*Séminaire Lotharingien de Combinatoire* **80B** (2018)
Article #73, 12 pp.

*Proceedings of the 30th Conference on Formal Power Series and Algebraic Combinatorics (Hanover)*

# $P$-Partition Generating Function Equivalence of Naturally Labeled Posets

Ricky Ini Liu*[1] and Michael Weselcouch†[1]

[1] *Department of Mathematics, North Carolina State University*

**Abstract.** The *P-partition generating function* of a (naturally labeled) poset $P$ is a quasisymmetric function enumerating order-preserving maps from $P$ to $\mathbf{Z}^+$. Using the Hopf algebra of posets, we give necessary conditions for two posets to have the same generating function. In particular, we show that they must have the same number of antichains of each size and the same shape (as defined by Greene). We also discuss which shapes guarantee uniqueness of the $P$-partition generating function and give a method of constructing pairs of non-isomorphic posets with the same generating function.

**Keywords:** $P$-Partition, Quasisymmetric Function, Combinatorial Hopf Algebra

# 1 Introduction

For a finite poset $P$ (labeled with the ground set $[n] = \{1, 2, \ldots n\}$), the *P-partition generating function* $K_P(\mathbf{x})$ is a quasisymmetric function enumerating certain order-preserving maps from $P$ to $\mathbf{Z}^+$. The question of when two distinct posets can have the same $P$-partition generating function has been studied extensively in the case of skew Schur functions [2, 9, 10], by McNamara and Ward [8] for general labeled posets, and by Hasebe and Tsujie [7] for rooted trees. The goal of this paper is to consider the naturally labeled case, that is, to give necessary and sufficient conditions for when two naturally labeled posets have the same $P$-partition generating function. (We say that $P$ is *naturally labeled* if $x \preceq_P y$ implies $x \leq y$ as integers.)

In general, it is not true that a poset can be distinguished by its $P$-partition generating function. The smallest case in which two distinct naturally labeled posets have the same $P$-partition generating function is the two 7-element posets shown below. We will explore this example further in Section 5, where we give a general construction for non-isomorphic posets with the same generating function.

*riliu@ncsu.edu. Partially supported by NSF grant DMS-1700302.
†mweselc@ncsu.edu. Partially supported by NSF grant DMS-1700302.

We will use tools from the combinatorial Hopf algebra structure on posets due to Schmitt [11] (see also [1]) to prove that if $K_P(\mathbf{x}) = K_Q(\mathbf{x})$, then for all triples $(k, i, j)$, $P$ and $Q$ must have the same number of $k$-element order ideals that have $i$ maximal elements and whose complement has $j$ minimal elements. As a result of our proof, one can compute certain coefficients in the fundamental quasisymmetric function expansion of $K_P(\mathbf{x})$ explicitly in terms of the number of such ideals.

We will also show that if $K_P(\mathbf{x}) = K_Q(\mathbf{x})$, then $P$ and $Q$ must have the same shape. Here, the *shape* of a finite poset, denoted $\mathrm{sh}(P)$, is the partition $\lambda$ whose conjugate partition $\lambda'$ satisfies

$$\lambda'_1 + \lambda'_2 + \cdots + \lambda'_i = a_i,$$

where $a_i$ is the largest number of elements in a union of $i$ antichains of $P$. In fact, we will prove a stronger statement, namely that if the support of $K_P(\mathbf{x})$ and $K_Q(\mathbf{x})$ in the fundamental quasisymmetric function basis is the same, then $P$ and $Q$ must have the same shape. This suggests the following question: for which partitions $\lambda$ does $\mathrm{sh}(P) = \lambda$ guarantee that $P$ is uniquely determined by $K_P(\mathbf{x})$?

We show that if $\mathrm{sh}(P)$ has at most two parts, is a hook shape, or has the form $\mathrm{sh}(P) = (\lambda_1, 2, 1, \ldots, 1)$, then $K_P(\mathbf{x}) = K_Q(\mathbf{x})$ implies $P \cong Q$. Conversely, we show that if $\mathrm{sh}(P)$ contains $(3, 3, 1)$ or $(2, 2, 2, 2)$, then $K_P(\mathbf{x}) = K_Q(\mathbf{x})$ does not necessarily imply $P \cong Q$ by constructing two distinct posets of this shape with the same generating function. It remains to be answered what happens when $\mathrm{sh}(P) = (\lambda_1, 2, 2, 1, \ldots, 1)$.

In Section 2 we will give some preliminary information; in Section 3 we state some necessary conditions for two posets to have the same generating function; in Section 4 we discuss when the shape of a poset ensures that its generating function is unique; and in Section 5 we give a general construction for pairs of posets with the same generating function.

