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Whitney labelings and 0-Hecke algebra actions on graded posets

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Abstract. In the proceedings of FPSAC 2017, the authors introduced the notion of Whitney duality of graded posets. Two graded posets are Whitney dual if their Whitney numbers of the first and second kind are (up to a sign) switched. In this extended abstract, we present new results in the study of Whitney duals. We present new types of edge and chain-edge labelings of graded posets which we call Whitney labelings. We prove that every graded poset with a Whitney labeling has a Whitney dual and we show how to explicitly construct a Whitney dual using a technique that involves quotient posets. As an application, we explicitly construct a Whitney dual for the lattice of noncrossing partitions. We also show that a graded poset P with a Whitney labeling admits a local action of the 0-Hecke algebra on the set of maximal chains of P. The characteristic of the associated representation is Ehrenborg's flag quasisymmetric function of P.

Keywords: Whitney numbers, 0-Hecke algebra actions, Flag Quasisymmetric function

1 Introduction

Throughout this extended abstract all *partially ordered sets (or posets)* considered will be finite, graded, and contain a minimum element (denoted by $\hat{0}$). Moreover, P will denote a finite graded poset with a $\hat{0}$, with a rank function denoted by ρ and $\mu(x,y)$ will denote the *Möbius function* of the interval $[x,y]$ in P. For background on posets the reader may visit [7, Chapter 3] and [9].

This extended abstract is mainly concerned with two invariants that we can associate to a graded poset P and that play an important role in many areas of combinatorics. These invariants are the Whitney numbers of the first and second kind. The k^{th} *Whitney number of the first kind*, $w_k(P)$, is defined by

$$w_k(P) = \sum_{\rho(x)=k} \mu(\hat{0}, x), \tag{1.1}$$

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and the k^{th} *Whitney number of the second kind*, $W_k(P)$, is defined by

$$W_k(P) = |\{x \in P \mid \rho(x) = k\}|. \tag{1.2}$$

González D'León and Wachs noticed, while studying a poset of weighted partitions in [4], the existence of several familiar pairs of graded posets (P, Q) whose Whitney numbers of first and second kind (up to sign) were switched with respect to each other. This phenomenon motivated the following definition in [3].

Definition 1.1. [3] Let P and Q be graded posets. We say that P and Q are *Whitney duals* if for all $k \geq 0$ we have that

$$|w_k(P)| = W_k(Q) \text{ and } |w_k(Q)| = W_k(P). \tag{1.3}$$

Remark 1.2. Note that according to this definition for any graded poset P, a Whitney dual Q is not necessarily unique. Indeed, the authors show in the full version of this extended abstract ([2]) examples of posets with multiple nonisomorphic Whitney duals.

In [3], the authors proved that every geometric lattice has a Whitney dual. One example of this is the lattice of set partitions Π_n, which was shown in [3] to be Whitney dual to a poset of increasing spanning forests.

In this extended abstract we present new types of edge and chain-edge labelings that we call *Whitney labelings*. The main result that relates these new types of labelings with Whitney duality is the following.

Theorem 1.3. *Let P be a graded poset with a Whitney labeling λ. Then P has a Whitney dual. Moreover, using λ we can explicitly construct a Whitney dual $Q_\lambda(P)$ of P.*

In addition to giving sufficient conditions for Whitney duality, we are also able to show the following connection between Whitney labelings and representation theory.

Theorem 1.4. *Let P be a graded poset with a Whitney labeling λ. Then there exists a local 0-Hecke algebra action on the set of maximal chains of P. The characteristic of the associated representation is Ehrenborg's flag quasisymmetric function.*

This extended abstract is organized as follows. In Section 2 we describe the main ingredients used in the proof of Theorem 1.3. In Section 3 we present an example of a poset with a Whitney labeling, namely the lattice of noncrossing partitions $\mathcal{NC}_n$. We also give a combinatorial description of its Whitney dual $Q_{\lambda_{\mathcal{NC}}}(\mathcal{NC}_n)$. In Section 4 we discuss the steps involved in the proof of Theorem 1.4. We also show that the action described in Theorem 1.4 can be transported to an action on the set of maximal chains of the Whitney dual $Q_\lambda(P)$. In this case, the characteristic of this action is Ehrenborg's flag quasisymmetric function of $Q_\lambda(P)$ after ω (the well-known involution in the ring of quasisymmetric functions) is applied. We then use results of McNamara [5] to prove some structural properties of the posets $Q_\lambda(P)$.

