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Delta operators at $q = 1$ and polyominoes

Angela Hicks*[1] and Marino Romero†[2]

[1] *Department of Mathematics, Lehigh University, Bethlehem, PA*
[2] *Department of Mathematics, University of California, San Diego, La Jolla, CA*

Abstract. For a symmetric function G, the Delta operator Δ_G is defined via its action on modified Macdonald polynomials by setting $\Delta_G \widetilde{H}_\mu = G[B_\mu]\widetilde{H}_\mu$, where B_μ is a polynomial in q and t. Previous work by Haglund, Remmel, Wilson conjectures a combinatorial interpretation for $\Delta_{e_k} e_n$, generalizing the Shuffle Theorem. Here, we prove combinatorial interpretations for $\Delta_{m_\lambda} e_n|_{q=1}$ and $\Delta_{s_\lambda} e_n|_{q=1}$, expressing each as weighted sum over (parallelogram) polyominoes in a rectangle, and provide an explicit combinatorial interpretation for their elementary and Schur function expansions.

Keywords: shuffle conjecture, Macdonald Polynomials, delta conjecture, polyominoes

1 Introduction

Several recent works study the symmetric function expression $\Delta_G e_n$ (as defined just below) for various symmetric functions G. One is the well known Shuffle Theorem ([10],[8]), recently proven by Carlsson and Mellit ([3]), which describes $\Delta_{e_n} e_n$ (or ∇e_n as it is more frequently written) as a weighted sum of parking functions and connects the result (via work of Mark Haiman ([11])) to the Frobenius characteristic of the space of diagonal harmonics. Another is the Delta Conjecture of Haglund, Remmel, and Wilson ([9]), which gives a combinatorial interpretation for the closely related symmetric function $\Delta_{e_k} e_n$. (In particular, the Delta Conjecture has been proved when $q = 1$ ([14]) or $q = 0$ ([7]).) In this work, we provide combinatorial interpretations for the more general $\Delta_{m_\lambda} e_n$ and $\Delta_{s_\lambda} e_n$ at $q = 1$.

The operator Δ_G is defined using plethystic notation. Given a symmetric function G and an expression $E(t_1, t_2, \ldots,)$, the plethystic evaluation $G[E]$ is defined as the image of G under the homomorphism mapping the power symmetric function $p_k \mapsto E(t_1^k, t_2^k, \ldots)$. (A complete exposition of plethysm is found in [12], while [2] and [6] contain many well known plethystic identities.) We let Δ_G ([2]) be the Delta operator on symmetric functions (with coefficients in $\mathbb{C}(q,t)$), which acts on the modified Macdonald basis $\{\widetilde{\mathrm{H}}_\mu\}_\mu$ by

$$\Delta_G \widetilde{\mathrm{H}}_\mu = G[B_\mu] \widetilde{\mathrm{H}}_\mu \text{, where } B_\mu = \sum_{(i,j)\in\mu} q^i t^j.$$

*anh316@lehigh.edu
†mar007@ucsd.edu. Partially supported by NSF Grant 1700233.

The sum is over all cells in μ, where we represent μ by its French diagram, drawn in the first quadrant with lower left corner at $(0,0)$, and (i,j) is the lower left corner of a lattice cell in μ. The expression $G[B_\mu]$ can be understood as the evaluation of G at all $q^i t^j$ where $(i,j) \in \mu$ (given in any order). Thus for example $B_{3,1} = 1 + q + q^2 + t$ and $e_4[B_{3,1}] = e_4(1,q,q^2,t) = q^3 t$.

Inspired by [1], which relates a particular coefficient of ∇e_n to the q,t-Narayana polynomials—a weighted sum of polyominoes—along with a conjecture of Adriano Garsia, our main result is that the set of polyominoes $P_{\lambda,\mu}$ (see Section 4) whose steps are indexed by partitions λ and μ give

Theorem 1.1.
$$\Delta_{m_\lambda} e_n \Big|_{q=1} = \sum_{\mu \vdash n} e_\mu \sum_{T \in P_{\lambda,\mu}} t^{\mathrm{area}(T)}. \tag{1.1}$$

$\Delta_{m_\lambda} e_n$ is not generally Schur positive, but in [2] it was conjectured that $\Delta_{s_\lambda} e_n$ is Schur positive; thus we end with a similar combinatorial formula for $\Delta_{s_\lambda} e_n|_{q=1}$. (One can also show these polynomials are symmetric in q and t, so this of course gives expansions when $t = 1$ as well.)

Section 2 transforms the left hand side of (1.1) to a weighted sum of ad hoc combinatorial objects, Section 3 gives a sign reversing involution reducing the number of summands, Section 4 gives a bijection to polyominoes in the right hand side of (1.1), and Section 5 gives a related expression for $\Delta_{s_\lambda} e_n|_{q=1}$.