# 2 Preliminaries

We begin with some preliminaries about posets, quasisymmetric functions, and Hopf algebras. For more information, see [6, 8, 12].

## 2.1 Posets and $P$-partitions

Let $P = (P, \prec)$ be a finite poset. A *labeling* of $P$ is a bijection $\omega \colon P \to \{1, 2, \ldots, n\}$.

**Definition 2.1.** For a labeled poset $(P, \omega)$, a *$(P, \omega)$-partition* is a map $\sigma\colon P \to \mathbf{Z}^+$ that satisfies the following:

(a) If $x \preceq y$, then $\sigma(x) \leq \sigma(y)$.

(b) If $x \preceq y$ and $\omega(x) > \omega(y)$, then $\sigma(x) < \sigma(y)$.

**Definition 2.2.** The *$(P, \omega)$-partition generating function* $K_{(P,\omega)}(x_1, x_2, \dots)$ for a labeled poset $(P, \omega)$ is given by

$$K_{(P,\omega)}(x_1, x_2, \dots) = \sum_{(P,\omega)\text{-partition } \sigma} x_1^{|\sigma^{-1}(1)|} x_2^{|\sigma^{-1}(2)|} \cdots,$$

where the sum ranges over all $(P, \omega)$-partitions $\sigma$.

A labeled poset $(P, \omega)$ is equivalent to a poset $P$ with ground set $[n]$. Hence we may refer to the generating function $K_{(P,\omega)}(x_1, x_2, \dots)$ as $K_P(x_1, x_2, \dots)$ or $K_P(\mathbf{x})$ if the choice of $\omega$ is implicit.

In this paper, we will usually restrict our attention to the case when $P$ is *naturally labeled*, that is, when $\omega$ is an order-preserving map. In this case, $K_P(\mathbf{x})$ does not depend on our choice of natural labeling but only on the underlying structure of $P$.

A *linear extension* of a poset $P$ with ground set $[n]$ is a permutation $\sigma$ of $[n]$ that respects the relations in $P$, that is, if $x \preceq y$, then $\sigma^{-1}(x) \leq \sigma^{-1}(y)$. The set of all linear extensions of $P$ is denoted $\mathcal{L}(P)$.

## 2.2 Compositions

A *composition* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of $n$ is a finite sequence of positive integers summing to $n$. The compositions of $n$ are in bijection with the subsets of $[n-1]$ in the following way: for any composition $\alpha$, define

$$D(\alpha) = \{\alpha_1, \quad \alpha_1 + \alpha_2, \quad \dots, \quad \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\} \subseteq [n-1].$$

Likewise, for any subset $S = \{s_1, s_2, \dots, s_{k-1}\} \subseteq [n-1]$ with $s_1 < s_2 < \cdots < s_{k-1}$, we can define the composition

$$\mathrm{co}(S) = (s_1, \quad s_2 - s_1, \quad s_3 - s_2, \quad \dots, \quad s_{k-1} - s_{k-2}, \quad n - s_{k-1}).$$

## 2.3 Quasisymmetric Functions

A *quasisymmetric function* in the variables $x_1, x_2, \dots$ (with coefficients in $\mathbf{C}$) is a formal power series $f(\mathbf{x}) \in \mathbf{C}[[\mathbf{x}]]$ of bounded degree such that, for any composition $\alpha$, the coefficient of $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_k^{\alpha_k}$ equals the coefficient of $x_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\cdots x_{i_k}^{\alpha_k}$ whenever $i_1 < i_2 < \cdots < i_k$. We denote the algebra of quasisymmetric functions by QSym.

The *fundamental quasisymmetric function basis* $\{L_\alpha\}$ is indexed by compositions $\alpha$ and is given by

$$L_\alpha = \sum_{\substack{i_1 \le \cdots \le i_n \\ i_s < i_{s+1} \text{ if } s \in D(\alpha)}} x_{i_1}x_{i_2}\cdots x_{i_n}.$$

For any labeled poset $P$ (on the ground set $[n]$), $K_P(\mathbf{x})$ is a quasisymmetric function, and we can express it in terms of the fundamental basis $\{L_\alpha\}$ using the linear extensions of $P$. For any linear extension $\sigma \in \mathcal{L}(P)$, define the *descent set* of $\sigma$ to be $\operatorname{des}(\sigma) = \{i \mid \sigma(i) > \sigma(i+1)\}$. We abbreviate $\operatorname{co}(\operatorname{des}(\sigma))$ by $\operatorname{co}(\sigma)$.

**Theorem 2.3** ([4, 12])**.** *Let $P$ be a (labeled) poset on $[n]$. Then*

$$K_P(\mathbf{x}) = \sum_{\sigma \in \mathcal{L}(P)} L_{\operatorname{co}(\sigma)}.$$

In other words, the descent sets of the linear extensions of $P$ determine its $P$-partition generating function.

