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A parking function interpretation for $\nabla m_{n,1^k}$

Emily Sergel*[1]

[1]*Mathematics Department, University of Pennsylvania, Philadelphia, PA, USA*

Abstract. The modified Macdonald polynomials introduced by Garsia and Haiman (1996) have many remarkable combinatorial properties. One such class of properties involves applying the ∇ operator of Bergeron and Garsia (1999) to basic symmetric functions. The first discovery of this type was the Shuffle Conjecture of Haglund, Haiman, Loehr, Remmel, and Ulyanov (2005), which relates the expression ∇e_n to parking functions. A refinement of this conjecture, called the Compositional Shuffle Conjecture, was introduced by Haglund, Morse, and Zabrocki (2012) and proved by Carlsson and Mellit (2015).

We give a symmetric function identity relating hook monomial symmetric functions to the operators used in the Compositional Shuffle Conjecture. This implies a parking function interpretation for nabla of a hook monomial symmetric function, as well as LLT positivity. We show that our identity is a q-analog of the expansion of a hook monomial into complete homogeneous symmetric functions given by Kulikauskas and Remmel (2006). We use this connection to conjecture a model for expanding ∇m_λ in this way when λ is not a hook.

Keywords: parking functions, Shuffle Conjecture, bi-brick permutations

1 Introduction

In 1988, Macdonald [15] introduced a new basis for the ring of symmetric functions. (See Macdonald [16] for an introduction to symmetric function theory.) Later Garsia and Haiman [5] modified this basis to form the modified Macdonald polynomial basis $\{\widetilde{H}_\mu[X;q,t]\}$. They sought a representation-theoretic interpretation for this basis, which led them to study a number of remarkable S_n bi-modules. Among these was the module of Diagonal Harmonics. They conjectured a formula for its Frobenius characteristic $DH_n[X;q,t]$ and Haiman [10] later proved their conjecture using algebraic geometry. However, this formula is not obviously Schur positive or even polynomial.

Bergeron and Garsia [1] noted that the formula of Garsia and Haiman was very close to the modified Macdonald expansion of e_n. Inspired by this similarity, they defined the linear symmetric function operator ∇, which acts by $\nabla \widetilde{H}_\mu = t^{n(\mu)} q^{n(\mu')} \widetilde{H}_\mu$. In this language, $DH_n[X;q,t] = \nabla e_n$. In [9], Haglund, Haiman, Loehr, Remmel and Ulyanov

*esergel@math.upenn.edu. Emily Sergel was partially supported by NSF grant DMS-1603681.

discovered a combinatorial interpretation for ∇e_n in terms of parking functions (defined below). Their conjecture is known as the Shuffle Conjecture, and was only recently proved.

In 1966, Konheim and Weiss [11] introduced parking functions to study a combinatorial problem involving cars parking on a one-way street. While they thought of parking functions as functions, for our purposes it is more helpful to follow the interpretation introduced by Garsia and Haiman [4]. A Dyck path in the $n \times n$ lattice is a path $(0,0)$ to (n,n) of North and East steps which stays weakly above the line $y = x$. A parking function is a Dyck path with labels $\{1,2,\ldots,n\}$ on North steps which are column-increasing. We write the labels of a parking function in the cell just East of each North step. The labels of a parking function are known as cars. For example, see Figure 1.

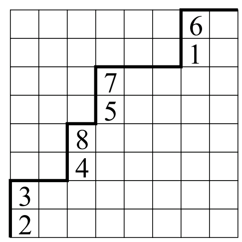

Figure 1: A parking function with 8 cars.

The symmetric function ∇e_n is a weighted sum of parking functions involving three statistics. The most natural of these statistics is the area - the number of full cells between the main diagonal $y = x$ and the underlying Dyck path. In Figure 1, the area is 6. The other two statistics use the notion of diagonals. Let the k-diagonal be the set of cells cut by the line $y = x + k$. In particular, the main diagonal or 0-diagonal consists of the cells cut by $y = x$. In Figure 1, there are 3 cars in the 0-diagonal, 4 cars in the 1-diagonal, and 1 car in the 2-diagonal.

