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The center of the twisted Heisenberg category, factorial P-Schur functions, and transition functions on the Schur graph

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Abstract. We establish an isomorphism between the center $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ of the twisted Heisenberg category of Cautis and Sussan and Γ, the subalgebra of the symmetric functions generated by odd power sums. We give a graphical description of Ivanov's factorial Schur P-functions as closed diagrams in $\mathcal{H}_{tw}$ and show that the curl generators of $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ correspond to two sets of generators of Γ discovered by Petrov which encode data related to up/down transition functions on the Schur graph. Our results are a twisted analogue of those of Kvinge, Licata, and Mitchell, which related the center of Khovanov's Heisenberg category to the algebra of shifted symmetric functions.

Keywords: Symmetric functions, Heisenberg categorification, spin representation theory of symmetric groups

1 Introduction

In [8], Khovanov describes a linear monoidal category $\mathcal{H}$ which conjecturally categorifies the Heisenberg algebra. The morphisms of $\mathcal{H}$ are governed by a graphical calculus of planar diagrams. This category has connections to many interesting areas of representation theory and combinatorics. The center of $\mathcal{H}$, which is the algebra $\mathrm{End}_{\mathcal{H}}(\mathbb{1})$ of endomorphisms of the monoidal identity, was shown in [10] to be isomorphic to the algebra of shifted symmetric functions Λ^* of Okounkov and Olshanskii [12].

A twisted version of Khovanov's Heisenberg category was defined in [3]. The twisted Heisenberg category $\mathcal{H}_{tw}$ is a $\mathbb{C}$-linear additive monoidal category, with an additional $\mathbb{Z}/2\mathbb{Z}$-grading. It conjecturally categorifies the twisted Heisenberg algebra. The center of $\mathcal{H}_{tw}$, $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$, was studied in [13]. There it was shown that as a commutative $\mathbb{C}$-algebra, $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1}) \cong \mathbb{C}[d_0, d_2, d_4, \dots] \cong \mathbb{C}[\bar{d}_2, \bar{d}_4, \bar{d}_6, \dots]$, where d_{2k} and $\bar{d}_{2k}$ correspond

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to clockwise and counterclockwise curls respectively. While symmetric groups played a central role for $\mathcal{H}$ in [8], finite Sergeev superalgebras $\{\mathbb{S}_n\}_{n\geq 0}$ (also known as finite Hecke–Clifford algebras of type A) play the central role for $\mathcal{H}_{tw}$. In particular, Cautis and Sussan construct a family of functors $\{F_n^{\mathcal{H}_{tw}}\}_{n\geq 0}$ from $\mathcal{H}_{tw}$ to bimodule categories of Sergeev algebras in order to categorify the Fock space representation of the twisted Heisenberg algebra. When restricted to $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$, each $F_n^{\mathcal{H}_{tw}}$ can be interpreted as a surjective algebra homomorphism $F_n^{\mathcal{H}_{tw}} : \mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1}) \twoheadrightarrow Z(\mathbb{S}_n)_{\bar{0}}$ where $Z(\mathbb{S}_n)_{\bar{0}}$ is the even center of $\mathbb{S}_n$.

In this paper, we study the combinatorial and representation theoretic properties of $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$. Our main result (Theorem 6.2) is an isomorphism $\varphi : \mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1}) \xrightarrow{\sim} \Gamma$, where Γ is a subalgebra of the algebra of symmetric functions $\Gamma = \mathbb{C}[p_1, p_3, p_5, \ldots]$ (Γ is sometimes known as the algebra of supersymmetric [5] or doubly symmetric [14] functions). The construction of φ relies on the fact that there are embeddings of both $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ and Γ into the algebra of functions on strict partitions, $\mathrm{Fun}(\mathcal{SP}, \mathbb{C})$. In our proof of Theorem 6.2 we identify the images of certain algebraically independent generators of these algebras in $\mathrm{Fun}(\mathcal{SP}, \mathbb{C})$ – the closures of n-cycles from $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ and inhomogeneous analogues of odd power sums $\mathfrak{p}_{(n)}$ in Γ. The latter were first investigated by Ivanov in his study of the asymptotic behavior of characters of projective representations of symmetric groups [5].