2 Symmetric function expansions

The modified Macdonald polynomial specializes at $q = 1$ (see section Integral Forms, [13, p. 364] and [5]) to
$$\widetilde{\mathrm{H}}_\mu[X;1,t] = (t:t)_{\mu'} h_{\mu'}\left[\frac{X}{1-t}\right],$$
where μ' is the conjugate of μ, $(t:t)_a = \prod_{i=1}^a (1-t^i)$ is the t-Pochhammer symbol, and $(t:t)_\mu = \prod_{j=1}^{\ell(\mu)} (t:t)_{\mu_j}$. This means that an eigenoperator for the modified Macdonald polynomial becomes at $q = 1$ an eigenoperator of $h_{\mu'}[X/(1-t)]$. This gives
$$\Delta_G h_\mu \left[\frac{X}{1-t}\right] \Big|_{q=1} = G\left[\sum_i [\mu_i]_t\right] h_\mu\left[\frac{X}{1-t}\right].$$

To apply Delta operators to e_n at $q = 1$, we can use the Cauchy formula (where f_η is the forgotten symmetric function) to show both:
$$e_n = \sum_{\eta \vdash n} h_\eta \left[\frac{X}{1-t}\right] f_\eta[1-t] \text{ and } h_\eta\left[\frac{X}{1-t}\right] = \sum_{\nu^i \vdash \eta_i} e_{\nu^1} \cdots e_{\nu^{\ell(\eta)}}[X] f_{\nu^1} \cdots f_{\nu^{\ell(\eta)}}\left[\frac{1}{1-t}\right].$$

In the following, we use $R(\mu)$ for the set of compositions which rearrange to μ and $\mathrm{PR}(\mu)$ (respectively $\mathrm{CR}(\mu)$) for the set of ordered lists of possibly empty partitions (respectively compositions), whose concatenation is in $R(\mu)$. For any G,

$$\begin{aligned}\Delta_G e_n|_{q=1} &= \sum_{\eta \vdash n} f_\eta[1-t] G\left[\sum [\eta_i]_t\right] \sum_{\nu^i \vdash \eta_i} e_{\nu^1} \cdots e_{\nu^{\ell(\eta)}}[X] f_{\nu^1} \cdots f_{\nu^{\ell(\eta)}}\left[\frac{1}{1-t}\right] \\ &= \sum_{\mu \vdash n} e_\mu[X] \sum_{\eta \vdash n} G\left[\sum [\eta_i]_t\right] \sum_{\substack{(\nu^1,\dots,\nu^{\ell(\eta)}) \in \mathrm{PR}(\mu) \\ \nu^i \vdash \eta_i}} f_\eta[1-t] f_{\nu^1} \cdots f_{\nu^{\ell(\eta)}}\left[\frac{1}{1-t}\right].\end{aligned}$$

2.1 A forgotten basis expansion

Our first goal will be to describe the product of forgotten symmetric functions. By the Cauchy formula and a plethystic addition formula, we have two expansions:

$$e_n[X(Y+Z)] = \sum_{\nu \vdash n} h_\nu[X] f_\nu[Y+Z], \tag{2.1}$$

$$e_n[X(Y+Z)] = \sum_{i=0}^{n} e_{n-i}[XY] e_i[XZ] = \sum_{i=0}^{n} \sum_{\alpha \vdash n-i} \sum_{\beta \vdash i} h_\alpha[X] f_\alpha[Y] h_\beta[X] f_\beta[Z]. \tag{2.2}$$

Comparing the coefficients of $h_\nu[X]$,

$$f_\nu[Y+Z] = \sum_{(\alpha,\beta) \in \mathrm{PR}(\nu)} f_\alpha[Y] f_\beta[Z]. \tag{2.3}$$

From the Jacobi–Trudi determinant (see details in [14]), we get $f_\nu[1] = (-1)^{|\nu|-\ell(\nu)}|R(\nu)|$. Using (2.3) repeatedly, treating each term x_i as a new alphabet, we get a formula for the forgotten symmetric functions (see [4] for an alternate combinatorial description):

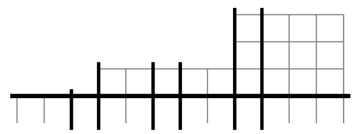
$$f_\nu[X] = (-1)^{|\nu|-\ell(\nu)} \sum_{(\alpha^1,\alpha^2,\dots) \in \mathrm{CR}(\nu)} x_1^{|\alpha^1|} x_2^{|\alpha^2|} \cdots. \tag{2.4}$$

Substituting $x_i = t^i$ for all i, we get a formula for $f_\nu[1/(1-t)]$. For a composition γ in $R(\nu)$ define $T \in CC_\gamma$, the Column Composition Tableau of horizontal composition (hcomp) γ:

1. Begin with a "bottom" row of size $|\nu|$ and draw a horizontal line segment above.
2. Draw vertical line segments between the columns to denote γ, adding lines after the first γ_1 columns, after the next γ_2, and so on.
3. To each set of columns corresponding to some γ_i, add a weakly increasing (as i increases) number of squares. All the columns with $j-1$ boxes will together form a subsequence of γ, which we call α^j.