The word σ of a parking function is the permutation obtained by reading cars from highest to lowest diagonal and right to left within each diagonal. In Figure 1, $\sigma = 7\,6\,5\,8\,3\,1\,4\,2$. Recall that the ides of a permutation σ is the descent set of σ^{-1}. Alternatively, it is the set of i so that $i+1$ occurs left of i in σ. In the example, $\text{ides}(\sigma) = \{2,4,5,6\}$. Then each parking function PF is weighted by the quasi-symmetric function $F_{\text{ides}(PF)}$. Here if $S \subset \{1,2,\ldots,n-1\}$, F_S is the following degree-n fundamental quasi-symmetric function defined by Gessel [6].

$$F_S = \sum_{\substack{0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \\ i \in S \Rightarrow a_i < a_{i+1}}} x_{a_1} x_{a_2} \ldots x_{a_n}$$

Finally, the dinv of a parking function counts certain inversions in σ. If two cars are in the same diagonal and the larger occurs further right, we say they create a primary diagonal inversion. If two cars are in adjacent diagonals so that the larger car is higher and further left, they create a secondary diagonal inversion. The dinv of a parking function is the total number of primary and secondary diagonal inversions. In Figure 1, for example, cars 3 and 5 make a primary diagonal inversion, while cars 1 and 3 make a secondary diagonal inversion. In total, there are five primary diagonal inversions and four secondary diagonal inversions in our example. Hence dinv $= 9$.

Let $\mathcal{P}F_n$ be the set of all parking functions on n cars. Then the classical Shuffle Conjecture of Haglund, Morse and Zabrocki [9] states

$$\nabla e_n = \sum_{PF \in \mathcal{P}F_n} t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} F_{\mathrm{ides}(PF)}.$$

In [8], Haglund, Morse and Zabrocki refined the Shuffle Conjecture using the following plethystic symmetric function operators, C_a for non-negative integers a. For any symmetric function P,

$$\mathrm{C}_a\ P[X] \;=\; \left(-\frac{1}{q}\right)^{a-1} P\Big[X - \frac{1-1/q}{z}\Big] \sum_{m \geq 0} z^m h_m[X]\,\Big|_{z^a}$$

where $f|_{z^a}$ is the coefficient of z^a when f is expanded as a formal power series in z. (For an introduction to plethystic notation see Loehr and Remmel [14].) Their refinement of the Shuffle Conjecture, which is stated below, was recently proved by Carlsson and Mellit [2]. Here $\mathrm{comp}(PF)$ is the composition of n giving the distances between points (i,i) on PF's underlying path. For example, the parking function in Figure 1 has comp $= (2,4,2)$.

Theorem 1.1 (Carlsson–Mellit)**.** *For all compositions* $\rho \models n$*,*

$$\nabla\, \mathrm{C}_{\rho_1} \cdots \mathrm{C}_{\rho_k}\, 1 = \sum_{\substack{PF \in \mathcal{P}F_n \\ \mathrm{comp}(PF)=\rho}} t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} F_{\mathrm{ides}(PF)}.$$

We will use the shorthand $\mathrm{C}_\alpha = \mathrm{C}_{\alpha_1} \cdots \mathrm{C}_{\alpha_k}$ for a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$. Haglund, Morse and Zabrocki showed that

$$e_n = \sum_{\alpha \models n} \mathrm{C}_\alpha\, 1.$$

This, together with the Compositional Shuffle Conjecture implies the classical Shuffle Conjecture of Haglund, Haiman, Loehr, Remmel and Ulyanov. In fact, the Compositional Shuffle Conjecture can be used as a tool for finding and proving combinatorial interpretations of images under the ∇ operator. If some symmetric function f can be expanded positively using the C operators applied to 1, then ∇f can be interpreted as a weighted sum of parking functions. Additionally, Haglund, Haiman, Loehr, Remmel and Ulyanov showed that the weighted sum of all parking functions with the same supporting Dyck path can be interpreted as an LLT polynomial. These polynomials, introduced by Lascoux, Leclerc and Thibon [13] are well-studied symmetric functions that are believed to be Schur-positive. Indeed, Grojnowski and Haiman claim to have proved the positivity conjecture in an unpublished manuscript [7]. (It is an open problem to give the Schur expansion of an LLT polynomial.) Hence a positive "C expansion" of f implies the Schur positivity of ∇f. However, the family $\{C_\alpha\, 1\}_{\alpha \models n}$ does not form a basis for the ring of symmetric functions – rather the subcollection $\{C_\lambda\, 1\}_{\lambda \vdash n}$ does. But simply expanding f in terms of this basis may not yield a positive or even polynomial expansion, even when a nice expansion into the full collection does exist.

