilings of $\tau_{S,A}$ which we denote $\Delta_0$ and $\Delta_0'$ (the horizontal sections of $\Delta_0$ and $\Delta_0'$ are shown in Figure 6 on the right). Then $\Delta$ and $\Delta'$ differ by a mutation if and only if they coincide on the complement of $\tau_{S,A}$ and their restrictions to $\tau_{S,A}$ are $\Delta_0$ and $\Delta_0'$ respectively.

As Figure 6 suggests, *performing a mutation on $\Delta$ is equivalent to performing moves (M1), (M2), and (M3) on the corresponding plabic graphs*.

**Remark 3.1.** For plabic graphs $G$ with $\pi_G = \sigma^{(k,n)}$, our construction allows to deduce Postnikov's Theorem 2.6 from Ziegler's [17, Theorem 4.1(G)] where he shows that the higher Bruhat order $B(n,k)$ is a graded poset with a unique minimal and maximal elements. Indeed, fine zonotopal tilings of $\mathcal{Z}(n,3)$ correspond to elements of $B(n,2)$, and their mutations correspond to covering relations in $B(n,2)$. We note however that our proof uses the results of [12] which, in turn, rely on Theorem 2.6. Thus we do not give an independent proof of Theorem 2.6.

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