The cells above columns corresponding to α^j thus correspond to $x_j^{|\alpha^j|}$ in some summand in (2.4), so we let $w(T) = |\alpha^2| + 2 \cdot |\alpha^3| + 3 \cdot |\alpha^4| + \cdots$, the number of squares above the bottom row. We use $\mathrm{fc}(T)$ to denote the number of cells in the first column of T. ($\mathrm{fc}(T) \neq 0$ if and only if $\alpha^1 = ()$, the empty composition.)

For example, when $\nu = (3,2,2,2,1,1,1)$, we may start with $\gamma = (2,1,2,1,2,1,3) \in R(v)$ and add boxes to indicate the compositions $\alpha^1 = (2,1)$, $\alpha^2 = (2,1,2)$, $\alpha^3 = ()$, and $\alpha^4 = (1,3)$. Then T is given by



with $w(T) = 17$ and $\mathrm{fc}(T) = 0$. Let $\overline{CC_\gamma}$ be the subset of CC_γ where $\mathrm{fc}(T) = 0$. Then

$$F_\nu = (-1)^{|\nu|-\ell(\nu)} f_\nu \left[\frac{1}{1-t}\right] = \sum_{\gamma \in R(\nu)} \sum_{T \in CC_\gamma} t^{w(T)}, \text{ and}$$

$$\overline{F_\nu} = (1-t^{|\nu|})(-1)^{|\nu|-\ell(\nu)} f_\nu \left[\frac{1}{1-t}\right] = \sum_{\gamma \in R(\nu)} \sum_{T \in \overline{CC_\gamma}} t^{w(T)}.$$

Additionally, we will need the fact that

$$f_\eta[1-t] = (-1)^{|\eta|-\ell(\eta)} \sum_{\beta \in R(\eta)} (1-t^{\beta_1}).$$

The derivation, which we omit for this extended abstract, is similar to (2.4) and follows from work in [14].

We can therefore interpret the product of forgotten functions as a generating series for sequences of column composition tableaux:

$$\begin{aligned} &\sum_{\substack{(\nu^1,\ldots,\nu^{\ell(\eta)}) \in \mathrm{PR}(\mu) \\ \nu^i \vdash \eta_i}} f_\eta[1-t] f_{\nu^1} \cdots f_{\nu^{\ell(\eta)}} \left[\frac{1}{1-t}\right] \\ &\qquad = (-1)^{\ell(\eta)-\ell(\mu)} \sum_{\beta \in R(\eta)} \sum_{\substack{\gamma^i \models \beta_i \\ (\gamma^1,\ldots,\gamma^{\ell(\eta)}) \in \mathrm{CR}(\mu)}} \overline{F_{\gamma^1}} F_{\gamma^2} \cdots F_{\gamma^{\ell(\gamma)}}. \end{aligned}$$

Next, setting $G = m_\lambda$, we interpret

$$\left(\Delta_{m_\lambda} e_n|_{q=1}\right)\Big|_{e_\mu} = \sum_{\eta \vdash n} m_\lambda \left[\sum [\eta_i]_t\right] \sum_{\substack{(\nu^1,\ldots,\nu^{\ell(\eta)}) \in \mathrm{PR}(\mu) \\ \nu^i \vdash \eta_i}} f_\eta[1-t] f_{\nu^1} \cdots f_{\nu^{\ell(\eta)}} \left[\frac{1}{1-t}\right].$$

We construct the set of (λ,μ)-lists of labeled column composition tableaux $CC(\lambda,\mu)$:

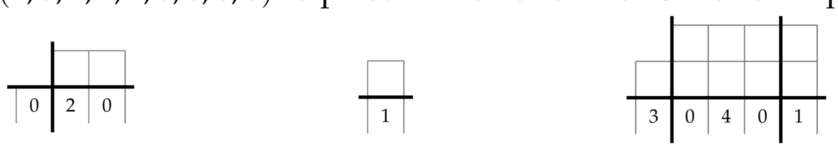
1. Select a partition $\eta \vdash n$ and $\beta \in R(\eta)$.
2. For $i = 1, \ldots, \ell(\eta)$, select $\gamma^i \models \beta_i$ such that $(\gamma^1, \ldots \gamma^{\ell(\gamma)}) \in \mathrm{CR}((\mu))$.
3. Select $T_1 \in \overline{CC_{\gamma^1}}$; and for $i = 2, \ldots, \ell(\eta)$, select $T_i \in CC_{\gamma^i}$.
4. Distribute a rearrangement of the list $\lambda_1, \ldots, \lambda_{\ell(\lambda)}, 0^{n-\ell(\lambda)}$ along the cells in the bottom rows of $T_1, \ldots, T_{\ell(\eta)}$ to obtain a sequence of labeled tableaux $(\mathrm{lT}_1, \ldots, \mathrm{lT}_{\ell(\eta)})$, which we denote by S. (Here and elsewhere, we use variable names beginning with 'l' for labeled objects, so lT is formed by labeling column composition tableau T.)