The remainder of this extended abstract is devoted to the problem of giving a positive polynomial C expansion for the monomial symmetric functions. We prove a nice C expansion for m_λ when λ is a hook shape. This formula interpolates between the expansion of e_n above and the expansion

$$p_n = \sum_{\alpha \models n} [\alpha_n]_q\, C_\alpha\, 1$$

proved by the author and Garsia (see [17]). We also show the connection between our formula and the combinatorial expansion for monomial symmetric functions into the complete homogeneous symmetric functions given by Kulikauskas and Remmel [12]. We conjecture that this connection can be expanded to all partitions λ, which gives a combinatorial model for an expansion of all m_λ into C operators.

2 A C expansion for hook monomials

In this section, we present a positive C expansion for m_λ when λ is a hook shape. Here the coefficient of each C operator is a polynomial in q computed algorithmically. We prove this formula inductively. In the next section, we will see that this polynomial enumerates the bi-brick permutations of Kulikauskas and Remmel [12].

Theorem 2.1. *Let $n \geq 2$ and $k \geq 1$. Then*

$$(-1)^{n-1} m_{n,1^k} = \sum_{a=1}^{n} \left(k+1+\textstyle\sum_{i=1}^{a-1} q^{n-i}\right) \sum_{\tau \models n-a} \sum_{b=0}^{k} \sum_{\rho \models k-b} C_\tau\, C_{a+b}\, C_\rho\, 1$$

Applying ∇ to this identity, together with the Compositional Shuffle Conjecture, gives a parking function interpretation for $(-1)^{n-1}\nabla m_{n,1^k}$.

Corollary 2.2. *Let $n \geq 2$ and $k \geq 1$. Then*

$$(-1)^{n-1}\nabla m_{n,1^k} = \sum_{PF \in \mathcal{PF}_n} \mathrm{qpoly}_{n,k}(PF)\, t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} F_{ides(\sigma(PF))}$$

where $\mathrm{qpoly}_{n,k}(PF)$ *is computed as follows. The constant term is* $k+1$*. If* PF *is not touching the main diagonal at* $(n-1, n-1)$*, add* q^{n-1}*. Otherwise stop. If* PF *is not touching the main diagonal at* $(n-2, n-2)$*, add* q^{n-2}*. Otherwise stop. Continue in this way until you stop or run out of points on the diagonal (this does not include the starting point).*

For example, if $n = 5$ and $k = 3$, then the parking function PF in Figure 1 has $\mathrm{qpoly}_{5,3}(PF) = 4 + q^3 + q^4$. This is because PF does not touch the main diagonal at $(4,4)$ or $(3,3)$ but it does touch at $(2,2)$. Note that $\mathrm{qpoly}_{n,k}(PF)$ does not depend on the placement of the cars in PF, only on the underlying Dyck path. Hence Corollary 2.2 expresses $\nabla m_{n,1^k}$ as a positive sum of LLT polynomials.

Before we sketch the proof of our theorem, we need a lemma. In [8], the authors observe that the C operators are closely related to the Bernstein operators. Let S_a be the operator that send $s_{\lambda_1,\ldots,\lambda_k}$ to $s_{a,\lambda_1,\ldots,\lambda_k}$ for any partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$. Then for any symmetric function P,

$$\mathrm{S}_a\, P[X] = P\left[X - \frac{1}{z}\right] \sum_{m \geq 0} z^m h_m[X]\Big|_{z^a}$$

While they use this relationship to express the S operators in terms of the C operators, the reverse can be accomplished in a similar way. Namely,

$$\mathrm{C}_m = (-1)^{m-1} \sum_{k \geq 0} \frac{\mathrm{S}_{m+k}\, h_k^{\perp}}{q^{m+k-1}}.$$

Here $h_k^{\perp}$ is the adjoint to multiplication by h_k with respect to the usual scalar product. We omit the proof of this identity here. Applying the above to a Schur function gives the necessary lemma.