If $K_P(\mathbf{x}) = \sum_\alpha c_\alpha L_\alpha$, then we define the *support* of $K_P(\mathbf{x})$ to be

$$\operatorname{supp}(K_P(\mathbf{x})) = \{\alpha \mid c_\alpha \neq 0\}.$$

## 2.4 Antichains and Shape

An *antichain* is a subset $A$ of a poset $P$ such that any two elements of $A$ are incomparable. The *width* of $P$ is the size of its longest antichain.

The following Duality Theorem due to Greene allows one to associate to any poset a partition called its *shape*. For $k \ge 0$, let $a_k$ (resp. $c_k$) denote the maximum cardinality of a union of $k$ antichains (resp. chains) in $P$. Let $\lambda_k = c_k - c_{k-1}$ and $\tilde{\lambda}_k = a_k - a_{k-1}$ for $k \ge 1$.

**Theorem 2.4** (Greene [5], [3])**.** *For any finite poset $P$, the sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \dots)$ are weakly decreasing and form conjugate partitions of the number $n = |P|$.*

The partition $\lambda$ is called the *shape* of $P$. Note that the number of nonzero parts in $\operatorname{sh}(P)$ equals the width of $P$.

## 2.5 Poset of Order Ideals

An *order ideal* of a poset $P$ is a subset $I$ such that $x \in I$ implies $y \in I$ for $y \le x$. The set of all order ideals of $P$, ordered by inclusion, forms a poset that we will denote $J(P)$. In fact, $J(P)$ is a finite ranked distributive lattice. The rank of an ideal is the number of elements in the ideal.

Each ideal $I$ in $J(P)$ covers a number of elements equal to the number of maximal elements of $I$, and $I$ is covered by a number of elements equal to the number of minimal elements of $P \setminus I$. Let $\mathrm{anti}_{k,i,j}(P)$ be the number of $k$-element ideals $I$ of $P$ such that $I$ has $i$ maximal elements and such that $P \setminus I$ has $j$ minimal elements. Equivalently, $\mathrm{anti}_{k,i,j}(P)$ is the number of rank $k$ elements of $J(P)$ that cover $i$ elements and are covered by $j$ elements.

If there is only one element of a certain rank in $J(P)$, then $P$ can be expressed as $P = Q \oplus R$, where $\oplus$ is *ordinal sum*. (By definition, in the ordinal sum $P = Q \oplus R$, $x \preceq_P y$ if and only if $x \preceq_Q y$, $x \preceq_R y$, or $x \in Q$ and $y \in R$.)

**Definition 2.5.** A finite poset $P$ is *irreducible* if $P = Q \oplus R$ implies that either $Q \cong P$ or $R \cong P$.

Each poset has a unique (up to isomorphism) ordinal sum decomposition, $P = P_1 \oplus P_2 \oplus \cdots \oplus P_k$ with $P_i$ irreducible and $|P_i| = n_i$ for $i = 1, \ldots, k$. The ranks in which $J(P)$ has exactly one element are $0, n_1, n_1 + n_2, \ldots, n_1 + n_2 + \cdots + n_k$. Linear extensions of $P$ can be broken up into $k$ parts: the first $n_1$ elements form a linear extension of $P_1$, the next $n_2$ elements form a linear extension of $P_2$ and so on. In the case when $P$ is naturally labeled, if elements $a$ and $b$ form a descent in a linear extension of $P$, then $a$ and $b$ must both be in $P_i$ for some $i$. This means that no linear extension of $P$ has a descent in the locations $n_1, n_1 + n_2, \ldots,$ and $n_1 + n_2 + \cdots + n_k$.

**Lemma 2.6.** *Suppose $P$ has an ordinal sum decomposition $P = P_1 \oplus P_2 \oplus \cdots \oplus P_k$ and $Q$ has an ordinal sum decomposition $Q = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_j$. If $K_P(\mathbf{x}) = K_Q(\mathbf{x})$ then $k = j$ and $K_{P_i}(\mathbf{x}) = K_{Q_i}(\mathbf{x})$ for $i = 1, \ldots, k$.*

*Proof.* This follows immediately from the fact that linear extensions can only have descents in their irreducible parts. $\square$

