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Hypergraphic polytopes: combinatorial properties and antipode

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Abstract. Given a hypergraph G, its hypergraphic polytope P_G is the Minkowski sum of simplices corresponding to each hyperedge of G. Using a notion of orientation on G, we prove that the faces of P_G are in bijective correspondence with acyclic orientations of G. This allows us to give a geometric understanding of the antipode in a cocommutative Hopf algebra of hypergraphs. We also give a characterization of when a hypergraphic polytope is a simple polytope. The correspondence between faces and acyclic orientations is used to prove some combinatorial properties of nestohedra and generalized Pitman–Stanley polytopes.

Keywords: combinatorial Hopf algebra, hypergraph, generalized permutahedron

1 Introduction

We let $[n] = \{1, 2, \ldots, n\}$ for any positive integer n. The standard basis of $\mathbb{R}^n$ will be denoted by $e_1, e_2, \ldots, e_n$. Let 2^V denote the collection of subsets of a finite set V, and let

$$\mathbf{HG}[V] = \left\{ G \subseteq 2^V \mid U \in G \text{ implies } |U| \geq 2 \right\}.$$

An element $G \in \mathbf{HG}[V]$ is a *hypergraph on* V. We will write $\mathbf{HG}[n]$ in place of $\mathbf{HG}[[n]]$. For any $U \subseteq [n]$ we let $\Delta_U \subset \mathbb{R}^n$ denote the simplex which is the convex hull of the vectors $\{e_i : i \in U\}$. Given any hypergraph G on $[n]$ we define the *hypergraphic polytope* associated to G to be the Minkowski sum

$$P_G := \sum_{U \in G} \Delta_U.$$

In the special case that G is a simple graph, the hypergraphic polytope is the *graphic zonotope*. In general, hypergraphic polytopes belong to a class of polytopes known as *generalized permutahedra* [8]. Consider the following example.

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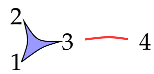
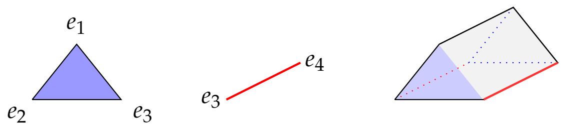
Example 1.1. For the hypergraph $G =$ 

, we have



Δ_{123} $\quad$ Δ_{34} $\quad$ $P_G = \Delta_{123} + \Delta_{34}$

which is a 3-dimensional polytope in $\mathbb{R}^4$.

Our work here is motivated by recent developments in combinatorial Hopf algebras and monoids. Cancellation free formulas for the antipodes in Hopf algebras of both graphs and simplicial complexes have been given [2, 3, 7]. In each case the antipode formula is given in terms of acyclic orientations. Aguiar and Ardila have considered a Hopf monoid of generalized permutahedra and provided a formula for the antipode in this Hopf monoid [1]. The antipode in the Hopf monoid of generalized permutahedra be used to recover the known antipode formulas for graphs and simplicial complexes.[1]

In [4] the first two authors have given an antipode formula in a cocommutative Hopf monoid of hypergraphs in terms of a notion of acyclic orientations. Moreover, it is shown that understanding the antipode for hypergraphs leads to understanding the antipode for a large class of Hopf monoids known as *linearized Hopf monoids*. However, the antipode formula is not completely cancellation free. One of our main results is providing a new geometric understanding of the antipode in terms of faces of hypergraphic polytopes. This allows for the coefficients of the antipode of a hypergraph G to be computed as the Euler characteristic of certain faces of P_G.

In this extended abstract we will exposit various results on hypergraphic polytopes. Our main tool will be a correspondence between faces of the polytope and acyclic orientations. This tool will allow us to give an interpretation of the antipode in a Hopf algebra on hypergraphs in terms of the hypergraphic polytope. The correspondence between faces and acyclic orientations can be used to obtain combinatorial information about hypergraphic polytopes. We are able to use it to give a proof that *nestohedra* are simple polytopes. We also use the correspondence to compute the f-vectors of a family of polytopes generalizing the *Pitman–Stanley polytope*.

2 Acyclic orientations and faces

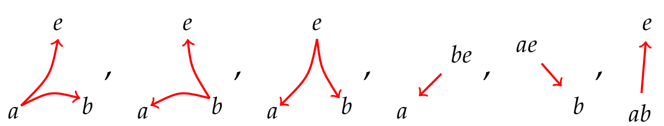
In this section we will give the correspondence between faces and acyclic orientations. We first recall some definitions from [4].