Lemma 2.3. *For all* $m \geq 0$ *and all partitions* λ*,*

$$q^m\, \mathrm{C}_m\, s_\lambda = \sum_{\mu} \frac{s_{m+|\lambda/\mu|,\mu}}{q^{|\lambda/\mu|-1}}$$

where the sum is over all partitions $\mu \subseteq \lambda$ *for which the diagram* λ/μ *is a horizontal strip (i.e., contains no two boxes in the same column).*

Proof sketch for Theorem 2.1. Note that our identity is equivalent to

$$\begin{aligned}
&(-1)^{n-1} m_{n,1^k} \\
&\quad = (k+1)e_{n+k} + \sum_{i=1}^{n-2} q^i\, \mathrm{C}_i \left((-1)^{n-i+1} m_{n-i,1^k} - (k+1) e_{n-i+k} \right) + q\,[n-1]_q \sum_{i=n}^{n-k} \mathrm{C}_i\, e_{n-i+k} \\
&\quad = (k+1)e_{n+k} + \sum_{i=1}^{n-2} q^i\, \mathrm{C}_i \left((-1)^{n-i+1} m_{n-i,1^k} - (k+1) e_{n-i+k} \right) + (-1)^{n-1} \frac{[n-1]_q}{q^{n-2}} s_{n,1^k}
\end{aligned}$$

Induct on n with k fixed. The above clearly holds when $n = 2$. Now suppose $N > 2$ and that the identity holds for all $n < N$. A combinatorial expansion for any monomial symmetric function into the Schur basis is given by Eğecioğlu and Remmel [3] in terms of rim hook tabloids. However, we only use the special case of a hook here, which appears in [16].

$$\langle m_{n,1^k}, s_\mu \rangle = \begin{cases} (-1)^{n+1}(k+1) & \text{if } \mu = 1^{n+k} \\ (-1)^{n-l} & \text{if } \mu = (l, 2^j, 1^i) \text{ for } j + l \leq n \\ 0 & \text{otherwise.} \end{cases}$$

This, together with the above lemma can be used to compute the Schur expansion of the right hand side. It is then routine (but technical) to show that this matches the Schur expansion of the left hand side. It is slightly easier to see that the right hand side is independent of q. We will see in the next section that at $q = 1$ our identity collapses to the complete homogeneous expansion of $m_{n,1^k}$.

□

3 Bi-brick permutations and a general model for m_λ

We begin with a simple observation brought to our attention during a very productive conversation with Adriano Garsia.

Lemma 3.1. *For any composition α,*

$$\mathrm{C}_\alpha 1\Big|_{q=1} = (-1)^{|\alpha| - l(\alpha)}\, h_\alpha$$

Proof. Note that when $q = 1$, the plethystic shift in the definition of the C operators disappears. Hence we are just left with

$$\mathrm{C}_k f\Big|_{q=1} = (-1)^{k-1}\, h_k f.$$

□

Hence a symmetric function f can only have a positive C-expansion if its expansion into the h-basis $f = \sum_\lambda c_\lambda h_\lambda$ has coefficients c_λ with signs $(-1)^{|\lambda| - \ell(\lambda)}$. If f does have such a C-expansion, then the coefficients of all α rearranging to λ sum to a q-analog of c_λ.

Therefore we are interested in the h-expansion of monomial symmetric functions. Kulikauskas and Remmel [12] give a combinatorial interpretation for the desired expansion. The details are quite technical, but we give a brief overview here along will some illustrative examples. Essentially, the coefficient of h_μ in m_λ counts (with the appropriate sign) what Kulikauskas and Remmel call "bi-brick permutations." These objects are

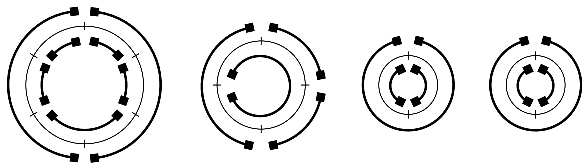

Figure 2: A bi-brick permutation of size 14.

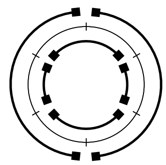

Figure 3: A cycle with rotational symmetry.

analogous to permutations written in cycle notation. They consists of products of cycles decorated inside and outside with "bricks." The lengths of the inner bricks form the parts of λ, while the outer bricks form μ. Rotational symmetry is forbidden. For example, the object in Figure 2 is a bi-brick permutation. The cycle shown in Figure 3, on the other hand, is not allowed in any bi-brick permutation. This is because it is unchanged when rotated 180 degrees.

Bi-brick permutations are in bijection with multisets of Lyndon words in the alphabet $\{B < L < N < U\}$. The bijection is as follows: Consider each cycle individually. Suppose the total length of the cycle is n. Start at any of the n segments with w initialized to the empty word and work clockwise. At each segment, add a letter to the end of w according to which kind of brick(s) start at that segment: B if both kinds start, L if only an outer brick (contributing to λ) starts, U if only an inner brick (contributing to μ) starts, and N if neither kind of brick starts. When the cycle is complete, w will be a word of length n. Since the initial cycle has no rotational symmetry, w is not a power of a shorter word. Hence there is exactly one Lyndon word in its rotational orbit. Take this Lyndon word to the be image of the given cycle. This process is clearly invertible since the letters of the word describe the starting cycle. Applying this process to each cycle of the bi-brick permutation in Figure 2 gives $\{BUULUU, LLLU, BU, BU\}$. However, for the cycle in Figure 3 we obtain $LUULUU$ which has no Lyndon word in its rotational orbit. This is due to the rotational symmetry of the starting cycle.