These new labels correspond to the weight $m_\lambda\,[\sum[\eta_i]_t]$, so define

$$w(\mathrm{lT}_i) = w(T_i) + \sum_{\text{label } j \text{ in } \mathrm{lT}_i} j \cdot (\text{ the number of cells to the right of } j \text{ in } \mathrm{lT}_i \text{ }),$$

and $w(S) = w(\mathrm{lT}_1) + \cdots + w(\mathrm{lT}_{\ell(\eta)})$. The sign of S is given by $\mathrm{sign}(S) = (-1)^{\ell(\eta)-\ell(\mu)}$.

For example, suppose $\eta = (5,3,1)$, $\mu = (3,2,1,1,1,1)$, and $\lambda = (4,3,2,1,1)$. Suppose we select: $\beta = (3,1,5)$, then select $\gamma^1 \models 3, \gamma^2 \models 1, \gamma^3 \models 5$: $(1,2), (1)$, and $(1,3,1)$ respectively. Finally, we select $T_1 \in \overline{CC_{(1,2)}}, T_2 \in CC_{(1)}$ and $T_3 \in CC_{(1,3,1)}$ and a rearrangement of $\lambda, 0^{n-\ell(\lambda)} = (4,3,2,1,1,0,0,0,0)$ to place in the bottom rows. For example, let $S =$



(2.5)

We can calculate the weight of this list by first counting the number of cells above the bottom rows. In this case we would have $2, 1$, and $5 + 4 = 9$ respectively. The weight corresponding to the new labels would be:

$$0(2) + 2(1) + 0(0) = 2, \quad 1(0) = 0, \quad \text{and} \quad 3(4) + 0(3) + 4(2) + 0(1) + 1(0) = 20.$$

Thus $w(S) = (2 + 1 + 9) + (2 + 0 + 20) = 34$ and $\mathrm{sign}(S) = (-1)^{(3-6)} = -1$.

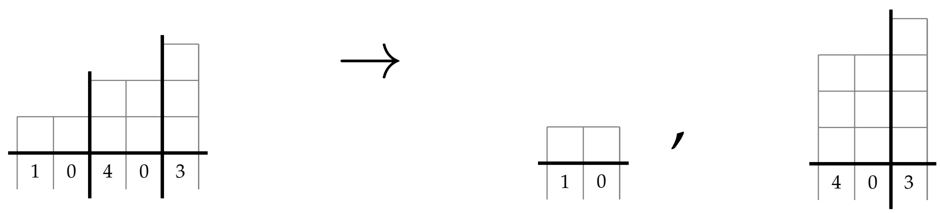
In conclusion, we have:

$$\Delta_{m_\lambda} e_n \Big|_{q=1} = \sum_{\mu \vdash n} e_\mu \left(\sum_{S \in CC(\lambda,\mu)} \mathrm{sign}(S) t^{w(S)} \right). \tag{2.6}$$

3 A weight-preserving, sign-reversing involution

We will find a sign reversing involution $\phi : CC(\lambda, \mu) \to CC(\lambda, \mu)$ which simplifies the right hand side of (2.6). The goal will be to only sum over a subset of the $CC(\lambda, \mu)$ where each lT_i has no vertical bars. In addition to continuing to use $\mathrm{fc}(\mathrm{lT}_i)$ for the number of cells in the first column of each labeled column composition tableau, we will use $|\mathrm{lT}_i|$ for the sum of its labels, so that in (2.5), we have $\mathrm{fc}(\mathrm{lT}_1) = 0$ and $|\mathrm{lT}_1| = 0 + 2 + 0 = 2$.

First, we will define the "separation" operation. Let γ be a composition of length larger than 1, and let lT be a labeled column composition tableau of horizontal composition γ. Then split lT at the first vertical line segment, dividing it into two labeled column composition tableaux lS_1 and lS_2. Modify lS_2 by adding $|\mathrm{lS}_1|$ cells to each of its columns. Here is an example:



(3.1)

Note that $w(\mathrm{lT}) = w(\mathrm{lS}_1) + w(\mathrm{lS}_2)$; the additional cells in $w(\mathrm{lS}_2)$ exactly make up for the loss of weight by the labels in lS_1. Moreover, since lT must have weakly increasing numbers of squares from one column to the next, before we add the new cells we have $\mathrm{fc}(\mathrm{lS}_1) \leq \mathrm{fc}(\mathrm{lS}_2)$ and thus after we add the cells,

$$\mathrm{fc}(\mathrm{lS}_1) + |\mathrm{lS}_1| \leq \mathrm{fc}(\mathrm{lS}_2).$$

To form the inverse operation, let lS_1 and lS_2 be given, where lS_1 consists of a single part (so there are no vertical line segments) and $\mathrm{fc}(\mathrm{lS}_1) + |\mathrm{lS}_1| \leq \mathrm{fc}(\mathrm{lS}_2)$. Then we say that we can "join" lS_1 and lS_2, producing a single labeled column composition tableau lT, by first removing $|\mathrm{lS}_1|$ cells from each column of lS_2, then concatenating lS_1 on the left with a vertical line segment. For example, note that this describes the inverse of (3.1).