[1]To avoid confusion we remark that the Hopf algebra on hypergraphs we define in Section 3 is a cocommutative Hopf algebra. In [1, Section 20] a different Hopf structure, which is not cocommutative, on hypergraphs is considered.

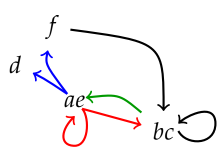
Definition 2.1 (Orientation)**.** Given a hypergraph G, an *orientation* $(\mathfrak{a}, \mathfrak{b})$ of a hyperedge $U \in G$ is an ordered set partition $(\mathfrak{a}, \mathfrak{b})$ of U. We will refer to $\mathfrak{a}$ as the *head* of the orientation. If $|U| = n$, then there are a total of $2^n - 2$ possible orientations. An *orientation of G* is an orientation of all its hyperedges. Given an orientation $\mathcal{O}$ on G, we say that $(\mathfrak{a}, \mathfrak{b}) \in \mathcal{O}$ if it is the orientation of a hyperedge U in G.

In general, given a hypergraph G on the vertex set V and an orientation $\mathcal{O}$ of G, we construct a directed multigraph $G/\mathcal{O}$ on vertex set $V/\mathcal{O}$ as follows. We let $V/\mathcal{O}$ be the set of equivalence classes of the equivalence relation on V defined by the transitive closure of the relation $a \sim a'$ if $a, a' \in \mathfrak{a}$ for some head $\mathfrak{a}$ of $\mathcal{O}$. For each oriented hyperedge $(\mathfrak{a}, \mathfrak{b})$ of $\mathcal{O}$, we have $|\mathfrak{b}|$ oriented edges $([\mathfrak{a}], [b])$ in $G/\mathcal{O}$ where $[\mathfrak{a}], [b] \in V/\mathcal{O}$ are equivalence classes and $b \in \mathfrak{b}$. Let us now consider an example.

Example 2.2. With $G = \big\{\{b,c\}, \{a,b,e\}, \{a,d,e,f\}, \{b,c,e\}, \{f,c\}\big\}$, we can orient the edge $U = \{a,b,e\}$ in $2^3 - 2 = 6$ different ways; three with a head of size 1: $(\{a\},\{b,e\})$, $(\{b\},\{a,e\})$, $(\{e\},\{a,b\})$, and three with a head of size 2: $(\{b,e\},\{a\})$, $(\{a,e\},\{b\})$, $(\{a,b\},\{e\})$. We represent this graphically as follows:

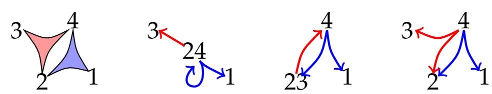


To orient G, we have to make a choice of orientation for each hyperedge. For example we can choose $\mathcal{O} = \big\{(\{b\},\{c\}), (\{a\},\{b,e\}), (\{a,e\},\{d,f\}), (\{b,c\},\{e\}), (\{f\},\{c\})\big\}$ and we represent this as

$G/\mathcal{O} =$ 

Definition 2.3 (Acyclic orientation)**.** An orientation $\mathcal{O}$ of G is *acyclic* if the oriented multigraph $G/\mathcal{O}$ has no cycles.

Example 2.4. Let $G = \big\{\{1,2,4\}, \{2,3,4\}\big\}$ be a hypergraph on $V = \{1,2,3,4\}$. As we can see the orientations $\mathcal{O} = \big\{(\{4\},\{1,2\}), (\{2,4\},\{3\})\big\}$ and $\mathcal{O}' = \big\{(\{4\},\{1,2\}), (\{2,3\},\{4\})\big\}$ are not acyclic, but $\mathcal{O}'' = \big\{(\{4\},\{1,2\}), (\{4\},\{2,3\})\big\}$ is acyclic:



G $\quad G/\mathcal{O}$ $\quad G/\mathcal{O}'$ $\quad G/\mathcal{O}''$

Out of the possible 36 orientations of G only 20 are acyclic:

$\{(\{4\},\{1,2\}),(\{4\},\{2,3\})\}$; $\{(\{4\},\{1,2\}),(\{3\},\{2,4\})\}$; $\{(\{4\},\{1,2\}),(\{3,4\},\{2\})\}$; $\{(\{2\},\{1,4\}),(\{3\},\{2,4\})\}$;
$\{(\{2\},\{1,4\}),(\{2\},\{3,4\})\}$; $\{(\{2\},\{1,4\}),(\{2,3\},\{4\})\}$; $\{(\{1\},\{2,4\}),(\{4\},\{2,3\})\}$; $\{(\{1\},\{2,4\}),(\{3\},\{2,4\})\}$;
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$\{(\{1,2\},\{4\}),(\{3\},\{2,4\})\}$; $\{(\{1,2\},\{4\}),(\{2\},\{3,4\})\}$; $\{(\{1,2\},\{4\}),(\{2,3\},\{4\})\}$; $\{(\{1,4\},\{2\}),(\{4\},\{2,3\})\}$;
$\{(\{1,4\},\{2\}),(\{3\},\{2,4\})\}$; $\{(\{1,4\},\{2\}),(\{3,4\},\{2\})\}$; $\{(\{2,4\},\{1\}),(\{3\},\{2,4\})\}$; $\{(\{2,4\},\{1\}),(\{2,4\},\{3\})\}$.

For a hypergraph G of vertex set V we let $\mathfrak{O}(G)$ denote the set of all acyclic orientations of G. Notice that if $|V| = n$, this set decomposes as

$$\mathfrak{O}(G) = \biguplus_{k=0}^{n-1} \mathfrak{O}_k(G)$$

where

$$\mathfrak{O}_k(G) := \{\mathcal{O} \in \mathfrak{O}(G) : |V(G/\mathcal{O})| = |V(G)| - k\}.$$

Also, for any subset $I \subseteq V$ define $G\big|_I = \{U \in G : U \subseteq I\}$ which is the hypergraph obtained by restricting G to I. If $A = A_1/A_2/\cdots/A_\ell$ is a set partition of $[n]$ we define

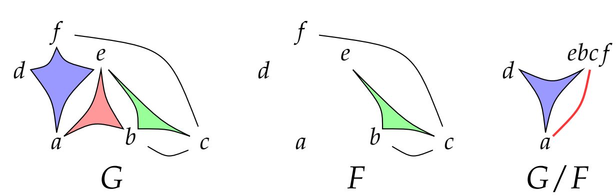
$$G\big|_A := G\big|_{A_1} \uplus G\big|_{A_2} \uplus \cdots \uplus G\big|_{A_\ell}.$$

Definition 2.5 (Flats)**.** For a hypergraph G of vertex set V, given a set partition A of V we say that $G\big|_A$ is a *flat* of G. The set of all flats of G is denoted by $Flats(G)$.

The reader familiar with the lattice of flats of a matroid should observe that in the case G is a simple graph this definition of flat agrees with the definition of flat when considering the graphic matroid. The nonstandard definition of flat used here is natural from the point of view of the Hopf algebra which will be defined in Section 3.

Given a $F \in Flats(G)$, let $A = (A_1, A_2, \ldots, A_\ell)$ be a finest set composition such that $F = G\big|_A$. Any permutation of the parts of A gives the same flat F and the set partition $A_1/A_2/\cdots/A_\ell$ is unique and well defined. We denote by G/F the hypergraph we obtain from G by contracting all the hyperedges in F which is a hypergraph on vertex set $V/F = \{A_1, A_2, \cdots, A_\ell\}$.

Example 2.6. Let $G = \{\{a,d,f,e\},\{a,b,e\},\{b,c,e\},\{c,f\}\}$ be a hypergraph on $V = \{a,b,c,d,e,f\}$ and $F = \{\{b,c,e\},\{c,f\}\} \in Flats(G)$. Then $V/F = \{\{a\},\{b,c,e,f\},\{d\}\}$ and the hypergraph, flat, and contraction can be visualized as:



We are now ready to state the main theorem.

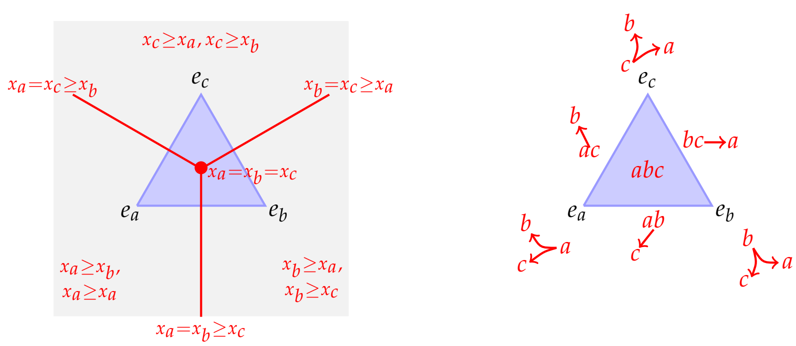
Theorem 2.7. *For any hypergraph G on n vertices, the faces of P_G are indexed by*

$$\bigcup_{F \in Flats(G)} \mathfrak{O}(G/F).$$

Moreover, an orientation $\mathcal{O} \in \mathfrak{O}(G/F)$ where $(G/F)/\mathcal{O}$ has k vertices corresponds to a face of P_G of dimension $n-k$.