## 2.6 Hopf Algebra

Let $\mathcal{J}$ denote the set of all finite distributive lattices up to isomorphism. The free $\mathbf{C}$-module, $\mathbf{C}[\mathcal{J}]$, whose basis consists of isomorphism classes of distributive lattices $[J] \in \mathcal{J}$, can be given a Hopf algebra structure known as the *reduced incidence Hopf algebra* [11]. Multiplication and comultiplication are defined as follows:

$$\nabla([J_1] \otimes [J_2]) := [J_1 \times J_2],$$
$$\Delta[J] := \sum_{x \in J} [\hat{0}, x] \otimes [x, \hat{1}]$$

where $[a, b] = \{x \in J \mid a \leq x \leq b\}$. In fact, the reduced incidence Hopf algebra can be made into a combinatorial Hopf algebra after choosing an appropriate character. A *combinatorial Hopf algebra* $\mathcal{H}$ is a graded connected Hopf algebra over a field $\mathbf{k}$ equipped with a character (multiplicative linear function) $\zeta\colon \mathcal{H} \to \mathbf{k}$ [1]. We define the character of the reduced incidence Hopf algebra to be the map $\zeta\colon \mathbf{C}[\mathcal{J}] \to \mathbf{C}$ defined on basis elements by $\zeta([J]) = 1$ for all $J$ and extended linearly.

These functions can similarly be defined on the free $\mathbf{C}$-module, $\mathbf{C}[\mathcal{P}]$, whose basis consists of isomorphism classes of posets $P$ in $\mathcal{P}$, the set of all finite posets:

$$\nabla([P_1] \otimes [P_2]) := [P_1 \cup P_2],$$
$$\Delta[P] := \sum_I [I] \otimes [P \setminus I],$$

where the sum runs over all order ideals $I$ of $P$. The corresponding character of $\mathbf{C}[\mathcal{P}]$ is $\zeta_{\mathcal{P}}\colon \mathbf{C}[\mathcal{P}] \to \mathbf{C}$ defined by $\zeta_{\mathcal{P}}(P) = 1$ for all $P$, extended linearly. These functions all commute with the map $J$ that sends $P$ to $J(P)$, so $J$ is a Hopf isomorphism.

We can define the *graded comultiplication* $\Delta_{k,n-k}[P]$ by

$$\Delta_{k,n-k}[P] := \sum_{\substack{I \subseteq P \\ |I|=k}} [I] \otimes [P \setminus I].$$

The map $K\colon \mathcal{P} \to \mathrm{QSym}$ that sends $P$ to the $P$-partition generating function $K_P(\mathbf{x})$ is the unique Hopf morphism that satisfies $\zeta_{\mathcal{P}} = \zeta_{\mathcal{Q}} \circ K$, where the character $\zeta_Q$ for QSym is the linear function that sends $L_n$ to 1 and all other $L_\alpha$ to 0.

# 3 Necessary Conditions

In this section, we will describe various necessary conditions for two naturally labeled posets to have the same partition generating function.

## 3.1 Order ideals and antichains

Recall that $\mathrm{anti}_{k,i,j}(P)$ is defined to be the number of $k$-element order ideals of $P$ that cover $i$ elements in $J(P)$ and are covered by $j$ elements.

**Theorem 3.1.** *If* $K_P(\mathbf{x}) = K_Q(\mathbf{x})$ *then* $\mathrm{anti}_{k,i,j}(P) = \mathrm{anti}_{k,i,j}(Q)$ *for all triples* $k, i, j$.

*Proof sketch.* There exists a linear function $\max_i\colon \mathrm{QSym} \to \mathbf{Z}$ such that

$$\max_i(K_P(\mathbf{x})) = \begin{cases} 1 & \text{if } P \text{ has exactly } i \text{ maximal elements,} \\ 0 & \text{otherwise.} \end{cases}$$

Explicitly, on the fundamental basis $\{L_\alpha\}$,

$$\max_i(L_\alpha) = \begin{cases} (-1)^{(k-i+1)}\binom{k}{i-1} & \text{if } \alpha = (n-k, \underbrace{1, 1, \ldots, 1}_{k}), \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, there exists a linear function $\min_i \colon \mathrm{QSym} \to \mathbf{Z}$ satisfying $\min_i(K_P(\mathbf{x})) = 1$ if $P$ has exactly $i$ minimal elements, and 0 otherwise.

Then

$$\mathrm{anti}_{k,i,j}(P) = ((\max_i \otimes \min_j) \circ \Delta_{k,n-k})(K_P(\mathbf{x})),$$

which depends only on $K_P(\mathbf{x})$. □

This shows that $\mathrm{anti}_{k,i,j}(P)$ can be expressed as a linear combination of the coefficients of the fundamental basis expansion of $K_P(\mathbf{x})$. In fact, if we order the compositions in lexicographic order, then the leading coefficient of $\mathrm{anti}_{k,i,j}(P)$ is $c_{\alpha(k,i,j)}(P)$, where

$$\begin{aligned} \alpha(k,i,j) &= \mathrm{co}([k-i+1, k+j-1] \setminus \{k\}) \\ &= (k-i+1, \underbrace{1,\ldots,1}_{i-2}, 2, \underbrace{1,\ldots,1}_{j-2}, n-k-j-1), \end{aligned}$$

and all coefficients contributing to any $\mathrm{anti}_{k,i,j}(P)$ have this form. One can then deduce the following result.