For any bi-brick permutation Π, let $\mu(\Pi)$ be the partition of the lengths of Π's inner bricks. We also associate a composition to Π whose parts are a reordering of the lengths of Π's outer bricks. That is, for a cycle C of length n whose Lyndon word contains a B, let $\alpha(C) = (i_2 - i_1, i_3 - i_2, \ldots, n + i_1 - i_k)$ where $i_1, i_2, \ldots, i_k$ are the locations of all B's and L's in C's Lyndon word. (Note: i_1 is always 1.) If C is a cycle whose Lyndon word does

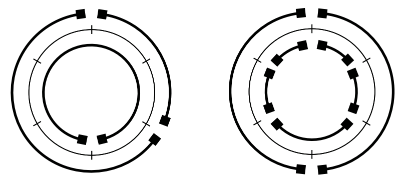

Figure 4: A bi-brick permutation with size 12.

not contain a B, rotate the inner blocks of C clockwise as many times as necessary until the resulting cycle $\widehat{C}$ has a B in its Lyndon word (i.e. until the ends of some inner and outer bricks line up.) Let $\alpha(C) = \alpha(\widehat{C})$. Finally for a bi-brick permutation Π, let $\alpha(\Pi)$ be the composition obtained by concatenating $\alpha(C)$ for all $C \in \Pi$ in reverse lexicographic order of Lyndon words. For example, the Lyndon word of the second cycle of Figure 2 does not contain a B. Rotating gives a new cycle with Lyndon word $BLLN$. Therefore the cycles of Figure 2 have α equal to $(3,3)$, $(1,1,2)$, (2), and (2), respectively, from left to right. Sorting them according to the Lyndon words of the original cycles gives $\alpha(\Pi) = (1,1,2,3,3,2,2)$.

It is important that when a cycle C's Lyndon word does not contain B, the order of α is read according to the Lyndon word of the rotated cycle $\widehat{C}$, yet C's Lyndon word is used for sorting. For example, in Figure 4, the first cycle C has Lyndon word $LNLUNN$ giving $\alpha = (4,2)$ while $\widehat{C}$'s Lyndon word $BNLNNN$ gives $\alpha = (2,4)$. The Lyndon word of the second cycle is $BUULUU$ which lies lexicographically between the Lyndon words of C and $\widehat{C}$. So for this bi-brick permutation Π, we have $\alpha(\Pi) = (2,4,3,3)$.

Based on experimental data, we make the following conjecture.

Conjecture 3.2. *Let μ be any partition. For each bi-brick permutation Π with $\mu(\Pi) = \mu$, there is a composition $\alpha(\Pi)$ and a non-negative integer* $\mathrm{stat}(\Pi)$ *so that*

$$(-1)^{|\mu|-\ell(\mu)} m_\mu = \sum_{\mu(\Pi)=\mu} q^{\mathrm{stat}(\Pi)}\, \mathrm{C}_{\alpha(\Pi)}\, 1.$$

Furthermore, it seems from the data that we can always do this in a way such that

- $\mathrm{stat}(\Pi)$ is the sum of $\mathrm{stat}(C)$ for the individual cycles C of Π (with multiplicity),
- $\mathrm{stat}(C) = 0$ for a single cycle C if and only if the Lyndon word corresponding to C contains the letter B, and
- $\mathrm{stat}(C) < n$ for every cycle C of length n.

We sketch the proof of this conjecture for the case when μ is a hook by showing that qpoly from Section 2 enumerates certain bi-brick permutations.