Define $\phi : CC(\lambda, \mu) \to CC(\lambda, \mu)$ by scanning $(\mathrm{lT}_1, \ldots, \mathrm{lT}_\ell)$, from left to right, for the first i such that either lT_i can be separated or lT_i and lT_{i+1} can be joined; separate or join accordingly. If no such i is found, leave the sequence fixed. We have

Proposition 3.1. *The map ϕ is a weight-preserving, sign-reversing involution.*

Proof. We have seen that the map is weight-preserving. Since we are changing the number of tableaux in the list by 1, we are reversing the sign. We need only check that it is an involution, which means we want to make sure we encounter the same i when we apply ϕ again. Recall we can separate lT_i if and only if lT_i is of horizontal composition ν where $l(\nu) > 1$ and we can join lS_i and lS_{i+1} if and only if lS_i is of horizontal composition (k) for some integer k and $\mathrm{fc}(\mathrm{lS}_i) + |\mathrm{lS}_i| \leq \mathrm{fc}(\mathrm{lS}_{i+1})$.

First, if $(\mathrm{lT}_1, \mathrm{lT}_2, \mathrm{lT}_3)$ produces $(\mathrm{lT}_1, \mathrm{lS})$ under ϕ, then lT_1 and lT_2 both have a single part, and lT_1 cannot join with lT_2, so $\mathrm{fc}(\mathrm{lT}_2) < \mathrm{fc}(\mathrm{lT}_1) + |\mathrm{lT}_1|$. Thus $\mathrm{fc}(\mathrm{lS}) = \mathrm{fc}(\mathrm{lT}_2) <$

$\mathrm{fc}(\mathrm{lT}_1) + |\mathrm{lT}_1|$, so lT_1 cannot join with lS as desired. If $(\mathrm{lT}_1, \mathrm{lS})$ produces $(\mathrm{lT}_1, \mathrm{lT}_2, \mathrm{lT}_3)$, then lT_1 must again have horizontal composition (k) and since lT_1 cannot join with lS, $\mathrm{fc}(\mathrm{lT}_2) = \mathrm{fc}(\mathrm{lS}) < \mathrm{fc}(\mathrm{lT}_1) + |\mathrm{lT}_1|$. Thus lT_1 and lT_2 cannot join.

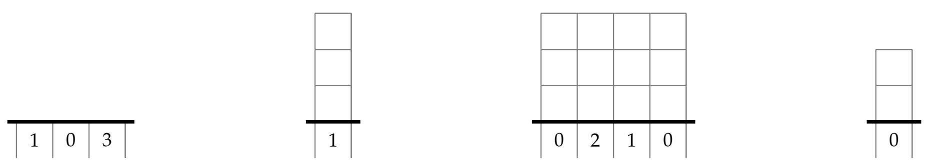
□

Let $M_{\lambda,\mu}$ give the set of fixed points $S = (\mathrm{lT}_1, \mathrm{lT}_2, \ldots, \mathrm{lT}_\ell)$ of ϕ, that is S such that:

1. For all i, $\mathrm{hcomp}(\mathrm{lT}_i)$ has a single part. Thus S is an element of $CC(\lambda, \mu)$, where in the first step of the construction, we select $\eta = \mu$ and $\mathrm{sign}(S) = (-1)^{\ell(\mu)-\ell(\mu)} = 1$.

2. For $i = 1, \ldots, \ell(\lambda) - 1$, $\mathrm{fc}(\mathrm{lT}_i) + |\mathrm{lT}_i| > \mathrm{fc}(\mathrm{lT}_{i+1})$. Also, $\mathrm{fc}(\mathrm{lT}_1) = 0$.

Let $M_{\lambda,\mu}$ denote the collection of fixed points. For example, let $\mu = (4,3,1,1)$ and $\lambda = (3,2,1,1,1)$. Then $(\mathrm{lT}_1, \mathrm{lT}_2, \mathrm{lT}_3, \mathrm{lT}_4) =$



is an element of $M_{\lambda,\mu}$. To check this, we see first that every tableau corresponds to a part of μ. To check the second condition, we start with $\mathrm{fc}(\mathrm{lT}_1) = 0$, $\mathrm{fc}(\mathrm{lT}_2) = 3$, $\mathrm{fc}(\mathrm{lT}_3) = 3$, $\mathrm{fc}(\mathrm{lT}_4) = 2$, $|\mathrm{lT}_1| = 4$, $|\mathrm{lT}_2| = 1$, $|\mathrm{lT}_3| = 3$, and $|\mathrm{lT}_4| = 0$. In summary, we have shown:

Theorem 3.2. *Let λ be a partition. Then*

$$\Delta_{m_\lambda} e_n \Big|_{q=1} = \sum_{\mu \vdash n} e_\mu \sum_{S \in M_{\lambda,\mu}} t^{w(S)}.$$