**Corollary 3.2.** *Let $c_\alpha(P)$ and $c_\alpha(Q)$ denote the coefficient of $L_\alpha$ in $K_P(\mathbf{x})$ and $K_Q(\mathbf{x})$, respectively. Then $\mathrm{anti}_{k,i,j}(P) = \mathrm{anti}_{k,i,j}(Q)$ for all triples $(k,i,j)$ if and only if $c_{\alpha(k,i,j)}(P) = c_{\alpha(k,i,j)}(Q)$ for all triples $(k,i,j)$.*

It follows that an easily counted property of $J(P)$ determines many of the coefficients in the fundamental basis expansion of $K_P(\mathbf{x})$.

We also obtain as a corollary the following result, conjectured by McNamara and Ward [8].

**Corollary 3.3.** *If $K_P(\mathbf{x}) = K_Q(\mathbf{x})$, then $P$ and $Q$ have the same number of antichains of size $i$ for all $i$.*

*Proof.* The number of antichains of size $i$ in $P$ is $\sum_{k,j} \mathrm{anti}_{k,i,j}(P)$. □

## 3.2 Shape

Next, we show that the shape of the poset $P$ is determined by $K_P(\mathbf{x})$, or more specifically, by its support.

**Theorem 3.4.** *If $\mathrm{supp}(K_P(\mathbf{x})) = \mathrm{supp}(K_Q(\mathbf{x}))$, then $\mathrm{sh}(P) = \mathrm{sh}(Q)$.*

*Proof sketch.* Since $P$ is naturally labeled, elements $i < j$ form an antichain in $P$ if and only if there exists a linear extension of $P$ in which $j$ appears immediately before $i$. This means that every descent in a linear extension of $P$ is formed by a 2-element antichain, and similarly, if there is a linear extension of $P$ that has $i$ consecutive descents, then these elements form an $(i+1)$-element antichain in $P$. Hence $P$ has $k$ disjoint antichains of total size $a_k$ if and only if there is a linear extension of $P$ that has $k$ decreasing runs of total size $a_k$, which can be determined from $\mathrm{supp}(K_P(\mathbf{x}))$. □

**Corollary 3.5.** *If* $K_P(\mathbf{x}) = K_Q(\mathbf{x})$, *then* $\mathrm{sh}(P) = \mathrm{sh}(Q)$.

*Proof.* Follows directly from the previous theorem. □

### 3.3 Jump

Let the *jump* of an element $x$, denoted $\mathrm{jump}(x)$, be the maximum number of relations in a saturated chain from $x$ down to a minimal element. McNamara and Ward [8] prove that if two posets have the same partition generating function, then they must have the same number of elements of jump $i$ for any $i$ using the following result.

**Theorem 3.6** ([8, Corollary 5.3])**.** *If $P$ and $Q$ have the same partition generating function, then so do the induced subposets consisting of elements of jump at least $i$.*

A similar argument gives the following result.

**Theorem 3.7.** *If $P$ and $Q$ have partition generating functions with the same support, then so do the induced subposets consisting of elements of jump at least $i$.*

We define the *upward jump* of an element $x$, denoted $\mathrm{up\text{-}jump}(x)$, to be the maximum number of relations in a saturated chain from $x$ up to a maximal element. We then define the *jump pair* of $x$ to be $(\mathrm{jump}(x), \mathrm{up\text{-}jump}(x))$.

**Theorem 3.8.** *If* $\mathrm{supp}(K_P(\mathbf{x})) = \mathrm{supp}(K_Q(\mathbf{x}))$, *then $P$ and $Q$ have the same number of elements with jump pair $(i, j)$ for all $i$ and $j$.*

*Proof sketch.* Let $P_{ij}$ be the induced subposet of $P$ consisting of all elements with jump at least $i$ and up-jump at least $j$. By the previous theorem and its dual, $\mathrm{supp}(K_{P_{ij}}(\mathbf{x}))$ is determined by $\mathrm{supp}(K_P(\mathbf{x}))$, hence so is $|P_{ij}|$. This implies the result since the number of elements with jump pair $(i, j)$ is $|P_{ij}| - |P_{i+1,j}| - |P_{i,j+1}| + |P_{i+1,j+1}|$. □

## 4 Uniqueness from shape

Since Theorem 3.4 shows that posets with the same generating function must have the same shape, one can ask for which shapes is a poset of that shape uniquely determined by its generating function. The next result shows that this is the case when $\lambda$ has at most two parts, that is, for posets of width 2.