In parallel to the surjective homomorphisms $\{F_n^{\mathcal{H}_{tw}}\}_{n\geq 0}$ from $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ to $\{Z(\mathbb{S}_n)_{\bar{0}}\}_{n\geq 0}$, for all $n \geq 0$ one can also construct surjective homomorphisms $F_n^{\Gamma} : \Gamma \twoheadrightarrow Z(\mathbb{S}_n)_{\bar{0}}$ [5]. Our isomorphism φ is canonical in the sense that it intertwines the pair $F_n^{\mathcal{H}_{tw}}$ and F_n^{Γ} for each $n \geq 0$.

$$\begin{tikzcd}
\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1}) \arrow[rr, leftrightarrow, "\varphi"] \arrow[dr, two heads, "F_n^{\mathcal{H}_{tw}}"'] & & \Gamma \arrow[dl, two heads, "F_n^{\Gamma}"] \\
& Z(\mathbb{S}_n)_{\bar{0}} &
\end{tikzcd}$$

One interesting feature of the center of the non-twisted Heisenberg category $\mathcal{H}$ is that as shifted symmetric functions, the curl generators are best understood in terms of moments of Kerov's transition and co-transition measures on Young diagrams [7]. These are tools used to answer probabilistic questions related to the asymptotic representation theory of symmetric groups. In this paper we show that this connection to probability theory extends to the twisted Heisenberg category as well. We identify the clockwise curl generators $\{d_{2k}\}_{k\geq 0}$ and counterclockwise curl generators $\{\bar{d}_{2k}\}_{k\geq 1}$ with two sets of generators for Γ discovered by Petrov [14], $\{\mathbf{g}_k^{\downarrow}\}_{k\geq 0}$ and $\{\mathbf{g}_k^{\uparrow}\}_{k\geq 0}$ respectively. The functions $\{\mathbf{g}_k^{\downarrow}\}_{k\geq 0}$ (respectively $\{\mathbf{g}_k^{\uparrow}\}_{k\geq 0}$) encode the down (resp. up) transition kernels for a Markov process on the graph of all strict partitions (also known as the Schur graph).

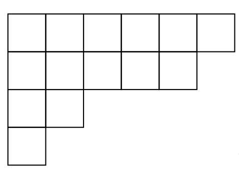
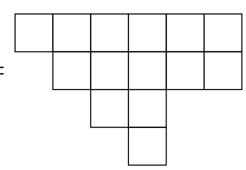
Hence, flipping from up to down (and vice-versa) in the probabilistic setting corresponds to flipping orientation of diagrams in $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$. This seems to be yet another indication of the "planar nature" of structures arising from asymptotic representation theory.

2 Strict Young diagrams and the Schur graph

Let $\mathcal{P}_n$ be the set of all partitions of n and set $\mathcal{P} := \bigcup_{n\geq 0} \mathcal{P}_n$. We freely identify a partition λ with its corresponding Young diagram. If $\lambda \in \mathcal{P}_n$ then we write $|\lambda| = n$. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_t) \in \mathcal{P}$ then we write $\mu \subset \lambda$ when $\mu_i \leq \lambda_i$ for all $i \geq 1$. A partition $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathcal{P}$ is called an *odd partition* if λ_i is odd for all $1 \leq i \leq r$. We denote the collection of odd partitions of n by $\mathcal{OP}_n$ and set $\mathcal{OP} := \bigcup_{n\geq 0} \mathcal{OP}_n$.

We call a partition $\lambda \in \mathcal{P}_n$ *strict* if all its nonzero parts are distinct. Let $\mathcal{SP}_n$ be the set of all strict partitions of n and set $\mathcal{SP} := \bigcup_{n\geq 0} \mathcal{SP}_n$. To a strict partition λ we can associate its *shifted Young diagram* $S(\lambda)$ which is obtained from the usual Young diagram (using English notation) by shifting all rows so that the ith row is shifted rightward by $(i-1)$ cells.

Example 2.1. *Let $\lambda = (6,5,2,1) \in \mathcal{SP}_{14}$. Then the Young diagram and shifted Young diagram, respectively, are given by*

$\lambda \quad =$  , $S(\lambda) \quad =$  .

For $\mu, \lambda \in \mathcal{SP}$, we write $\mu \nearrow \lambda$ (respectively $\mu \searrow \lambda$) when we can obtain λ from μ by adding (resp. removing) a single cell $\square$. Define $\kappa(\mu,\lambda)$ so that $\kappa(\mu,\lambda) = 2$ if $\mu \nearrow \lambda$ and $\ell(\lambda) = \ell(\mu)$, $\kappa(\mu,\lambda) = 1$ if $\mu \nearrow \lambda$ and $\ell(\lambda) = \ell(\mu)+1$, and 0 otherwise.

Definition 2.2. *The* Schur graph $\mathbb{G}$ *is the graded graph with vertex set equal to $\mathcal{SP}$, nth graded component equal to $\mathcal{SP}_n$, and the number of edges from μ to λ equal to $\kappa(\mu,\lambda)$.*

The version of $\mathbb{G}$ that we consider here is the same as that studied in [14].