4 A bijection

Next, we biject from $M_{\lambda,\mu}$ to a subset of the polyominoes. Let $P_{m,n}$ be the set of (parallelogram) polyominoes in the $m \times n$ rectangle, i.e. the collection of pairs (P, Q) of lattice paths consisting of North and East steps from $(0,0)$ to (m,n) such that

1. P and Q only touch at $(0,0)$ and (m,n).

2. P is above Q.

The above conditions force Q to begin with an East step. From here on, when we say the path Q, we mean the portion of Q which begins at $(1,0)$ and ends at (m,n). For $(P, Q) \in P_{m,n}$, let

$$\mathrm{area}((P,Q)) = (\text{the number of lattice cells between } P \text{ and } Q) \; - \; (m+n-1).$$

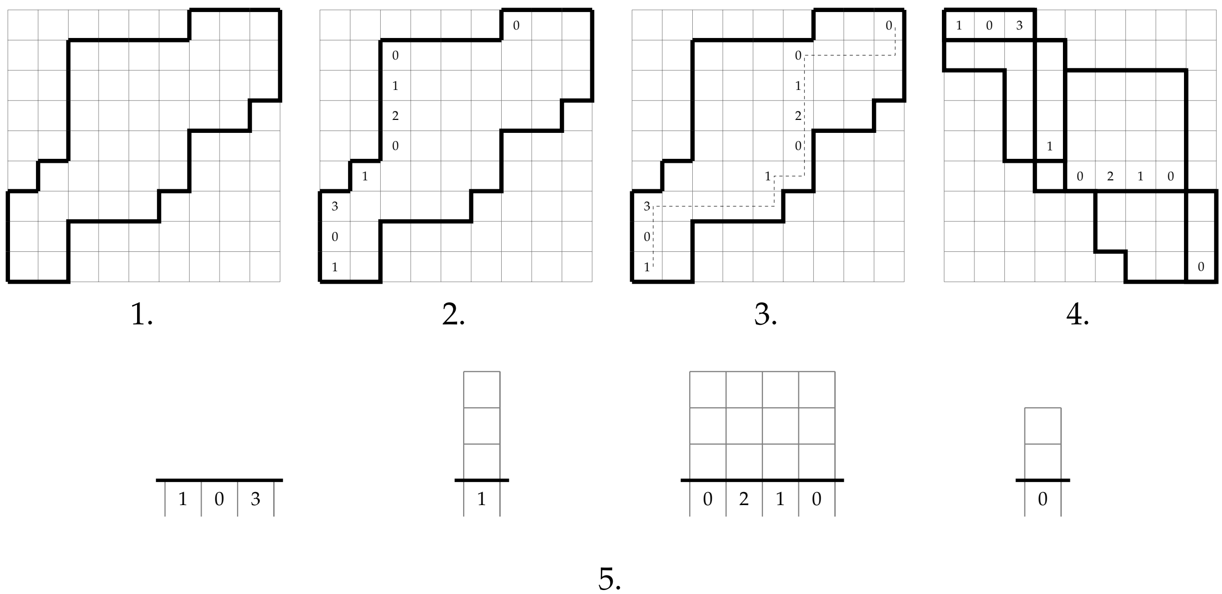


Figure 1: Constructing Ψ. The numbers correspond to steps.

Define colcomp(P) (respectively rowcomp(Q)) to be the composition whose ith part is the number of North (respectively East) steps in P (in Q) occurring along $x = i-1$ ($y = i-1$), where as usual we do not count the first step of Q. We can now define $P_{\lambda,\mu}$ of Theorem 1 to be the set of all $(P, Q) \in P_{|\lambda|+1,|\mu|}$ such that colcomp(P) rearranges to μ and rowcomp(Q) rearranges to λ.

For example, if (P, Q) is the first diagram in Figure 1, then (P, Q) is in $P_{9,9}$, area(P, Q) $= 24$, colcomp(P) $= (3, 1, 4, 0, 0, 0, 1, 0, 0)$, and rowcomp$(Q) = (1, 0, 3, 1, 0, 2, 1, 0, 0)$. Thus, $(P, Q) \in P_{(3,2,1,1,1),(4,3,1,1)}$.

The goal of this section is to prove the following theorem:

Theorem 4.1.

$$\sum_{S \in M_{\lambda,\mu}} t^{w(S)} = \sum_{P \in P_{\lambda,\mu}} t^{\mathrm{area}(P)}$$

We will do this by producing a weight-preserving bijection $\psi : P_{\lambda,\mu} \to M_{\lambda,\mu}$. See the example in Figure 1 as we work through the bijection. Let $(P, Q) \in P_{\lambda,\mu}$.

1. Draw (P, Q), as in Figure 1, step 1.

2. Place the sequence rowcomp(Q) along the North steps of P. Then for T in Figure 1, rowcomp$(Q) = (1, 0, 3, 1, 0, 2, 1, 0, 0)$. Thus we get Figure 1, step 2.