**Theorem 4.1.** *Let $P$ and $Q$ be width 2 posets. Then* $K_P(\mathbf{x}) = K_Q(\mathbf{x})$ *if and only if* $P \cong Q$.

*Proof idea.* It is enough to show that the result holds for irreducible width 2 posets. An irreducible width 2 poset must have exactly 2 minimal elements. Suppose $P$ has minimal elements $x_0$ and $y_0$, and let $P' = P \backslash \{x_0, y_0\}$. Then $K_{P'}(\mathbf{x})$ is determined from $K_P(\mathbf{x})$, so

$P'$ is determined up to isomorphism by induction. Thus it remains to determine how the minimal elements of $P$ compare to elements in $P'$.

To do this, we use the operations from the reduced incidence Hopf algebra. For instance, if there is a unique order ideal $I$ of $P$ isomorphic to a chain of size $a$, then

$$K_{P\setminus I}(\mathbf{x}) = (\zeta \otimes id)\Delta_{a,n-a}(K_P(\mathbf{x})) - (\zeta \otimes id)\Delta_{a-2,n-a+2}(K_{P'}(\mathbf{x})),$$

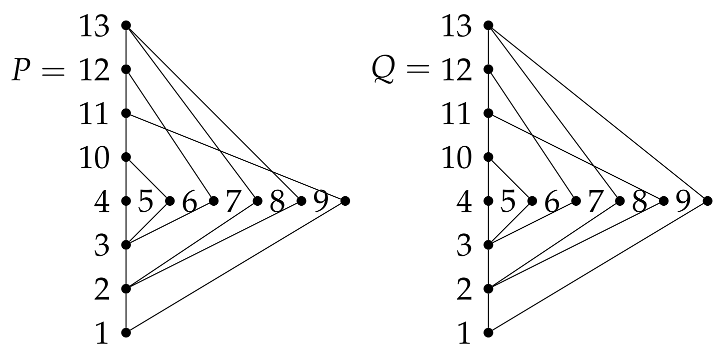
so $P \setminus I$ is determined up to isomorphism by induction. This is typically enough to determine the entire structure of $P$ since we can often choose $a$ so that $P \setminus I$ is the dual order ideal generated by either $x_0$ or $y_0$. (The complete proof involves several cases of this form.) □

A partition $\lambda$ is a *hook* if $\lambda_2 \leq 1$, i.e., $\lambda = (\lambda_1, 1, 1, \ldots, 1)$. If $sh(P)$ is a hook, then we say that the poset $P$ is *hook-shaped*.

**Theorem 4.2.** *If* $\mathrm{sh}(P)$ *is a hook and* $\mathrm{supp}(K_P(\mathbf{x})) = \mathrm{supp}(K_Q(\mathbf{x}))$*, then* $P \cong Q$*.*

*Proof sketch.* If $P$ is hook-shaped, then it is completely determined by the jump pairs of its elements. □

**Example 4.3.** Consider the following two hook shaped posets.




These posets have different generating functions because the element $9 \in P$ has jump 1 and upward jump 3, but no element of $Q$ does.

We also consider posets $P$ whose shape is nearly a hook, namely for which $\mathrm{sh}(P) = (\lambda_1, 2, 1, \ldots, 1)$.

**Theorem 4.4.** *Suppose* $\mathrm{sh}(P) = \mathrm{sh}(Q) = (\lambda_1, 2, 1, \ldots, 1)$*. Then* $K_P(\mathbf{x}) = K_Q(\mathbf{x})$ *if and only if* $P \cong Q$*.*

Although the posets in Theorem 4.4 are very similar to hook-shaped posets, this theorem requires much more care to prove due to various subtleties. For instance, Theorem 4.2 only requires that $\mathrm{supp}(P) = \mathrm{supp}(Q)$ whereas Theorem 4.4 requires that $K_P(\mathbf{x}) = K_Q(\mathbf{x})$. Indeed, there exist pairs of non-isomorphic posets of shape $(\lambda_1, 2, 1, \ldots, 1)$ whose generating functions have the same support (but which are necessarily different).

For most of the remaining shapes, we present a negative result in the next section.

# 5 Posets with the same $P$-partition generating function

In this section, we give a method for constructing distinct posets with the same generating function.

Given a poset $P$ and a pair of incomparable elements $(x, y)$, write $P + (x \prec y)$ for the poset obtained by adding the relation $x \prec y$ to $P$ (and taking the transitive closure).