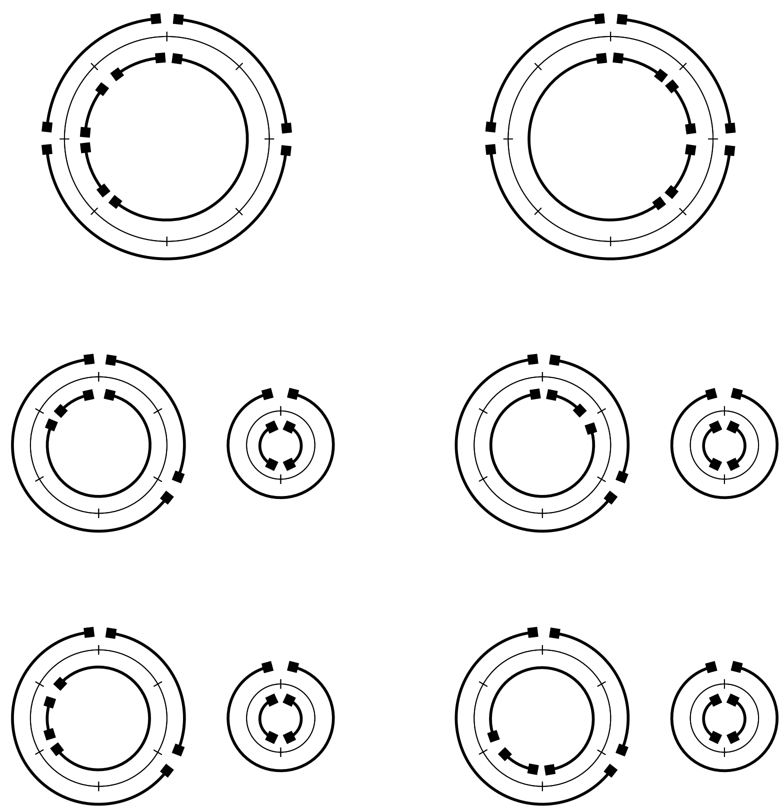

Figure 5: The bi-brick permutations corresponding to Figure 1 when $n = 5$ and $k = 3$.

Suppose that $\mu = (n, 1^k)$ and let α be any composition of size $n+k$. First we note that (up to rotation) there is only one way to arrange the inner bricks of size μ along a cycle of size $n+k$. Starting with this arrangement of inner bricks, choose any of the $k+1$ segments at which an inner brick starts. From this point, add outer bricks of sizes $\alpha_1, \alpha_2, \ldots$ in a clockwise fashion. Now look at the word in the alphabet $\{B < L < N < U\}$ obtained by reading clockwise from the previously chosen starting point. It may or may not be a Lyndon word. But we claim there is always a unique way to chop it into Lyndon words so that the pieces are in reverse lexicographic order. Chop it like so and take the bi-brick permutation corresponding to this multiset of Lyndon words. For example, when $\mu = (5, 1^3)$ and $\alpha = (2, 4, 2)$, we obtain the first four bi-brick permutations of Figure 5. Clearly each of these $k+1$ bi-brick permutations Π have $\alpha(\Pi) = \alpha$ and $\mu(\Pi) = \mu$. Furthermore, each cycle's Lyndon word contains a B. This is because the original large cycle's (not necessarily Lyndon) word started with B, each cut was only made before another B.

We claim that these are the only bi-brick permutations Π satisfying $\alpha(\Pi) = \alpha$ and $\mu(\Pi) = \mu$ in which each cycle's Lyndon word contains a B. We also claim that all the remaining bi-brick permutations Π with $\alpha(\Pi) = \alpha$ and $\mu(\Pi) = \mu$ are obtained by rotating the inner bricks of a single bi-brick permutation. In particular, let r and s be as small as possible so that $\sum_{i=1}^{r} \alpha_i = n + s$. Then the desired bi-brick permutation's inner brick of size n will lie on a cycle with outer bricks of size $\alpha_1, \ldots, \alpha_r$, all of whose s inner bricks of size 1 will be underneath the outer brick corresponding to α_r. Call this the main cycle. The remaining outer bricks of sizes $\alpha_{r+1}, \ldots, \alpha_{\ell(\alpha)}$ will each have it's own cycle with inner bricks all of size 1. Then starting with this special bi-brick permutation, we will rotate the inner bricks of the main cycle so that all s inner bricks of size 1 still lie underneath the outer brick corresponding to α_r. This gives the remaining terms of $\mathrm{qpoly}_{n,k}$. Continuing our previous example, we obtain the last two bi-brick permutations of Figure 5.

In the above algorithm, the implied value of stat is the number of segments covered by the inner brick of size n before (clockwise) the outer brick corresponding to α_r starts. Various modifications of this statistic have been tried for non-hook shapes without success. Already the shapes $\mu = (3, 2)$ and $\mu = (2, 2, 1)$ are troublesome.

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