3. Next, shift the integers by columns, keeping integers in the first column fixed, but otherwise moving the integers in a given column right until the bottom integer's

cell touches the bottom path Q. See Figure 1, step 3. Call the cells containing the integers, and the sequences of horizontal squares which connect them, the "middle path." See the dashed line in our running example.

4. Turn the diagram 90 degrees clockwise. Rotating (P, Q) gives Figure 1, step 4.

5. To get $\psi((P, Q)) = (\mathrm{lT}_1, \ldots, \mathrm{lT}_{\ell(\lambda)})$ we look at the cells bounded from above by the vertical segments of P (which will now be horizontal since we rotated the picture), and bounded from below by the middle path. The cells along the middle path are the labeled bottom rows of $\mathrm{lT}_1, \ldots, \mathrm{lT}_{\ell(\lambda)}$. Adding the cells between P and the middle path produces the element in Figure 1, step 5.

To show that this process is the required bijection we must show that the resulting element satisfies the conditions of $M_{\lambda,\mu}$ and that the process is invertible and weight preserving.

Proof. Briefly, we begin by inverting Ψ. Start with $(\mathrm{lT}_1, \ldots \mathrm{lT}_n)$ in $M_{\lambda,\mu}$. By construction, its labels give $\mathrm{rowcomp}(Q)$ which immediately determines Q which (as desired) rearranges to λ. Next, rotating each lT_i counterclockwise and stacking them from bottom to top will give our path P, if we can determine where to align the "right column" (formerly "bottom row") of each lT_i in our diagram to create the middle path. lT_1 must be left justified. For $i > 1$, each lT_i should be placed as far West as possible among all positions where its Southeast corner still touches the path Q. In particular, for all j, say lT_j's bottom right (respectively bottom left) corner is placed along the line $x = c_j$ (respectively $x = d_j$). Then $c_j = |\mathrm{lT}_1| + |\mathrm{lT}_2| + \cdots |\mathrm{lT}_{j-1}| + 1$. The resulting path P will fail to have $\mathrm{colcomp}(P)$ rearrange to μ (or worse, will fail to be a connected lattice path) only if the left column (formerly top row) of some lT_i is weakly East of the left of column of lT_{i+1} so that $d_i \geq d_{i+1}$. Since the bottom left corner of lT_j is by construction $|\,\mathrm{fc}(\mathrm{lT}_j)|$ left of its right corner, $d_j = c_j - \mathrm{fc}(\mathrm{lT}_j)$. Thus the left column of lT_i is weakly East of the left of column of lT_{i+1} if and only if

$$|\mathrm{lT}_1| + |\mathrm{lT}_2| + \cdots |\mathrm{lT}_{i-1}| + 1 - \mathrm{fc}(\mathrm{lT}_i) \geq |\mathrm{lT}_1| + |\mathrm{lT}_2| + \cdots |\mathrm{lT}_i| + 1 - \mathrm{fc}(\mathrm{lT}_{i+1}) \Leftrightarrow$$
$$\mathrm{fc}(\mathrm{lT}_{i+1}) \geq \mathrm{fc}(T_i) + |\mathrm{lT}_i|$$

This, of course, is exactly when $(\mathrm{lT}_1, \ldots \mathrm{lT}_n)$ was not a fixed point of our involution and thus was not in $M_{\lambda,\mu}$.

To see that ψ is weight preserving, first note that the middle path goes through $m + n - 1$ cells; thus area counts the number of cells between (P,Q) but not on the middle path. The cells West of the middle path clearly correspond to the cells above the bottom rows in $(\mathrm{lT}_1, \ldots \mathrm{lT}_n)$, so we need to show the cells South/East of the middle path correspond to the weight contributed by the labels in $(\mathrm{lT}_1, \ldots \mathrm{lT}_n)$. This is exactly the case by construction. □

This proves Theorem 4.1. Theorems 3.2 and 4.1 imply Theorem 1.1.

5 The main result

The previous theorem may now be rewritten as

Corollary 5.1. *Let ν be a partition. Then*

$$\Delta_{m_\nu} e_n\Big|_{q=1} = \sum_{\eta \vdash n} e_\eta \sum_{S \in P_{\nu,\eta}} t^{w(S)}.$$

To get a Schur function expansion of $\Delta_{s_\lambda} e_n|_{q=1}$, recall that for any $\zeta, \kappa \vdash n$, $\alpha \in R(\kappa)$, the Kostka number $K_{\zeta,\kappa} = K_{\zeta,\alpha}$ counts the set of semi-standard tableaux of shape ζ and content α, $T_{\zeta,\alpha}$. Also recall

$$e_\eta = \sum_{\mu' \vdash n} K_{\mu,\eta} s_{\mu'} \qquad \text{and} \qquad s_\lambda = \sum_{\nu \vdash n} K_{\lambda,\nu} m_\nu$$