2 Whitney labelings and Whitney duality

We say that $x \in P$ is *covered* by $y \in P$ and write $x \lessdot y$ if $x < y$ and there is no $z \in P$ such that $x < z < y$. The *Hasse diagram of P* is the directed graph on P whose directed edges are the covering relations $x \lessdot y$ in P. An *edge labeling or E-labeling* of P is a map $\lambda : \mathcal{E}(P) \to \Lambda$ where $\mathcal{E}(P)$ is the set of edges of the Hasse diagram of P and Λ is some other poset of labels. An edge labeling is said to be an *ER-labeling* if in every interval $[x, y]$ of P there is a unique *saturated or unrefinable chain* $\mathbf{c} : (x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{\ell-1} \lessdot x_\ell = y)$ that is *increasing*, i.e., such that

$$\lambda(x_0 \lessdot x_1) < \lambda(x_1 \lessdot x_2) < \cdots < \lambda(x_{\ell-1} \lessdot x_\ell).$$

We say that an edge labeling is an ER*-labeling if in each closed interval $[x, y]$ of P, there is a unique *ascent-free* saturated chain from x to y. That is, a chain $\mathbf{c}$ such that $\lambda(x_{i-1} \lessdot x_i) \not< \lambda(x_i \lessdot x_{i+1})$, for all $i = 1, \ldots, \ell - 1$.

A consequence of a theorem of Stanley (Theorem 3.14.2 in [7]) is that ER and ER*-labelings are useful for the determination of Möbius values and hence also of Whitney numbers. We summarize this relation in the following proposition.

Proposition 2.1. *Let P be a graded poset with an ER-labeling (ER*-labeling). Then $|w_k(P)|$ is the number of ascent-free (increasing) saturated chains starting at $\hat{0}$ of length k. Moreover, $W_k(P)$ is the number of increasing (ascent-free) saturated chains starting at $\hat{0}$ of length k.*

Let λ be an edge labeling of P. We say that λ has the *rank two switching property* if for every maximal chain $\mathbf{c} : (\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{k-1} \lessdot x_k)$ that has an increasing step $\lambda(x_{i-1} \lessdot x_i) < \lambda(x_i \lessdot x_{i+1})$ at rank i there is a unique maximal chain

$$\mathbf{c}' : (\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{i-1} \lessdot x_i' \lessdot x_{i+1} \lessdot \cdots \lessdot x_{k-1} \lessdot x_k),$$

whose labels are the same as the ones from $\mathbf{c}$ except that $\lambda(x_{i-1} \lessdot x_i') = \lambda(x_i \lessdot x_{i+1})$ and that $\lambda(x_i' \lessdot x_{i+1}) = \lambda(x_{i-1} \lessdot x_i)$. In this case, we say that the chain $\mathbf{c}'$ is obtained from $\mathbf{c}$ by a *quadratic exchange at rank i*.

Definition 2.2. An *EW-labeling* of P is an ER-labeling with the rank two switching property and with the property that in every interval $[x, y]$ of P each ascent-free maximal chain is determined uniquely by its sequence of labels from bottom to top.

Remark 2.3. In [2] the authors also define the notion of a *CW-labeling* that has the same implications with respect to Whitney numbers and Whitney duality as an EW-labeling, but where the underlying labeling is a chain-edge labeling (or C-labeling) as defined by Björner and Wachs (see [9] for the context about C-labelings). We will refer in general to a *Whitney labeling* in any result that applies to both types of labelings.

Remark 2.4. In [3] the authors defined the related concept of an $\overline{\text{EW}}$-labeling. The reason for the use of an overline in that article is that the conditions are far more restrictive and are just special cases of EW-labelings. While the definition of an $\overline{\text{EW}}$-labeling greatly simplifies the proofs of the theorems we present in this article, there are posets with EW-labelings, but no known $\overline{\text{EW}}$-labelings.

In addition to using edge labelings, we use the following notion of a quotient poset.

Definition 2.5. Let $\sim$ be an equivalence relation on a graded poset P such that if $x \sim y$, then $\rho(x) = \rho(y)$. We define the *quotient poset* $P/\sim$ to be the set of equivalence classes ordered by $X \leq Y$ if and only if there exists $x \in X$, $y \in Y$ and $z_1, z_2, \ldots, z_k \in P$ such that

$$x = z_0 \leq z_1 \sim z_2 \leq \cdots \leq z_{n-1} \sim z_k \leq z_{k+1} = y.$$

Definition 2.6. Given a graded poset P, let $C(P)$ denote the poset whose elements are saturated chains of P starting at $\hat{0}$ ordered by inclusion. We call $C(P)$ the *chain poset of* P. Suppose that λ is a Whitney labeling of P. Let $\sim_\lambda$ be the equivalence relation on $C(P)$ defined by $\mathbf{c} \sim_\lambda \mathbf{c}'$ if and only if there is a common chain $\mathbf{c}''$ that can be reached, after applying a sequence of quadratic exchanges, from both $\mathbf{c}$ and $\mathbf{c}'$. We will use $Q_\lambda(P)$ to denote $C(P)/\sim_\lambda$.