A *standard shifted Young diagram* of shape $\lambda \in \mathcal{SP}_n$ is a bijective labeling of the cells of $S(\lambda)$ by the integers $\{1, \ldots, n\}$ such that entries increase from left to right across rows and down columns. Let g_λ be the number of standard shifted Young diagrams of shape λ. g_λ can be computed explicitly as

$$g_\lambda = \frac{n!}{\lambda_1!\lambda_2!\ldots\lambda_r!} \prod_{1\leq i\leq j\leq \ell(\lambda)} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}.$$

Following [14] we denote the number of paths from $\emptyset$ to λ in $\mathbb{G}$ by $h(\lambda)$. We have $h(\lambda) = 2^{|\lambda|-\ell(\lambda)} g_\lambda$.

In [2] Borodin and Olshanskii used coherent families of measures on partitions to construct infinite-dimensional diffusion processes. Petrov studied analogous processes on strict partitions [14]. We review some basic definitions from the latter below.

The *down transition function* $p^\downarrow : \mathbb{G} \times \mathbb{G} \to \mathbb{Q}$ on $\mathbb{G}$ is defined so that for $\lambda, \mu \in \mathcal{SP}$,

$$p^\downarrow(\lambda,\mu) := \frac{h(\mu)}{h(\lambda)} \kappa(\mu,\lambda). \tag{2.1}$$

In particular, when restricted to $\mathcal{SP}_n$ the function $p^\downarrow$ gives a Markov transition kernel from $\mathbb{G}_n$ to $\mathbb{G}_{n-1}$.

To $p^\downarrow$ defined in (2.1) and the system of Plancherel measures on $\mathbb{G}$ (see [14]) one can associate up transition functions which take the form

$$p^\uparrow(\mu,\lambda) = \frac{h(\lambda)}{h(\mu)(|\mu|+1)}$$

when $\mu \nearrow \lambda$ and $p^\uparrow(\mu,\lambda) = 0$ otherwise.

In the next section we will make a connection between induction and restriction of simple Sergeev modules and $p^\uparrow$, $p^\downarrow$.

3 The Sergeev algebra

Let S_n be the symmetric group on n elements with $s_1, s_2, \ldots, s_{n-1}$ the Coxeter generators of S_n. Recall that the Clifford algebra $\mathcal{C}\ell_n$ with n generators is the unital associative algebra $\mathcal{C}\ell_n := \mathbb{C}\langle c_1, \ldots, c_n \mid c_i^2 = -1, c_ic_j = -c_jc_i \text{ for } i \neq j\rangle$.

Definition 3.1. *The* finite Sergeev algebra, *(also known as the* finite Hecke–Clifford algebra of type A*) is* $\mathbb{S}_n \cong \mathcal{C}\ell_n \rtimes \mathbb{C}[S_n]$ *where* S_n *acts on the Clifford generators by permuting indices, i.e.* $s_ic_i = c_{i+1}s_i$*,* $s_ic_{i+1} = c_is_i$*, and* $s_ic_j = c_js_i$ *for* $j \neq i,\ i+1$.

$\mathbb{S}_n$ is a superalgebra via the $\mathbb{Z}/2\mathbb{Z}$-grading in which the S_n generators are even and the $\mathcal{C}\ell_n$ generators are odd. For homogeneous element $x \in \mathbb{S}_n$ we write $|x|$ for the degree of x. The Sergeev algebras form a tower of superalgebras via the *standard embedding* $\mathbb{S}_{n-1} \hookrightarrow \mathbb{S}_n$ which sends $s_i \mapsto s_i$ and $c_i \mapsto c_i$. Henceforth when we mention a module of the Sergeev algebra, we mean a supermodule.

$\mathbb{S}_n$ has analogs to the classical Jucys–Murphy elements of $\mathbb{C}[S_n]$. These elements $\{J_i\}_{i=1}^n$, which we also call Jucys–Murphy elements are defined by $J_1 := 0$ and $J_k := \sum_{j=1}^{n-1}(1+c_jc_k)(j,k)$. They generate a commutative subalgebra of $\mathbb{S}_n$ and their spectra have a combinatorial interpretation analogous to that of the classical Jucys–Murphy elements [9]. The algebras $\{\mathbb{S}_n\}_{n\geq 0}$ are semi-simple with a well-studied representation theory (see [9] and [15]).

Theorem 3.2 ([9])**.** *The set of simple $\mathbb{S}_n$-modules are indexed by $\mathcal{SP}_n$.*

Let $\delta : \mathcal{SP} \to \{0,1\}$ be defined by $\delta(\lambda) = \ell(\lambda) \bmod 2$.

Theorem 3.3. *Let $\lambda \in \mathcal{SP}_n$. Then* $\dim(L^\lambda) = 2^{n-\frac{\ell(\lambda)-\delta(\lambda)}{2}} g_\lambda$.