**Lemma 5.1.** *Suppose $R$ is a finite poset with an automorphism $\phi\colon R \to R$. Let $e = (e_1, e_2)$ and $f = (f_1, f_2)$ be two pairs of incomparable elements of $R$ such that in $R + (f_2 \prec f_1)$, both $e_1 \prec e_2$ and $\phi(e_1) \prec \phi(e_2)$. Then $K_P(\mathbf{x}) = K_Q(\mathbf{x})$, where*

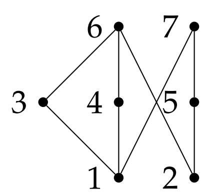
$$P = R + (f_1 \prec f_2) + (e_1 \prec e_2), \text{ and}$$
$$Q = R + (f_1 \prec f_2) + (\phi(e_1) \prec \phi(e_2)),$$

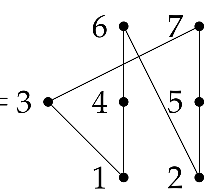
*assuming both are naturally labeled.*

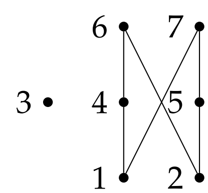
*Proof sketch.* The linear extensions of $P$ are precisely those of $R + (e_1 \prec e_2)$ except for those of $R + (e_1 \prec e_2) + (f_2 \prec f_1) = R + (f_2 \prec f_1)$. Similarly, the linear extensions of $Q$ are those of $R + (\phi(e_1) \prec \phi(e_2)) \cong R + (e_1 \prec e_2)$ except for those of $R + (f_2 \prec f_1)$. Hence $K_P(\mathbf{x})$ and $K_Q(\mathbf{x})$ are both equal to the difference of $K_{R+(e_1 \prec e_2)}(\mathbf{x})$ and $K_{R+(f_2 \prec f_1)}(\mathbf{x})$ (taking care that $R + (f_2 \prec f_1)$ is not naturally labeled). $\square$

(One can also formulate a more general version of this result in which more relations are added.)

**Example 5.2.** Consider the following 7-element posets. The posets $P_7$ and $Q_7$ are not isomorphic but they are $K$-equivalent.

$P_7 =$ 

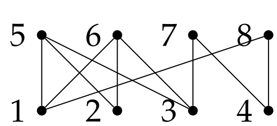
$Q_7 =$ 

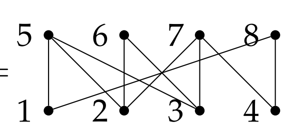
$R =$ 

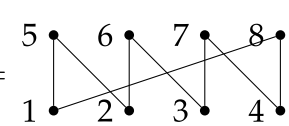

We can express $P_7$ and $Q_7$ in terms of the poset $R$ with a nontrivial automorphism along with some additional covering relations.

The automorphism $\phi$ is the map that fixes 3 and swaps the two chains, $e = (3, 6)$, and $f = (1, 3)$. Note that $\phi(e) = (3, 7)$, and adding the relation $3 \prec 1$ to $R$ makes both $3 \prec 6$ and $3 \prec 7$. By Lemma 5.1, we have $K_{P_7}(\mathbf{x}) = K_{Q_7}(\mathbf{x})$.

**Example 5.3.** Consider the following 8-element posets.

$P_8 =$ 

$Q_8 =$ 

$R =$ 

The automorphism $\phi$ is the permutation $(1234)(5678)$, $e = (1,6)$, $\phi(e) = (2,7)$, and $f = (3,5)$. Adding the relation $5 \prec 3$ to $R$ implies both $1 \prec 6$ and $2 \prec 7$, so once again, Lemma 5.1 implies $K_{P_8}(\mathbf{x}) = K_{Q_8}(\mathbf{x})$.

Observe that the posets in Example 5.2 have shape $(3,3,1)$, and the posets in Example 5.3 have shape $(2,2,2,2)$. We can generalize these examples to make pairs of posets of larger shapes that are $K$-equivalent.

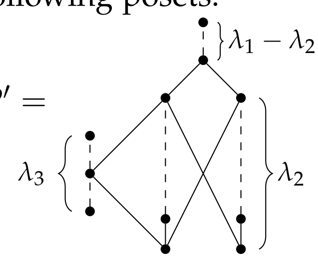
**Theorem 5.4.** *For all partitions $\lambda$ with $\lambda \supset (3,3,1)$ or $\lambda \supset (2,2,2,2)$, there exist posets $P$ and $Q$ such that $P \not\cong Q$, $\mathrm{sh}(P) = \mathrm{sh}(Q) = \lambda$, and $K_P(\mathbf{x}) = K_Q(\mathbf{x})$.*