We now describe the main steps taken in the proof of Theorem 1.3. The complete details of the proof can be found in the full version of this article ([2]). We first define an edge labeling λ^* on $Q_\lambda(P)$ and prove that this labeling is an ER*-labeling. We then show that there is a label preserving bijection between the saturated chains of length k starting at $\hat{0}$ in P and the saturated chains of length k starting at $\hat{0}$ in $Q_\lambda(P)$. These steps, together with Proposition 2.1, imply that P and $Q_\lambda(P)$ are Whitney duals.

To define the labeling on $Q_\lambda(P)$, let $S(\mathbf{c})$ be the multiset of labels on the chain $\mathbf{c}$. Note that, by the definition of a quadratic exchange, $\mathbf{c} \sim_\lambda \mathbf{c}'$ implies $S(\mathbf{c}) = S(\mathbf{c}')$. In light of this, we will use $S(X)$ to denote the multiset of labels along any chain in the equivalence class $X \in Q_\lambda(P)$. If $X \lessdot Y$ in $Q_\lambda(P)$ then there exists a unique element in $S(Y) \setminus S(X)$. Define the edge labeling λ^* on $Q_\lambda(P)$ by

$$\lambda^*(X \lessdot Y) = S(Y) \setminus S(X). \tag{2.1}$$

Proposition 2.7. *Let P be a graded poset and let λ be a Whitney labeling of P. Then the labeling λ^* of $Q_\lambda(P)$ given by Equation (2.1) is an ER* labeling.*

Proposition 2.8. *Let λ be a Whitney labeling of P. There is a label preserving bijection from the set of saturated chains from $[\hat{0}]$ of length k in $Q_\lambda(P)$ and the set of saturated chains from $\hat{0}$ of length k in P. In particular, there is a label preserving bijection $\mathcal{M}_{Q_\lambda(P)} \to \mathcal{M}_P$, where $\mathcal{M}_P$ denotes the set of maximal chains of P.*

We omit the proofs of Propositions 2.7 and 2.8 because they are rather technical to be part of this extended abstract (see [2] for details). As mentioned earlier, these two propositions together with Proposition 2.1 imply Theorem 1.3.

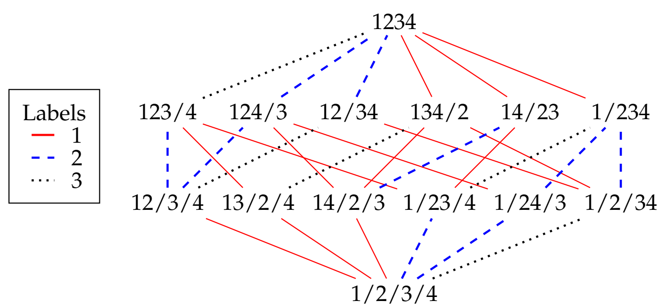


Figure 1: $\mathcal{NC}_4$.

2.1 A simpler description of $Q_\lambda(P)$

Even though Theorem 1.3 gives a constructive proof for the existence of Whitney duals, the description of a Whitney dual as a quotient is somewhat unsatisfying combinatorially. We now give a more combinatorial description of this quotient. We will use this description in Section 3 to describe a Whitney dual of the lattice of noncrossing partitions.

Let Λ be any poset and let w be a word with letters in the alphabet Λ. Assume that whenever $w_i < w_{i+1}$ we are allowed to do exchanges on w of the form

$$w_1 w_2 \cdots w_{i-1} w_i w_{i+1} w_{i+2} \cdots w_n \xrightarrow{i} w_1 w_2 \cdots w_{i-1} w_{i+1} w_i w_{i+2} \cdots w_n.$$

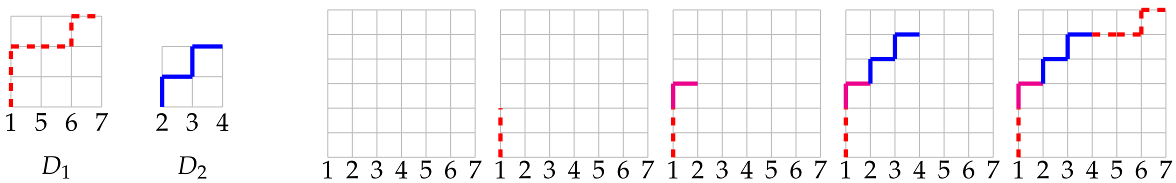
It is not hard to check that there is a unique ascent-free word w' that is related to w in this manner. We define $\mathrm{sort}(w) := w'$ to be this unique ascent-free word.