The proof of Theorem 2.7 is omitted in this extended abstract and will appear in the full version. The proof works by considering the normal fan to the polytope P_G. We illustrate the theorem now with an example.

Example 2.8. For the hypergraph $G = \{\{a,b,c\}\}$ on $V = \{a,b,c\}$, the hypergraphic polytope P_G is a 2-dimensional simplex in $\mathbb{R}^3$. In this case $Flats(G) = \{\{\}, G\}$. The polytope P_G is shown both with its normal fan, and with its faces labeled by $(G/F)/\mathcal{O}$ for $F \in Flats(G)$ and $\mathcal{O} \in \mathfrak{O}(G)$:



3 A Hopf algebra of hypergraphs

Given two hypergraphs $G, G' \in \mathbf{HG}[V]$, we say the G and G' are isomorphic if there exists a permutation $\sigma \colon V \to V$ such that $G' = \{\sigma(U) \mid U \in G\}$. In this case we write $G \sim G'$. Let H be the graded vector space

$$H = \bigoplus_{n \geq 0} H_n = \bigoplus_{n \geq 0} \mathbb{Q}\mathbf{HG}[n]/_{\sim}.$$

That is, for each $n \geq 0$, we consider $H_n = \mathbb{Q}\mathbf{HG}[n]/_{\sim}$ the linear span of isomorphism classes of hypergraphs on $[n]$. This space has a structure of graded Hopf algebra given by the following operations.

Multiplication: Let $\uparrow_n^m \colon [n] \to \{1+m, \ldots, n+m\}$ be the map that sends $i \in [n]$ to $i+m$. This induces a map from $\mathbf{HG}[n]$ to $\mathbf{HG}[\{1+m, \ldots, n+m\}]$ where

$$G^{\uparrow_n^m} = \{\{i+m : i \in U\} \mid U \in G\}.$$

For all $m, n \geq 0$, we have well defined associative linear operations $\mu_{m,n} \colon H_m \otimes H_n \to H_{m+n}$ given by

$$\mu_{m,n}(G_1 \otimes G_2) = G_1 \cup G_2^{\uparrow_n^m},$$

for $G_1 \in \mathbf{HG}[m]$ and $G_2 \in \mathbf{HG}[n]$. This operation extends to equivalence classes of hypergraphs, in particular, it is commutative since

$$(G_1 \cup G_2^{\uparrow_n^m}) \sim (G_2 \cup G_1^{\uparrow_m^n}).$$

and thus $\mu = \sum_{m,n} \mu_{m,n} \colon H \otimes H \to H$ defines a graded, associative, commutative multiplication on H. The unit u for this operation is given by the unique hypergraph $\emptyset \in \mathbf{HG}[0]$.

Comultiplication: Given $K \subseteq [n]$ let $k = |K|$ and let $St \colon K \to [k]$ be the unique order preserving bijection between K and $[k]$. Given a hypergraph $G \in \mathbf{HG}[n]$ we let

$$G\big|_K = \{U \in G \mid U \subseteq K\} \in \mathbf{HG}[K].$$

We can then use the map St to get a hypergraph $St(G\big|_K) \in \mathbf{HG}[k]$. For all $m, n \geq 0$, we now have a well defined coassociative linear operation $\Delta_{m,n} \colon H_{m+n} \to H_m \otimes H_n$ given by

$$\Delta_{m,n}(G) = \sum_{\substack{K \cup L = [m+n] \\ |K|=m,\ |L|=n}} St(G\big|_K) \otimes St(G\big|_L),$$

for $G \in \mathbf{HG}[m+n]$. This operation is clearly cocommutative. We have that $\Delta = \sum_{m,n} \Delta_{m,n} \colon H \to H \otimes H$ defines a graded, coassociative, cocommutative comultiplication on H. The counit for this operation is given by the map $\epsilon \colon H \to \mathbb{Q}$ defined by

$$\epsilon(G) = \begin{cases} 1 & \text{if } G = \emptyset \in \mathbf{HG}[0], \\ 0 & \text{otherwise.} \end{cases}$$

The structure $(H, \mu, u, \Delta, \epsilon)$ gives a structure of graded, connected, commutative and cocommutative bialgebra on H. In the next section we recall that there is a unique antipode $S \colon H \to H$ that gives a structure of graded, connected, commutative and cocommutative Hopf algebra on H.