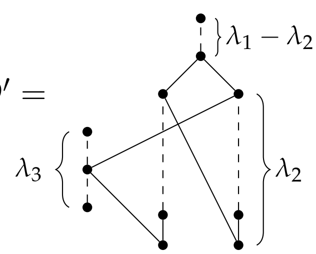
*Proof sketch.* We base our construction off of the posets $P_7$ and $Q_7$ from Example 5.2 and posets $P_8$ and $Q_8$ from Example 5.3. Observe that if $\mathrm{sh}(P) = \mu = (\mu_1, \ldots, \mu_k)$ and $\mathrm{sh}(Q) = \nu = (\nu_1, \ldots, \nu_l)$, then

$$\mathrm{sh}(P \oplus Q) = \mu + \nu = (\mu_1 + \nu_1, \mu_2 + \nu_2, \ldots),$$
$$\mathrm{sh}(P \cup Q) = \mu \cup \nu = (\mu_1' + \nu_1', \mu_2' + \nu_2', \ldots)',$$

(where $\mu'$ and $\nu'$ are the conjugate partitions of $\mu$ and $\nu$, respectively).

Suppose first that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition that contains $(3,3,1)$. Consider the following posets:

$P' =$ 


$Q' =$ 


Since $\lambda_2 \geq 3$, $P' \not\cong Q'$. It follows from Lemma 5.1 that $P'$ and $Q'$ are $K$-equivalent. Note that $\mathrm{sh}(P') = \mathrm{sh}(Q') = (\lambda_1, \lambda_2, \lambda_3)$. Taking the disjoint union of either $P'$ or $Q'$ with disjoint chains of lengths $\lambda_4$, $\lambda_5$, … gives the result.

If $\lambda \supset (2,2,2,2)$, then we can take the ordinal sum of either $P_8$ or $Q_8$ with a union of chains of sizes $\lambda_1 - 2$, …, $\lambda_4 - 2$, and then take the disjoint union with a union of chains of sizes $\lambda_5$, $\lambda_6$, …. □

The only remaining shapes for which it is not known whether there exist non-isomorphic $K$-equivalent posets are those of the form $(\lambda_1, 2, 2, 1, 1, 1, \ldots)$.

**Question 5.5.** Do there exist non-isomorphic posets of shape $\lambda$ with $\lambda_2 = \lambda_3 = 2$ and $\lambda_4 < 2$ that are $K$-equivalent?

# References


[1] M. Aguiar, N. Bergeron, and F. Sottile. “Combinatorial Hopf algebras and generalized Dehn-Sommerville relations”. *Compos. Math.* **142**.1 (2006), pp. 1–30. URL.

[2] L. Billera, H. Thomas, and S. van Willigenburg. “Decomposable compositions, symmetric quasisymmetric functions and equality of ribbon Schur functions”. *Adv. Math.* **204**.1 (2006), pp. 204–240. DOI: 10.1016/j.aim.2005.05.014.

[3] T. Britz and S. Fomin. “Finite posets and Ferrers shapes”. *Adv. Math.* **158**.1 (Mar. 2001), pp. 86–127. DOI: 10.1006/aima.2000.1966.

[4] I. Gessel. “Multipartite $P$-partitions and inner products of skew Schur functions”. *Contemp. Math.* **34** (1984), pp. 289–301. DOI: 10.1090/conm/034.

[5] C. Greene. “Some partitions associated with a partially ordered set”. *J. Combin. Theory Ser. A* **20**.1 (1976), pp. 69–79. DOI: 10.1016/0097-3165(76)90078-9.

[6] D. Grinberg and V. Reiner. “Hopf Algebras in Combinatorics”. 2014. arXiv: 1409.8356.

[7] T. Hasebe and S. Tsujie. “Order Quasisymmetric Functions Distinguish Rooted Trees”. *J. Algebraic Combin.* **46**.3–4 (2017), pp. 499–515. DOI: 10.1007/s10801-017-0761-7.

[8] P. McNamara and R. Ward. “Equality of $P$-Partition Generating Functions”. *Ann. Comb.* **18**.3 (2014), pp. 489–514. DOI: 10.1007/s00026-014-0236-7.

[9] P. McNamara and S. van Willigenburg. “Towards a combinatorial classification of skew Schur functions”. *Trans. Amer. Math. Soc.* **361**.8 (2009), pp. 4437–4470. DOI: 10.1090/S0002-9947-09-04683-2.

[10] V. Reiner, K. Shaw, and S. van Willigenburg. “Coincidences among skew Schur functions”. *Adv. Math.* **216**.1 (2007), pp. 118–152. DOI: 10.1016/j.aim.2007.05.006.

[11] W. Schmitt. “Incidence Hopf algebras”. *J. Pure Appl. Algebra* **96**.3 (1994), pp. 299–330. DOI: 10.1016/0022-4049(94)90105-8.

[12] R. Stanley. *Enumerative combinatorics. Vol. 2*. Vol. 62. Cambridge Studies in Advanced Mathematics. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. Cambridge University Press, Cambridge, 1999, pp. xii+581. DOI: 10.1017/CBO9780511609589.