Definition 2.9. Let P be a poset with a Whitney labeling λ. Let $R_\lambda(P)$ be the poset whose elements are pairs (x, w) where $x \in P$ and w is the word of labels that uniquely determines an ascent-free saturated chain $\mathbf{c}$ in $[\hat{0}, x]$; and such that $(x, w) \lessdot (y, u)$ whenever $x \lessdot y$ and $u = \mathrm{sort}(w\lambda(\mathbf{c}, x \lessdot y))$ (wv here means concatenation of the words w and v).

Theorem 2.10. *If λ is a Whitney labeling of P, then $R_\lambda(P) \cong Q_\lambda(P)$.*

3 A Whitney labeling of the noncrossing partition lattice

We say a partition $\pi = B_1/B_2/\cdots/B_k$ of $[n]$ is *noncrossing* if there are no $a < b < c < d$ such that $a, c \in B_i$ and $b, d \in B_j$ for some $i \neq j$. For example, 124/35/67 is not a noncrossing partition since $2 < 3 < 4 < 5$ and $\{2,4\}$ and $\{3,5\}$ are in two different blocks, but 127/45/36 is noncrossing. The *noncrossing partition lattice*, denoted $\mathcal{NC}_n$, is the set of noncrossing partitions of $[n]$ ordered by refinement. As the name suggest, $\mathcal{NC}_n$ is a lattice and has many nice combinatorial properties (see [8] for more information). Figure 1 depicts $\mathcal{NC}_4$.



(a) Labeled Dyck paths (b) Steps involved in merging D_1 and D_2.

Figure 2

In [8], Stanley found a beautiful connection between $\mathcal{NC}_n$ and a set of combinatorial objects known as parking functions. A *parking function* of length n is a sequence of n positive integers $(p_1, p_2, \ldots, p_n)$ with the property that when rearranged in weakly increasing order $p_{i_1} \leq p_{i_2} \leq \cdots \leq p_{i_n}$, we have that $p_{i_k} \leq k$ for all k. In [8], an edge labeling of $\mathcal{NC}_n$ is defined with the property that the words of labels along the maximal chains are exactly the parking functions of length $n-1$. To describe this labeling of Stanley, first note that the cover relation in $\mathcal{NC}_n$ is given by merging exactly two blocks together. Suppose that σ is obtained from π by merging B_i and B_j, where $\min B_i < \min B_j$. Then define

$$\lambda_{\mathcal{NC}}(\pi \lessdot \sigma) = \max\{a \in B_i \mid a < \min B_j\}. \tag{3.1}$$

Figure 1 depicts this labeling for $\mathcal{NC}_4$ encoded using line styles and colors.

Theorem 3.1. *The labeling $\lambda_{\mathcal{NC}}$ is an EW-labeling of $\mathcal{NC}_n$. Hence $Q_\lambda(\mathcal{NC}_n)$ is a Whitney dual of $\mathcal{NC}_n$.*

Idea of the proof. One can check that $\lambda_{\mathcal{NC}}$ satisfies that in each interval of $\mathcal{NC}_n$ there is a unique strictly increasing chain and so $\lambda_{\mathcal{NC}}$ is an ER-labeling. The work in [8] considers a local action of the symmetric group $\mathfrak{S}_{n-1}$ on the set of maximal chains of $\mathcal{NC}_n$. Showing that this action is well-defined also implies that $\lambda_{\mathcal{NC}}$ has the rank two switching property (see section 4 of [8]). The fact that maximal chains are uniquely determined by parking functions implies that ascent-free chains are determined by their sequences of labels in each interval. □

We will now use Theorem 2.10 to provide a more familiar combinatorial description of the Whitney dual $Q_{\lambda_{\mathcal{NC}}}(\mathcal{NC}_n)$ of $\mathcal{NC}_n$. Recall that a Dyck path of order n is a lattice path from $(0,0)$ to (n,n) that never goes below the line $y = x$ and only takes steps in the directions of the vectors $(1,0)$ (East) and $(0,1)$ (North). We will consider Dyck paths D that come with a special labeling. Given an increasing sequence $b_1 < b_2 < \cdots < b_{n+1}$ of positive integers, we label the point $(i-1, 0)$ of D by b_i. In Figure 2a we illustrate two labeled Dyck paths.