We can now relate $p^\downarrow(\cdot,\cdot)$ and $p^\uparrow(\cdot,\cdot)$ to the representation theory of the algebras $\{\mathbb{S}_n\}_{n\geq 0}$. If N and M are $\mathbb{S}_n$-modules we write $[M:N] := \dim(\mathrm{Hom}_{\mathbb{S}_n}(M,N))$.

Proposition 3.4. *Let $\lambda \in \mathcal{SP}_n$, $\mu \in \mathcal{SP}_{n-1}$. Then*

$$p^\downarrow(\lambda,\mu) = \frac{[\mathrm{Res}^{\mathbb{S}_n}_{\mathbb{S}_{n-1}} L^\lambda : L^\mu]\dim(L^\mu)}{\dim(L^\lambda)} \text{ and } 2^{\delta(\lambda)-\delta(\mu)} p^\uparrow(\mu,\lambda) = \frac{[\mathrm{Ind}^{\mathbb{S}_n}_{\mathbb{S}_{n-1}} L^\mu : L^\lambda]\dim(L^\lambda)}{\dim(\mathrm{Ind}^{\mathbb{S}_n}_{\mathbb{S}_{n-1}} L^\mu)}.$$

For $\lambda \in \mathcal{SP}_n$, we write χ^λ for the character corresponding to the simple $\mathbb{S}_n$-module L^λ. The *normalized character* $\widetilde{\chi}^\lambda$ is defined such that for $x \in \mathbb{S}_n$

$$\widetilde{\chi}^\lambda(x) := \frac{\chi^\lambda(x)}{\chi^\lambda(1)}.$$

3.1 The center of $\mathbb{S}_n$

As a superalgebra the center of $\mathbb{S}_n$ breaks up into even and odd components of supercommutative elements, $Z(\mathbb{S}_n) = Z(\mathbb{S}_n)_{\bar{0}} \oplus Z(\mathbb{S}_n)_{\bar{1}}$. We will focus on $Z(\mathbb{S}_n)_{\bar{0}}$, which corresponds to the center of $\mathbb{S}_n$ after the $(\mathbb{Z}/2\mathbb{Z})$-grading has been forgotten. In [5], Ivanov constructs a basis for $Z(\mathbb{S}_n)_{\bar{0}}$ indexed by $\mathcal{OP}_n$, we denote this basis by $\{C_\mu\}_{\mu\in\mathcal{OP}_n}$. These elements are the analog of conjugacy class sums.

We now define a scaled version of Ivanov's basis which naturally appears from the Fock space representation of the twisted Heisenberg category. For $\mu = (\mu_1,\ldots,\mu_t) \in \mathcal{OP}_k$ and $k \leq n$, define $\sigma_{\mu;n} \in S_n$ to be the permutation

$$\sigma_{\mu;n} := \omega_0(s_1\ldots s_{\mu_1-1})(s_{\mu_1+1}\ldots s_{\mu_1+\mu_2-1})\ldots(s_{k-\mu_t+1}\ldots s_{k-1})\omega_0^{-1}.$$

where ω_0 is the longest permutation in S_n by Coxeter length.

Definition 3.5. *For $k \leq n$ and $\mu \in \mathcal{OP}_k$, define*

$$A_{\mu;n} := \sum_{x\in\mathcal{LC}^n_{n-k}} x\sigma_{\mu;n}x^{-1}.$$

where $\mathcal{LC}^n_{n-k} := \{\, (s_{i_n}\ldots s_{n-1}c_n^{\epsilon_n})(s_{i_{n-1}}\ldots s_{n-2}c_{n-1}^{\epsilon_{n-1}})\ldots(s_{i_{n-k+1}}\ldots s_{n-k}c_{n-k+1}^{\epsilon_{n-k+1}})$
$\mid 1 \leq i_j \leq j,\ \epsilon_j \in \{0,1\}\,\}$.

The set $\mathcal{LC}^n_{n-k}$ consists of minimal left coset representatives of S_{n-k} in S_n multiplied by additional Clifford generators.

Proposition 3.6. *The set $\{A_{\mu;n} \mid \mu \in \mathcal{OP}_n\}$ is a linear basis of $Z(\mathbb{S}_n)_{\bar{0}}$.*

For a simple representation L^λ of $\mathbb{S}_n$, the corresponding normalized character $\tilde{\chi}^\lambda$ is a homomorphism when restricted to $Z(\mathbb{S}_n)_{\bar{0}}$. We have the following formula for the values of these characters on the above basis.

Proposition 3.7. *Let $\mu \in \mathcal{OP}_k$ and $\lambda \in \mathcal{SP}_n$ for $k \leq n$. Then*

$$\widetilde{\chi}^\lambda(A_{\mu;n}) = 2^k n^{\downarrow k} \frac{\chi^\lambda(\sigma_{\mu;n})}{\chi^\lambda(1)}.$$

Another basis for $Z(\mathbb{S}_n)_{\bar{0}}$ is given by the set of central idempotents $\{e_\lambda \mid \lambda \in \mathcal{SP}_n\}$ of $\mathbb{S}_n$ corresponding to the simple $\mathbb{S}_n$-modules.