For any graded connected bialgebra H the existence of the antipode map $S \colon H \to H$ is guaranteed and it can be computed using Takeuchi's formula [10] as follows. For any $x \in H_n$ with $n \geq 1$,

$$S(x) = \sum_{\alpha \models n} (-1)^{\ell(\alpha)} \mu_\alpha \Delta_\alpha(x). \tag{3.1}$$

Here, for $\ell(\alpha) = 1$, we have $\mu_\alpha = \Delta_\alpha = \mathrm{Id}$ the identity map on H_n, and for $\alpha = (a_1, \ldots, a_k)$ with $k > 1$,

$$\mu_\alpha = \mu_{a_1, n-a_1}(\mathrm{Id} \otimes \mu_{a_2,\ldots,a_k}) \qquad \text{and} \qquad \Delta_\alpha = (\mathrm{Id} \otimes \Delta_{a_2,\ldots,a_k})\Delta_{a_1, n-a_1}.$$

Given a set composition $A \models I$ we get an integer composition using cardinalities: $\alpha(A) = (|A_1|, |A_2|, \ldots, |A_k|) \models |I|$ and $\ell(A) = \ell(\alpha(A))$. In the case of hypergraphs, for $G \in \mathbf{HG}[n]$, the antipode formula gives

$$S(G) = \sum_{A \models [n]} (-1)^{\ell(A)} \mu_{\alpha(A)} \big(St(G|_{A_1}) \otimes \cdots \otimes St(G|_{A_k})\big)$$

But up to a permutation of $[n]$, we have that

$$\mu_{\alpha(A)} \big(St(G|_{A_1}) \otimes \cdots \otimes St(G|_{A_k})\big) \quad \sim \quad G|_{A_1} \cup G|_{A_2} \cup \cdots \cup G|_{A_k}.$$

We denote the right hand side by $G|_A = G|_{A_1} \cup G|_{A_2} \cup \cdots \cup G|_{A_k}$ and the antipode formula in this case is

$$S(G) = \sum_{A \models [n]} (-1)^{\ell(A)} G|_A \tag{3.2}$$

which contains lots of cancellations. In [4] a new formula using acyclic orientations of hypergraphs which refines Equation (3.2) is given. The antipode formula is

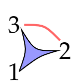
$$S(G) = \sum_{F \in Flats(G)} a(G/F) F$$

where

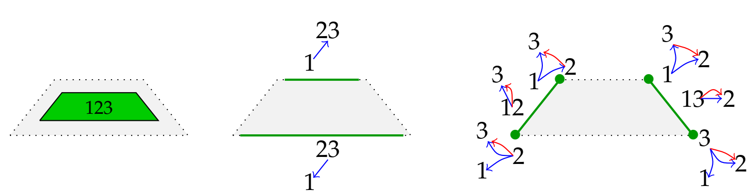
$$a(G/F) = \sum_{\mathcal{O} \in \mathfrak{O}(G/F)} (-1)^{|([n]/F)/\mathcal{O}|}$$

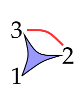
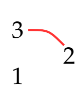
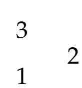
for $G \in \mathbf{HG}[n]$. This refined formula still contains some cancellations. The following corollary, which follows from Theorem 2.7, gives a geometric reason why these cancellations are still present.

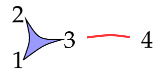
Corollary 3.1. *For a hypergraph $G \in \mathbf{HG}[n]$, the coefficient of a flat $F \in Flats(G)$ in $S(G)$ is $a(G/F)$, and $(-1)^n a(G/F)$ is the Euler characteristic of the complex given by the union of the faces of P_G indexed by the acyclic orientations of G/F.*

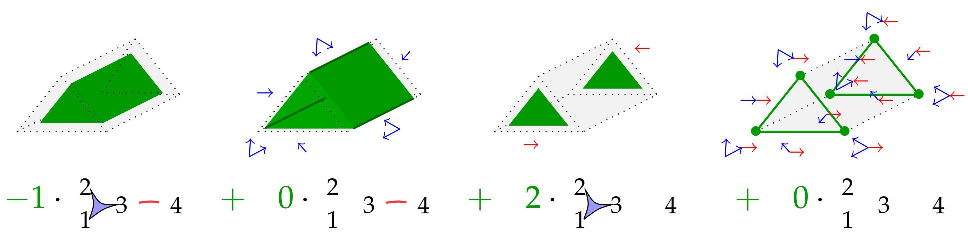
Example 3.2. For $G =$ 

, the flats of G are $G, \{\{2,3\}\}, \varnothing$. The coefficient of each flat F in $S(G)$ is given by the Euler characteristic of the faces of $P_G = \mathbf{\Delta}_{123} + \mathbf{\Delta}_{23}$ indexed by acyclic orientations of G/F:



$S(G) = \quad -1 \cdot$  $\quad + \quad 2 \cdot$  $\quad - \quad 2 \cdot$ 

Example 3.3. For the hypergraph $G =$  in Example 1.1, the flats are of G are G,$\{\{3,4\}\}$,$\{\{1,2,3\}\}$ and $\emptyset$. Thus, we have that $S(G)$ is given by



Antipode formula for a hypergraph G is complicated by the fact that for a given flat F two acyclic orientations $\mathcal{O}$ and $\mathcal{O}'$ of G/F can correspond to faces of different dimensions in P_G. Notice the Example 3.3 shows that not all flats of G need not occur in the antipode as $a(G/F)$ may be zero for a certain $F \in Flats(G)$. This pathology disappears if one only considers hypergraphs G such that $\{i,j\} \in G$ whenever $i,j \in U \in G$ with $i \neq j$. Thus, if one considers simple graphs or abstract simplicial complexes a cancellation free formula for the antipode can be given and each flat will be present with nonzero coefficient. For simple graphs a cancellation free formula was originally given by Humpert and Martin [7]. Using the technique of sign reversing involutions Benedetti and Sagan, as well as Bergeron and Ceballos, were able to give cancellation free formulas of antipodes using acyclic orientations of graphs [3, 5]. In the case of simplicial complexes a cancellation free formula was given by Benedetti, Hallam, and Machacek [2]. Aguiar and Ardila have computed a cancellation free formula for the antipode in the Hopf monoid of generalized permutahedra for which the antipode of graphs and simplicial complexes can be deduced [1].

4 Simple polytopes

For any polytope P let $P^{(1)}$ denote the 1-skeleton of P which is a graph consisting of the 0-dimensional faces and 1-dimensional faces of P. If P is a d-dimensional polytope, then P is called *simple* if and only if $P^{(1)}$ is a d-regular graph. Recall that a graph is d-regular if every vertex is incident on exactly d edges.

By Theorem 2.7 the vertex set of $P_G^{(1)}$ is in one-to-one correspondence with $\mathfrak{O}_0(G)$ and the edge set is in one-to-one correspondence with

$$\mathfrak{O}_1(G) \uplus \biguplus_{\substack{e \in G \\ |e|=2}} \mathfrak{O}_0(G/e).$$

Any directed acyclic graph D determines a poset on the vertices of D, and we let $\text{Hasse}(D)$ denote the *Hasse diagram* of this poset. We will think of $\text{Hasse}(D)$ as a directed acyclic graph which is a subgraph of D containing only the edges which give covering relations in the poset determined by D. In other words, we take $\mathrm{Hasse}(D)$ to be the *transitive reduction* of D.

Lemma 4.1. *Let G be a hypergraph. The vertex corresponding to $\mathcal{O} \in \mathfrak{O}_0(G)$ is incident on the edge corresponding to*

$$\mathcal{O}' \in \mathfrak{O}_1(G) \uplus \biguplus_{\substack{e \in G \\ |e|=2}} \mathfrak{O}_0(G/e)$$

if and only if $\mathrm{Hasse}(G/\mathcal{O}')$ *is obtained from* $\mathrm{Hasse}(G/\mathcal{O})$ *by contracting an edge.*

Proof. In terms of the normal fan of P_G, the acyclic orientation $\mathcal{O}$ corresponds to some cone. Contracting an edge (a, b) of the Hasse diagram replaces an inequality $x_a \geq x_b$ defining this cone with an equality $x_a = x_b$. Contracting an edge of $G/\mathcal{O}$ that is not an edge in $\mathrm{Hasse}(G/\mathcal{O})$ results in a graph which is not acyclic. The lemma then follows from Theorem 2.7 since faces of the polytope P_G, equivalently cones in the normal fan, are in one-to-one correspondence with by acyclic orientations. □

Theorem 4.2. *Let G be a hypergraph. The polytope P_G is a simple polytope if and only if for every $\mathcal{O} \in \mathfrak{O}_0(G)$ the Hasse diagram* $\mathrm{Hasse}(G/\mathcal{O})$ *is a forest.*