We now define a process of "merging" two labeled Dyck paths D_1 and D_2 to obtain a new labeled Dyck path D. Suppose that D_1 and D_2 have disjoint and noncrossing label sets $B = \{b_1, b_2, \ldots, b_j\}$ and $C = \{c_1, c_2, \ldots, c_k\}$, where both sets are written in increasing order and $b_1 < c_1$. Since the sets are noncrossing then there exists an i such that $b_i < c_1 < c_2 < \cdots < c_k < b_{i+1}$ (where we use the convention $b_{j+1} = \infty$). Then, the new lattice path D, will be a path from $(0,0)$ to $(j+k, j+k)$ whose labels along the bottom row are $b_1, b_2, \ldots b_i, c_1, c_2, \ldots, c_k, b_{i+1}, \ldots, b_j$. From left to right until we reach the vertical line labeled b_i, D looks exactly the same as D_1. In the line labeled b_i in D we add all the north steps that D_1 had originally at b_i plus one additional north step followed by an additional east step from the line labeled b_i to the line labeled c_1. Then we glue D_2 where we left off in the line labeled c_1. After we finish gluing D_2, we glue the remaining part of D_1 that goes from the line labeled b_i to the line labeled b_j. In Figure 2b we illustrate the step by step construction of the labeled Dyck path D obtained by merging the two labeled Dyck paths D_1 and D_2 of Figure 2a. Note that in this example, 1 is the largest element in $\{1,5,6,7\}$ smaller than all the elements of $\{2,3,4\}$.

In order to verify that the resulting labeled lattice path is also a labeled Dyck path (that is, it has the same number of north and east steps and is always above the diagonal), we rely on an equivalent definition of a Dyck path. A *ballot sequence* of length $2n$ is a $\{0,1\}$-string $s_1 s_2 \cdots s_{2n}$ with the same number of 1's and 0's and such that for every $i \in [2n]$ the subword $s_1 s_2 \cdots s_i$ has at least as many 1's as 0's. It is well-known that a lattice path that takes only north and east steps is a Dyck path if and only if the sequence obtained associating to each north step a 1 and to each east step a 0 is a ballot sequence. Relying on this equivalent definition, we see that in the resulting path D the number of north steps and east steps is equal and the construction never breaks the property that every preamble in D contains at least as many north steps as east steps. Hence D is a well-defined labeled Dyck path.

Let $\mathcal{NCD}yck_n$ be the set whose objects are collections of labeled Dyck paths such that their underlying sets of labels form a noncrossing partition of $[n]$. We provide $\mathcal{NCD}yck_n$ with a partial order by defining for $F, F' \in \mathcal{NCD}yck_n$ the cover relation $F \lessdot F'$ whenever F' can be obtained from F by merging exactly two of the labeled Dyck paths in F. Note here that each labeled Dyck path can be represented by its set of labels together with an exponent for each label. The exponent of an element i being the number of north steps in the vertical line labeled i in its Dyck path. This notation extends to the elements in $\mathcal{NCD}yck_n$. For example, we can denote the collection of Dyck paths in Figure 2a by $1^2 5^0 6^1 7^0 / 2^1 3^1 4^0$. In Figure 3 we illustrate $\mathcal{NCD}yck_4$.

Theorem 3.2. *For all $n \geq 1$, $Q_{\lambda_{\mathcal{NC}}}(\mathcal{NC}_n) \cong \mathcal{NCD}yck_n$.*

Idea of Proof. Theorem 2.10 characterizes the poset $Q_{\lambda_{\mathcal{NC}}}(\mathcal{NC}_n)$ as being isomorphic to the poset $R_{\lambda_{\mathcal{NC}}}(\mathcal{NC}_n)$ whose elements are pairs (π, w) where $\pi \in \mathcal{NC}_n$ and w is the word of labels of an ascent-free chain in $[\hat{0}, \pi]$. Since maximal chains are labeled with parking functions, when $\pi = \{[n]\}$ is the partition with a single block, we have that w is a weakly decreasing parking function of length $n-1$, which are known to be in bijection