Lemma 3.8. *For $\lambda \in \mathcal{SP}_n$, the central idempotent $e_\lambda \in \mathbb{S}_n$ corresponding to the simple representation L^λ can be written as*

$$e_\lambda = 2^{\frac{-\ell(\lambda)-\delta(\lambda)}{2}} \frac{g_\lambda}{n!} \sum_{\mu \in \mathcal{OP}_n} \chi^\lambda(\mu) C_\mu.$$

3.2 Interlacing coordinates for strict partitions

In [14] Petrov showed that shifted strict Young diagrams can be parametrized via their interlacing coordinates. For $\lambda \in \mathcal{SP}$, and $\square \in S(\lambda)$ with coordinates (i, j), the *content* of $\square$ is defined to be $\mathrm{cont}(\square) := i - j$. Note that when $\square$ comes from a shifted diagram, $\mathrm{cont}(\square)$ is always nonnegative.

Let $X(\lambda)$ be the set of contents for cells that we can add to $S(\lambda)$ to get another shifted strict partition and let $Y(\lambda)$ be the set of contents for cells that we can remove from $S(\lambda)$ to get another shifted strict partition. The set $(X(\lambda), Y(\lambda))$ uniquely characterizes $S(\lambda)$ and is called the *interlacing coordinates* (or *Kerov coordinates*) of $S(\lambda)$.

For $i \in \mathbb{Z}_{\geq 0}$, set $s(i) := i(i+1)$. In [14] Petrov defined functions

$$\mathbf{g}^\uparrow_k(\lambda) := \sum_{x \in X(\lambda)} p^\uparrow(\lambda, \lambda + \square(x)) s(x)^k$$

and

$$\mathbf{g}^\downarrow_{k+1}(\lambda) := 2|\lambda| \sum_{y \in Y(\lambda)} p^\downarrow(\lambda, \lambda - \square(y)) s(y)^k.$$

Note that $\mathbf{g}^\uparrow_k(\lambda)$ and $\mathbf{g}^\downarrow_k(\lambda)$ are the strict partition analogues to moments of Kerov's transition and co-transition measure [6], and will play a similar role to the one they played in [10].

We now give algebraic interpretations of $\mathbf{g}_k^{\uparrow}(\lambda)$ and $\mathbf{g}_k^{\downarrow}(\lambda)$ in the spirit of [1]. Let $\mathrm{pr}_{n-1} : \mathbb{S}_n \to \mathbb{S}_{n-1}$ be the linear map defined by projection onto the copy of $\mathbb{S}_{n-1} \subset \mathbb{S}_n$ indexed by $\{1, \dots, n-1\}$.

Proposition 3.9. *If $\lambda \in \mathcal{SP}_n$, then we have $\widetilde{\chi}^{\lambda}(\mathrm{pr}_n(J_{n+1}^{2k})) = \mathbf{g}_k^{\uparrow}(\lambda)$ and*

$$\widetilde{\chi}^{\lambda}\Big(\sum_{\mathcal{LC}_{n-1}^{n}} (-1)^{|x|} x J_n^{2k} x^{-1}\Big) = \mathbf{g}_{k+1}^{\downarrow}(\lambda).$$

4 The subalgebra Γ

We recall relevant facts about the algebra Γ following [11]. Let p_k be the kth power sum symmetric function, and for $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathcal{P}$ set $p_\lambda := \prod_{k=1}^{r} p_{\lambda_k}$. Define Γ to be the subalgebra of the symmetric functions generated by $\{p_{2k+1}\}_{k \geq 0}$.

Elements of Γ can be evaluated on partitions in the following way. For $f \in \Gamma$ and $\lambda \in \mathcal{P}$, set $f(\lambda) = f(\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}, 0, \dots)$. Then it is known that an element of Γ is uniquely determined by its values on $\mathcal{SP}$, giving an embedding of Γ into the algebra of functions on strict partitions $\mathrm{Fun}(\mathcal{SP}, \mathbb{C})$ (with pointwise multiplication).

The Schur P-functions $\{P_\lambda\}_{\lambda \in \mathcal{SP}_n}$ are an important basis of Γ. They are specializations of the Hall–Littlewood polynomials at $t = -1$. Define X_ρ^λ for $\lambda \in \mathcal{SP}_n$, $\rho \in \mathcal{OP}_n$, via the expansion $p_\rho = \sum_{\lambda \in \mathcal{SP}_n} X_\rho^\lambda P_\lambda$.

A "factorial" version of the Schur P-functions is defined in [4]. These elements of Γ are indexed by $\mathcal{SP}$ and satisfy the following properties.