Proof. If G has n vertices, then the dimension of P_G is $n - c$ where c is the number of connected components of G. Observe that $G/\mathcal{O}$ will have c connected components for any acyclic orientation $\mathcal{O}$, and hence $\mathrm{Hasse}(G/\mathcal{O})$ will also have c connected components. Now P_G is a simple polytope if and only if each vertex of P_G is incident on exactly $n - c$ edges of P_G. By Lemma 4.1 we know that the edges of the polytope P_G incident to the vertex corresponding to $\mathcal{O} \in \mathfrak{O}_0(G)$ are in bijective correspondence with the edges of $\mathrm{Hasse}(G/\mathcal{O})$. The theorem follows since $\mathrm{Hasse}(G/\mathcal{O})$ has $n - c$ edges if and only if it is a forest. □

4.1 Nestohedra

A *building set* $\mathcal{B}$ on $[n]$ is a collection of nonempty subsets of $[n]$ satisfying the following two conditions

(i) if $I, J \in \mathcal{B}$ and $I \cap J \neq \emptyset$, then $I \cup J \in \mathcal{B}$,

(ii) and $\{i\} \in \mathcal{B}$ for all $i \in [n]$.

The *nestohedron* $P_{\mathcal{B}}$ associated to a building set $\mathcal{B}$ defined to be the Minkowski sum

$$P_{\mathcal{B}} := \sum_{I \in \mathcal{B}} \Delta_I.$$

Given a building set $\mathcal{B}$ we will consider the hypergraph $G_{\mathcal{B}}$ with vertex set $[n]$ and hyperedge set consisting of $I \in \mathcal{B}$ such that $|I| \geq 2$. Note that the hypergraphic polytope $P_{G_{\mathcal{B}}}$ and the nestohedron $P_{\mathcal{B}}$ only differ by translation. Nestohedra are known to be simple polytopes from work of Postnikov [8, Theorem 7.4] as well as Feichtner and Sturmfels [6, Theorem 3.14]. We now provide another proof on this fact using Theorem 4.2.

Proposition 4.3. *Any nestohedron is a simple polytope.*

Proof. Let $\mathcal{B}$ be any building set and let $G = G_{\mathcal{B}}$. We will show for any $\mathcal{O} \in \mathfrak{O}_0(G)$ that $\mathrm{Hasse}(G/\mathcal{O})$ is a forest. The corollary will then follow from Theorem 4.2. In order for $\mathrm{Hasse}(G/\mathcal{O})$ to be a forest, we must not be able to find a cycle in the underlying undirected graph. Assume we could find a cycle in $\mathrm{Hasse}(G/\mathcal{O})$, then we could find vertices a, b, and c such that (a, c) and (b, c) are edges in $\mathrm{Hasse}(G/\mathcal{O})$. This would mean there are hyperedges I, J with $a, c \in I$ and $b, c \in J$ such that a is the source of I and b is the source of J in $\mathcal{O}$. Since $\mathcal{B}$ is a building set, this implies we must also have the hyperedge $I \cup J$. For $\mathcal{O}$ to be acyclic either a or b must be the source of $I \cup J$. However, this means either $a < b$ or $b < a$ which contradicts both (a, c) and (b, c) being edges in the Hasse diagram.

□

4.2 Pitman–Stanley polytopes

For any finite set of positive integers A we define the polytope

$$PS_A := \sum_{a \in A} \Delta_{[a]}.$$

The polytope $PS_{[n]}$ is the *Pitman–Stanley polytope* [9]. A *parking function* of length n is a sequence of nonnegative integers $\alpha = (a_1, a_2, \ldots, a_n)$ such that $b_i \leq i - 1$ where $b_1 \leq b_2 \leq \cdots \leq b_n$ is the increasing rearrangement of α. For any set of nonnegative integers B we define $\mathrm{Park}_{n,B}$ to be the collection of parking functions of length n which are sequences of integers from B. Given a finite set of positive integers A with $n = \max A$, define $\bar{A} := \{n - a : a \in A\}$. Let G be a connected hypergraph on n vertices. It follows from [8, Corollary 9.4] that the normalized volume of an hypergraphic polytope P_G is equal to the number of sequences $(e_1, e_2, \ldots, e_{n-1})$ of hyperedges of G such that $|e_{i_1} \cup e_{i_2} \cup \cdots \cup e_{i_k}| \geq k + 1$ for any distinct $i_i, i_2, \cdots, i_k$.