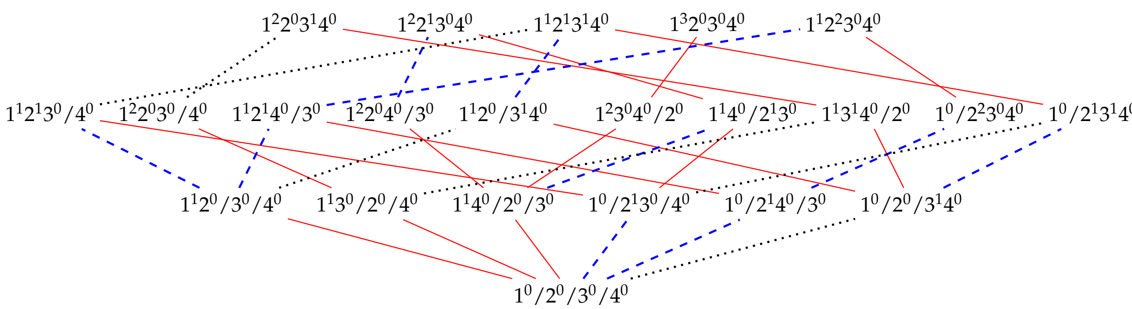


Figure 3: $\mathcal{NCD}yck_4$

with Dyck paths. The bijection assigns to a parking function with k_i occurrences of the label i the Dyck path with k_i north steps on the line $x = i - 1$. In our notation, the pairs $([4], w)$ are represented as $1^3 2^0 3^0 4^0$, $1^2 2^1 3^0 4^0$, $1^1 2^2 3^0 4^0$, $1^2 2^0 3^1 4^0$ and $1^1 2^1 3^1 4^0$. Now, it is not hard to see that any interval of the form $[\hat{0}, B_1/B_2/\cdots/B_k]$ is isomorphic to the product of smaller noncrossing partition lattices $\mathcal{NC}_{B_1} \times \mathcal{NC}_{B_2} \times \cdots \times \mathcal{NC}_{B_k}$, where $\mathcal{NC}_{B_j}$ is the lattice of noncrossing partitions of $B_j \subset [n]$. Note that the description of $\lambda_{\mathcal{NC}}$ given in Equation (3.1) is equivalent to the definition of merging labeled Dyck paths. Using this relation, the reader can also verify that a cover relation in $R_{\lambda_{\mathcal{NC}}}(\mathcal{NC}_n)$ corresponds to the merging of two labeled Dyck paths. □

It is interesting to note the well-known fact that the Möbius function value of $\mathcal{NC}_n$ is (up to a sign) the Catalan number C_{n-1}. This information is recovered here since $\mathcal{NCD}yck_n$ is a Whitney dual of $\mathcal{NC}_n$ and its maximal elements are Dyck paths of order $n-1$ which are Catalan objects.

We finish this section by noting that in the full version of this paper [2] we show that in addition to geometric lattices and the noncrossing partition lattice; the weighted partition poset studied by González D'León–Wachs [4] and the R*S-labelable posets studied by Simion–Stanley [6] (satisfying a certain consistency condition), all have Whitney labelings (and hence Whitney duals). The noncrossing partition lattices of types B and C, and Greene's poset of shuffles belong to this latter family.

4 $H_n(0)$-actions and Whitney labelings

In this section we describe an action of the 0-Hecke algebra on the maximal chains of a poset with a Whitney labeling. The techniques we describe here closely follow the work of Simion–Stanley [6] and McNamara [5].

Suppose that P is a graded poset of rank n. Moreover, suppose that λ is a Whitney labeling of P. We define maps $U_i : \mathcal{M}_P \to \mathcal{M}_P$ acting on the set $\mathcal{M}_P$ of maximal chains of P. For $\mathbf{c} \in \mathcal{M}_P$ define

$$U_i(\mathbf{c}) = \begin{cases} \mathbf{c}' & \text{if } \mathbf{c} \text{ has an ascent at position } i, \\ \mathbf{c} & \text{otherwise,} \end{cases}$$

where $\mathbf{c}'$ is the maximal chain obtained by applying a quadratic exchange at rank i to $\mathbf{c}$. As an example, consider the maximal chain $\mathbf{c} : (1/2/3/4 \lessdot 13/2/4 \lessdot 123/4 \lessdot 1234)$ in $\mathcal{NC}_4$ (see Figure 1). Since there is no ascent at rank 1, $U_1(\mathbf{c}) = \mathbf{c}$. However, there is an ascent at rank 2, and $U_2(\mathbf{c}) = 1/2/3/4 \lessdot 13/2/4 \lessdot 134/2 \lessdot 1234$.

The following proposition is a consequence of the definitions of the maps U_i's, the rank two switching property and the fact that ascent-free maximal chains are uniquely determined by their sequence of labels.