Proposition 4.1 ([5])**.** *Let $\lambda \in \mathcal{SP}$.*

1. *There exists $g \in \Gamma$ of degree less than $|\lambda|$ such that $P_\lambda^* = P_\lambda + g$.*
2. *The collection $\{P_\lambda^*\}_{\lambda \in \mathcal{SP}}$ is a linear basis of Γ.*

Let ψ be the linear map $\Gamma \to \Gamma$ that sends $P_\lambda \mapsto P_\lambda^*$ for all $\lambda \in \mathcal{SP}_n$. For any $\rho \in \mathcal{OP}$, define $\mathfrak{p}_\rho := \psi(p_\rho) \in \Gamma$. These functions were first studied in [5] where Ivanov proves that they satisfy the following properties.

Proposition 4.2 ([5])**.** *The family $\{\mathfrak{p}_\rho\}_{\rho \in \mathcal{OP}}$ forms a linear basis for Γ and for $\rho \in \mathcal{OP}_k$ and $\lambda \in \mathcal{SP}_n$*

$$\mathfrak{p}_\rho(\lambda) = 2^{k-\ell(\rho)} n^{\downarrow k} \frac{\chi^\lambda(\sigma_{\rho;n})}{\chi^\lambda(1^n)}$$

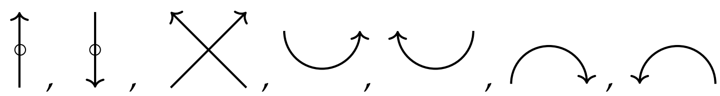
if $k \leq n$ and $\mathfrak{p}_\rho(\lambda) = 0$ otherwise. Further, $(\mathfrak{p}_{2k+1})_{k \geq 0}$ is an algebraically independent generating set for Γ.

Identifying Γ with its image in $\mathrm{Fun}(\mathcal{SP}, \mathbb{C})$ Petrov shows that $\{\mathbf{g}_k^{\uparrow}\}_{k \geq 0}$ and $\{\mathbf{g}_k^{\downarrow}\}_{k \geq 0}$ belong to Γ [14].

Proposition 4.3 ([14]). *Both* $\{\mathbf{g}_k^{\uparrow}\}_{k\geq 0}$ *and* $\{\mathbf{g}_k^{\downarrow}\}_{k\geq 0}$ *are algebraically independent sets of generators of* Γ.

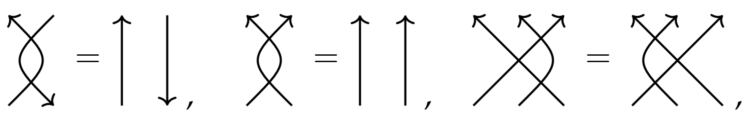
5 The twisted Heisenberg category

The twisted Heisenberg category $\mathcal{H}_{tw}$ was formulated by Cautis–Sussan in [3].The objects of $\mathcal{H}_{tw}$ are generated by P and Q, so that a generic object in $\mathcal{H}_{tw}$ is a direct sum of sequences of P's and Q's. We denote the empty sequence, which is the unit object of $\mathcal{H}_{tw}$, by $\mathbb{1}$. The morphisms of $\mathcal{H}_{tw}$ are generated by oriented planar diagrams up to boundary fixing isotopies, with generators

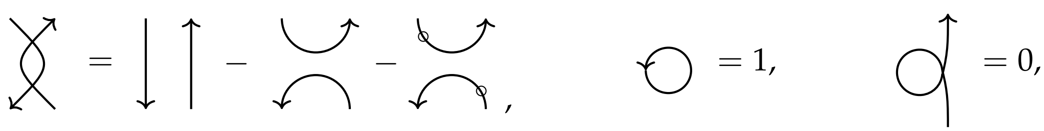


(5.1)

where the first diagram corresponds to a map $P \to P\{1\}$ and the second diagram corresponds to a map $Q \to Q\{1\}$, where $\{1\}$ denotes the $\mathbb{Z}/2\mathbb{Z}$-grading shift. These generators satisfy the following relations:



(5.2)



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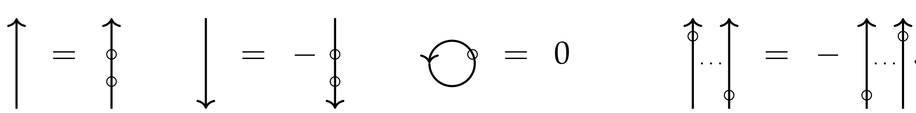
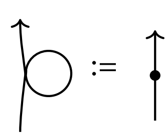
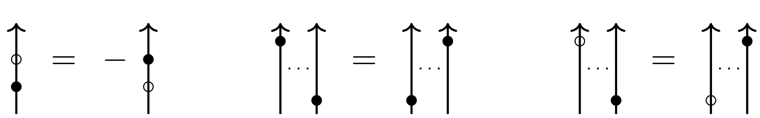
(5.4)



(5.5)

Generators commute in all other situations (for instance, hollow dots commute with crossings).