We have

$$\begin{aligned}
\Delta_{s_\lambda} e_n\Big|_{q=1} &= \sum_{\nu \vdash n} K_{\lambda,\nu} \Delta_{m_\nu} e_n\Big|_{q=1} \\
&= \sum_{\nu \vdash n} K_{\lambda,\nu} \sum_{\eta \vdash n} e_\eta \sum_{S \in P_{\nu,\eta}} t^{w(S)} \\
&= \sum_{\nu \vdash n} K_{\lambda,\nu} \sum_{\eta \vdash n} \sum_{\mu \vdash n} K_{\mu',\eta} s_\mu \sum_{S \in P_{\nu,\eta}} t^{w(S)} \\
&= \sum_{\mu \vdash n} s_\mu \sum_{\nu \vdash n} \sum_{\eta \vdash n} \sum_{S \in P_{\nu,\eta}} K_{\lambda,\nu} K_{\mu',\eta} t^{w(S)}.
\end{aligned}$$

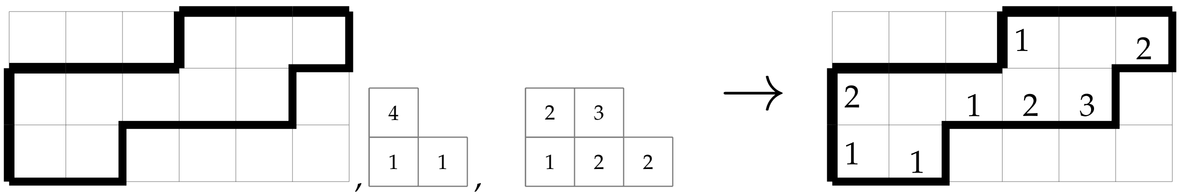
We can use lattice words to give a nice combinatorial interpretation of

$$\sum_{\nu \vdash n} \sum_{\eta \vdash n} \sum_{S \in P_{\nu,\eta}} K_{\lambda',\nu} K_{\mu',\eta} t^{w(S)},$$

using labeled polynomials (note that here we work with λ' for convenience).

We will call (lP, lQ) a labeled polyomino, if P (respectively Q) has positive integer labels beside each North step in P (respectively each East step except the first in Q) such that the integers in P (respectively Q) are increasing from South to North (respectively East to West). We use ccontent (P) (respectively ccontent(Q)) for the composition whose ith part gives the number of labels i along P (respectively Q).

If we start with $(P, Q) \in P_{\nu,\eta}$, let $\operatorname{colcomp}(P) = \alpha$ and $\operatorname{rowcomp}(Q) = \beta$ so that $\alpha \in R(\eta)$ and $\beta \in R(\nu)$. Then we can construct a labeled polyomino (lP, lQ), by the following: Select $T_1 \in T_{\mu',\alpha}$, and $T_2 \in K_{\lambda',\beta}$. Then working through the columns of T_1 from left to right, if i is in the cth column of T_i, place a c by a north step in the ith column of P (within a column, fill cells from South to North to force an increasing column condition). Repeat the process with Q and T_2, this time filling the cells above East steps in an analogous manner. Call the result $(\mathrm{lP}, \mathrm{lQ})$. For instance,



(5.1)

Note that we have exactly one label for each North step since $\mathrm{colcomp}(\mathrm{lP}) = \mathrm{content}(T_1)$. Similarly we have $\mathrm{rowcomp}(\mathrm{lQ}) = \mathrm{content}(T_2)$. Note that the conjugate of the shape of T_1, μ, is equal to $\mathrm{ccontent}(\mathrm{lP})$ and similarly $\mathrm{ccontent}(\mathrm{lQ})$ is equal to λ. Finally, note that if we read the labels from bottom to top along lP and right to left along lQ, each word must be a lattice word, since the columns of T_1 and T_2 are strictly increasing. Thus we call a labeled polyomino $(\mathrm{lP}, \mathrm{lQ})$ with $\mathrm{ccontent}(\mathrm{lP}) = \mu$ and $\mathrm{ccontent}(\mathrm{lQ}) = \lambda$ a lattice labeled polyomino in $\mathrm{llP}_{\lambda,\mu}$ if the word for each path is a lattice word. In particular we get

$$\sum_{\nu \vdash n} \sum_{\eta \vdash n} K_{\lambda,\nu} K_{\mu',\eta} \sum_{S \in P_{\nu,\eta}} t^{w(S)} = \sum_{P \in \mathrm{llP}_{\lambda',\mu}} t^{\mathrm{area}(P)}.$$

Theorem 5.2. *Let λ be a partition. Then*

$$\Delta_{s_\lambda} e_n \Big|_{q=1} = \sum_{\mu \vdash n} s_\mu \sum_{P \in llP_{\lambda',\mu}} t^{\mathrm{area}(P)}.$$

It is an interesting open question as to what second statistic on the labeled parallelogram polyominoes gives an interpretation for $\Delta_{s_\lambda} e_n$ (i.e. not just when $q = 1$). Several of the previous conjectures/theorems in [9] or [1] about $\Delta_{e_k} e_n$ can be seen to be consistent with this conjecture at $q = 1$, and thus provide clues in specific cases; we omit the details for this extended abstract.

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