Proposition 4.1. *The maps $U_1, U_2, \dots, U_{n-1}$ have the following properties:*

1. *For all $\mathbf{c} \in \mathcal{M}_P$, $U_i(\mathbf{c})$ and $\mathbf{c}$ are the same except possibly at rank i.*
2. *$U_i^2 = U_i$ for all i.*
3. *$U_iU_j = U_jU_i$ for all i, j such that $|i - j| > 1$.*
4. *$U_iU_{i+1}U_i = U_{i+1}U_iU_{i+1}$ for all i.*

The 0-Hecke algebra of type A is defined by abstract generators satisfying the same properties of those in Proposition 4.1. Thus the properties described in the proposition imply that there is an action of the generators of the 0-Hecke algebra $H_n(0)$ on the set $\mathcal{M}_P$. This action is said to be *local* since the chains $U_i(\mathbf{c})$ and $\mathbf{c}$ are the same except possibly at rank i. Moreover, this action gives rise to a representation of the 0-Hecke algebra on the space $\mathbb{C}\mathcal{M}_P$ linearly spanned by $\mathcal{M}_P$.

4.1 The characteristic of this action

We will use χ_P to denote the character of the representation of $H_n(0)$ on $\mathbb{C}\mathcal{M}_P$. It turns out that the characteristic of this representation is a well-known quasisymmetric function. Before we look at this characteristic, we need to review some material on quasisymmetric functions associated with posets.

Ehrenborg [1] introduced the following formal power series known as the *flag quasisymmetric function*. Given a graded poset P with a $\hat{0}$ and $\hat{1}$, it is defined by

$$F_P(\mathbf{x}) = F_P(x_1, x_2, \dots) := \sum_{\hat{0}=t_1 \le t_2 \le \cdots \le t_{k-1} < t_k = \hat{1}} x_1^{rk(t_0,t_1)} x_2^{rk(t_1,t_2)} \cdots x_k^{rk(t_{k-1},t_k)}$$

where the sum is over all multichains from $\hat{0}$ to $\hat{1}$ where $\hat{1}$ appears exactly once. As the name suggests, $F_P(\mathbf{x})$ belongs to the ring $\mathcal{QS}ym$ of quasisymmetric functions. In addition to being quasisymmetric, $F_P(\mathbf{x})$ also keeps track of the flag f-vector and the flag h-vector of P as we describe next.

Let P be a graded poset of rank n with a $\hat{0}$ and a $\hat{1}$. For $S \subseteq [n-1]$ define $\alpha_P(S)$ to be the number of maximal chains of the rank-selected subposet P_S, that is, the induced subposet of P generated by the elements with ranks in S. The function given by $\alpha_P : 2^{[n-1]} \to \mathbb{Z}$ is called the *flag f-vector* of P. We also define $\beta_P(S) = \sum_{T\subseteq S}(-1)^{|S\setminus T|}\alpha_P(T)$ known as the *flag h-vector* of P. The reason for the names flag f-vector and flag h-vector is that they refine the classical f-vector and h-vector of the order complex of P. See [7, Section 3.13] for more details.

When P has a $\hat{0}$ and $\hat{1}$, there is a nice relationship between $F_P(\mathbf{x})$ and $\beta_P(S)$. Indeed, it is well-known that if P has rank n, then

$$F_P(\mathbf{x}) = \sum_{S\subseteq[n-1]} \beta_P(S) L_{S,n}(\mathbf{x}),$$

where $L_{S,n}$ is Gessel's fundamental quasisymmetric function associated to $S \subseteq [n-1]$.

The original definition of $F_P(\mathbf{x})$ requires that P has a $\hat{1}$. We extend this definition to more general posets, defining

$$F_P(\mathbf{x}) = \sum_m F_{[\hat{0},m]}(\mathbf{x}),$$

where the sum is over all maximal elements m of P. Note that in the case that P has a $\hat{1}$, this is just Ehrenborg's classical definition.

In the representation theory of $H_n(0)$, it is known that there are 2^{n-1} irreducible representations, all of them one-dimensional and hence they can be indexed by subsets of $[n-1]$. We denote by χ_S the irreducible representation indexed by $S \subseteq [n-1]$. The *(quasisymmetric) characteristic* of χ_S is defined by $\mathrm{ch}(\chi_S) = L_{S,n}(\mathbf{x})$.

Theorem 4.2. *Let P be a graded poset of rank n and λ a Whitney labeling of P. The local $H_n(0)$-action previously described is such that*

$$\mathrm{ch}(\chi_P) = F_P(\mathbf{x}).$$

Remark 4.3. The proof of this theorem is almost identical to the proof in [5, Proposition 4.1]. The difference lies in that our definition of $F_P(\mathbf{x})$ involves possibly more than one maximal interval and in our case the operators U_i remove ascents instead of descents as in McNamara's work.