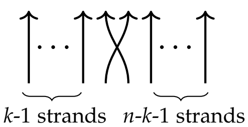
If we denote a right-twist curl by a dot  then we have the following relations:

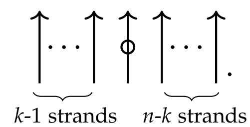


(5.6)

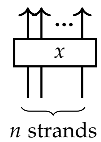
We also have dot sliding relations making $\mathrm{End}_{\mathcal{H}_{tw}}(P^m)$ isomorphic to the degenerate affine Hecke Clifford algebra of type A; see [13] for details.

Because of these relations, there are homomorphisms $\mathcal{T}_n : \mathbb{S}_n \to \mathrm{Hom}_{\mathcal{H}_{tw}}(P^n)$ which send

$s_k \xmapsto{\mathcal{T}_n}$ 

$c_k \xmapsto{\mathcal{T}_n}$ 

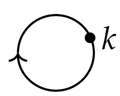
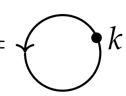
In order to simplify our diagrams we write the image of elements of $\mathbb{S}_n$ as

$\mathcal{T}_n(x) \quad =:$ 

5.1 Center of $\mathcal{H}_{tw}$

The center of a monoidal category $\mathcal{C}$ is defined to be the endomorphism algebra of the monoidal identity object $\mathbb{1}$ of $\mathcal{C}$, that is $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$. Thus $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ is by definition the commutative algebra of closed diagrams where multiplication of two closed diagrams corresponds to placing them next to each other.

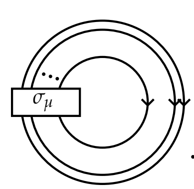
There is a natural generating set of $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ consisting of bubbles. Let d_k be the clockwise-oriented bubble decorated with k solid dots (i.e., with k right twist curls), and let $\overline{d}_k$ be the counterclockwise-oriented bubbles with k dots, as follows:

$d_k :=$  and $\bar{d}_k :=$  . (5.7)

Proposition 5.1 ([13]). *The elements $\{d_{2k}\}_{k\geq 0}$ and $\{\overline{d}_{2k}\}_{k\geq 1}$ are algebraically independent generators of $End_{\mathcal{H}_{tw}}(\mathbb{1})$, i.e. there is an isomorphism*

$$End_{\mathcal{H}_{tw}}(\mathbb{1}) \cong \mathbb{C}[d_0, d_2, d_4, \dots] \cong \mathbb{C}[\bar{d}_2, \bar{d}_4, \bar{d}_6, \dots].$$

Another natural set of diagrams in $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ comes from the closure of permutations under the maps $\mathcal{T}_n : \mathbb{S}_n \to \mathrm{Hom}_{\mathcal{H}_{tw}}(P^n)$. For $\mu \in \mathcal{OP}_n$ and σ_μ an element of S_n of cycle type μ, define

$\alpha_\mu \quad :=$ 

Set $\alpha_n := \alpha_{(n)}$. One can impose a grading on $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ by setting $\deg(d_0) = 0$ and $\deg(d_{2k}) = 2k+1$.

Lemma 5.2. *In terms of the grading defined above $\alpha_{2k+1} = d_{2k} + l.o.t.$. Furthermore, $End_{\mathcal{H}_{tw}}(\mathbb{1})$ is generated by $\{\alpha_{2k+1}\}_{k\geq 0}$ and these elements are algebraically independent. Finally, the set $\{\alpha_\rho\}_{\rho\in\mathcal{OP}}$ is a basis of $End_{\mathcal{H}_{tw}}(\mathbb{1})$.*

5.2 Closed diagrams as bimodule homomorphisms

Cautis and Sussan describe an action of $\mathcal{H}_{tw}$ on the category $\mathfrak{S}$ whose objects are compositions of induction and restriction functors between finite dimensional $\mathbb{S}_n$-modules [3, Section 6.3]. The morphisms in $\mathfrak{S}$ are certain natural transformations of these compositions of functors (or, equivalently, certain bimodule homomorphisms). After passing to the idempotent closure of $\mathcal{H}_{tw}$, this action becomes a categorified Fock space action of $\mathcal{H}_{tw}$ [3].

The action is defined via a family of functors $\{F_n^{\mathcal{H}_{tw}}\}_{n\geq 0}$. When restricted to $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$, $F_n^{\mathcal{H}_{tw}}$ defines a homomorphism from $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ to $Z(\mathbb{S}_n)_{\bar{0}}$. We describe the image of some of the generators of $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ from Section 5.1.