Proposition 4.4. *Consider $A = \{a_1 < a_2 < \cdots < a_k = n\}$ with $1 < a_1$. The polytope PS_A is a simple polytope with f-vector entries*

$$f_j = \sum_{\substack{(\alpha_1, \alpha_2, \ldots, \alpha_k) \\ 0 \leq \alpha_i \leq a_i - a_{i-1} \\ \alpha_1 + \alpha_2 + \cdots + \alpha_k = j}} \prod_{i=1}^{k} \binom{a_i - a_{i-1} + 1}{\alpha_i + 1}$$

where $a_0 = 1$. The normalized volume of PS_A is given by $\mathrm{Vol}(PS_A) = |\mathrm{Park}_{n-1,\bar{A}}|$.

Proof. Let $G = \{[a_1], [a_2], \cdots, [a_k]\}$. So, $PS_A = P_G$ is a hypergraphic polytope. The polytope PS_A is simple by Proposition 4.3 since G is the hypergraph of a building set.

Flats of G are $F_i = G\big|_{A_i}$ where $A_i = ([a_i], \{a_{i+1}\}, \dots, \{n\})$ for $0 \le i \le k$. The j-faces of $PS_{n,A}$ will correspond to acyclic orientations $\mathcal{O}$ of $G' = G/F$ where F is a flat and $G'/\mathcal{O}$ has $n - j$ vertices. Such orientations $\mathcal{O}$ can be described by sequences of sets $(S_1, S_2, \dots, S_k)$ where $S_i \subseteq ([a_i] \setminus [a_{i-1}]) \cup \{*\}$. Given such a sequence of sets we get an acyclic orientation by declaring the sources of the hyperedge $[a_i]$ to be S_i if $S_i \subseteq [a_i] \setminus [a_{i-1}]$ or otherwise $S_i \cup S_{i+1}$ if $* \in S_i$. If $S_i = [a_i]$ for some i, then let $i^* = \max\{i : S_i = [a_i]\}$. In this case the orientation constructed above is an acyclic orientation of G/F_{i^*}.

If we have an acyclic orientation $\mathcal{O}$ of G/F_{i^*} we use $\overline{[a_i]} = \{i^*, i^*+1, \dots, i\}$ to denote a representation the image of the hyperedge $[a_i]$ in the contraction for any $i > i^*$. We obtain a sequence of sets $(S_1, S_2, \dots, S_k)$ by letting $S_i = [a_i]$ for $1 \le i \le i^*$ and otherwise letting S_i be

- the sources of $\overline{[a_i]}$ in $\mathcal{O}$ if the sources of $\overline{[a_i]}$ are disjoint from the sources of $\overline{[a_{i-1}]}$,
- or alternatively the sources of $\overline{[a_i]}$ in $\mathcal{O}$ along with $*$ if the sources of $\overline{[a_i]}$ are not disjoint from the sources of $\overline{[a_{i-1}]}$.

This process is inverse to the process of constructing an acyclic orientation from a sequence of sets. We have used the fact that if $e \subset f$ are hyperedges, then in any acyclic orientation the sources of f must either contain all sources of e or must be disjoint from them. It is clear that there are

$$\sum_{\substack{(\alpha_1,\alpha_2,\dots,\alpha_k) \\ 0 \le \alpha_i \le a_i - a_{i-1} \\ \alpha_2+\alpha_3+\cdots+\alpha_k = j}} \prod_{i=1}^{k} \binom{a_i - a_{i-1} + 1}{\alpha_i + 1}$$

such sequences of sets. The result on the f-vector follows.

It remains to compute the volume of P_G. The volume of P_G is the number of sequences $(e_1, e_2, \dots, e_{n-1})$ of hyperedges of G such that $|e_{i_1} \cup e_{i_2} \cup \cdots \cup e_{i_k}| \ge k + 1$ for any distinct $i_i, i_2, \cdots, i_k$. We will exhibit a bijection between the set of such sequences and $\mathrm{Park}_{n-1,\bar{A}}$. We claim the map

$$(e_1, e_2, \dots, e_{n-1}) \mapsto (n - |e_1|, n - |e_2|, \dots, n - |e_{n-1}|)$$

gives this desired bijection between the sequences of hyperedges contributing to the volume of P_G and $\mathrm{Park}_{n-1,\bar{A}}$. The inverse map is

$$(a_1, a_2, \cdots, a_{n-1}) \mapsto (e_1, e_2, \dots, e_{n-1})$$

where e_i is the unique hyperedge in G with $|e_i| = n - a_i$. $\square$

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