Proposition 5.3. 1. *For $\mu \in \mathcal{OP}_k$,* $F_n^{\mathcal{H}_{tw}}(\alpha_\mu) = \begin{cases} A_{\mu;n} & \text{if } k \leq n \\ 0 & \text{otherwise.}\end{cases}$

2. *For $k \geq 0$,* $F_n^{\mathcal{H}_{tw}}(d_{2k}) = \sum_{x\in\mathcal{LC}_{n-1}^n} (-1)^{|x|} x J_n^{2k} x^{-1}$.

3. *For $k \geq 2$,* $F_n^{\mathcal{H}_{tw}}(\bar{d}_{2k}) = \mathrm{pr}_n(J_{n+1}^{2k})$.

6 An isomorphism between $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ and Γ

In this section we will establish the isomorphism between $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ and Γ. The key step in the construction of this map is identifying the elements of $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ with elements of $\mathrm{Fun}(\mathcal{SP}, \mathbb{C})$. To do this let $\lambda \in \mathcal{SP}_n$ and $x \in \mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$; we define $x(\lambda) = \widetilde{\chi}^\lambda(F_n^{\mathcal{H}_{tw}}(x))$. Because $F_n^{\mathcal{H}_{tw}}$ is a homomorphism on $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ which maps into $Z(\mathbb{S}_n)_{\bar{0}}$ and $\widetilde{\chi}^\lambda$ is a homomorphism when restricted to $Z(\mathbb{S}_n)_{\bar{0}}$, this defines a homomorphism into $\mathrm{Fun}(\mathcal{SP}, \mathbb{C})$.

Proposition 6.1. *For $\mu \in \mathcal{OP}_k$ and $\lambda \in \mathcal{SP}_n$ we have*

$$\alpha_\mu(\lambda) = \begin{cases} 2^k n^{\downarrow k} \frac{\chi^\lambda(\sigma_{\mu;n})}{\chi^\lambda(1)} & \text{if } k \leq n \\ 0 & \text{otherwise.}\end{cases}$$

Theorem 6.2. *There is an isomorphism $\varphi : \mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1}) \to \Gamma$ which sends $\alpha_\mu \mapsto 2^{\ell(\mu)}\mathfrak{p}_\mu$ for $\mu \in \mathcal{OP}_k$.*

Proof. Proposition 6.1 and Proposition 4.2 show that $2^{-\ell(\mu)}\alpha_\mu$ and $\mathfrak{p}_\mu$ map to the same function in $\mathrm{Fun}(\mathcal{SP}, \mathbb{C})$. The homomorphisms $\Gamma \to \mathrm{Fun}(\mathcal{SP}, \mathbb{C})$ and $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1}) \to \mathrm{Fun}(\mathcal{SP}, \mathbb{C})$ are both injective. By Lemma 5.2 $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ is generated by the algebraically independent elements $\{\alpha_k\}_{k\geq 0}$ and by Proposition 4.2, Γ is generated by the algebraically independent elements $\{\mathfrak{p}_k\}_{k\geq 0}$, it follows that the map that sends $\alpha_\mu \mapsto 2^{\ell(\mu)}\mathfrak{p}_\mu$ is an isomorphism. □

Theorem 6.3. *Let $\lambda \in \mathcal{SP}_n$. Under the isomorphism $\varphi : \mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1}) \to \Gamma$, the central idempotent e_λ of $\mathbb{S}_n$ maps to $2^n g_\lambda P^*_\lambda$.*

Moving in the opposite direction, we can also identify the elements of Γ corresponding to the generators $\{d_{2k}\}_{k\geq 0}$ and $\{\bar{d}_{2k}\}_{k\geq 1}$.

Theorem 6.4. *For $k \geq 0$, we have $\varphi(d_{2k}) = \mathbf{g}^{\downarrow}_{k+1}$ and $k \geq 1$ we have $\varphi(\bar{d}_{2k}) = \mathbf{g}^{\uparrow}_k$.*

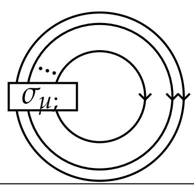
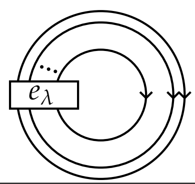
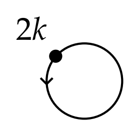
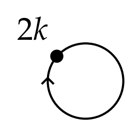
Γ	$\mathfrak{p}_\mu$	P^*_λ	$\mathbf{g}^{\uparrow}_k$	$\mathbf{g}^{\downarrow}_{k+1}$
Diagram in $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$	$\frac{1}{2^{\ell(\mu)}}$ 	$\frac{1}{2^n g_\lambda}$ 		

Table 1: A dictionary between Γ and diagrams in $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$.

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