Now suppose that λ is an ER-labeling of P and that P has a $\hat{0}$ and a $\hat{1}$. Recall that $\mathcal{M}_P$ denotes the set of maximal chains in P. For $\mathbf{c} : (x_0 \lessdot x_1 \lessdot x_2 \lessdot \cdots \lessdot x_n) \in \mathcal{M}_P$, the *descent set* of $\mathbf{c}$ is defined to be $D(\mathbf{c}) = \{i \mid \lambda(x_{i-1}, x_i) \not\lessdot \lambda(x_i, x_{i+1})\}$. It was shown by Stanley [7, Theorem 3.14.2] that $\beta_S(P)$ is the number of maximal chains with descent set S.

Example 4.4. We compute $F_P(\mathbf{x})$ for $P = \mathcal{NC}_4$. Recall from Section 3 that the labels on the set $\mathcal{M}_{\mathcal{NC}_4}$ of maximal chains of $\mathcal{NC}_4$ correspond to parking functions of length 3, see Figure 1. So $\mathcal{NC}_4$ has 16 maximal chains with label words given by $(1,1,1)$, the three permutations of each $(1,1,2)$ $(1,1,3)$ and $(1,2,2)$; and the six permutations of $(1,2,3)$. Considering the descent sets of each of these sequences we can compute that $\beta_{\mathcal{NC}_4}(\emptyset) = 1$, $\beta_{\mathcal{NC}_4}(\{1\}) = 5$, $\beta_{\mathcal{NC}_4}(\{2\}) = 5$ and $\beta_{\mathcal{NC}_4}(\{1,2\}) = 5$. Thus,

$$F_{\mathcal{NC}_4}(\mathbf{x}) = L_{\emptyset,3}(\mathbf{x}) + 5L_{\{1\},3}(\mathbf{x}) + 5L_{\{2\},3}(\mathbf{x}) + 5L_{\{1,2\},3}(\mathbf{x}).$$

The quasisymmetric function $F_{\mathcal{NC}_n}(\mathbf{x})$ is in fact symmetric. Stanley [8] showed that $\omega(F_{\mathcal{NC}_n}(\mathbf{x}))$ is *Haiman's Parking Function Symmetric Function* of n, where ω is the involution on the ring of quasisymmetric functions given by $\omega(L_{S,n}) = L_{S^c,n}$ where S^c is the complement of S in $[n-1]$.

A simple modification of Stanley's proof of the combinatorial description of the numbers $\beta_P(S)$, shows that if λ is an ER*-labeling of a poset P with $\hat{0}$ and $\hat{1}$, then $\beta_P(S) = \{\mathbf{c} \in \mathcal{M}_P \mid D(\mathbf{c}) = S^c\}$ for any $S \subseteq [n-1]$. With this information, one can show that $F_{\mathcal{NCD}yck_4}(\mathbf{x}) = 5L_{\emptyset,3}(\mathbf{x}) + 5L_{\{1\},3}(\mathbf{x}) + 5L_{\{2\},3}(\mathbf{x}) + L_{\{1,2\},3}(\mathbf{x})$. The reader may have noticed that $F_{\mathcal{NC}_4}(\mathbf{x}) = \omega(F_{\mathcal{NCD}yck_4}(\mathbf{x}))$. This is no coincidence as we will see.

Proposition 2.8 implies that there is a bijection between maximal chains of P and $Q_\lambda(P)$ which preserves labels. It follows that the local $H_n(0)$-action on $\mathcal{M}_P$ can also be transported to an $H_n(0)$-action on $\mathcal{M}_{Q_\lambda(P)}$. It turns out that this action on the maximal chains of $Q_\lambda(P)$ is also local.

Lemma 4.5. *Let P be a graded poset with a Whitney labeling λ. The 0-Hecke algebra action on $\mathcal{M}_{Q_\lambda(P)}$ is local.*

We denote by $\chi_{Q_\lambda(P)}$ the representation of $H_n(0)$ on $\mathbb{C}\mathcal{M}_{Q_\lambda(P)}$. Note that since the action on $Q_\lambda(P)$ is local, this representation restricts to representations χ_I on $\mathbb{C}\mathcal{M}_I$ where I is any maximal interval of $Q_\lambda(P)$. We obtain the following Proposition whose proof follows a similar idea of the one of Theorem 4.2, but noticing that $Q_{\lambda(P)}$ has an ER*-labeling instead of an ER-labeling.

Proposition 4.6. *Let P be a graded poset with a Whitney labeling λ. For any maximal interval I in $Q_\lambda(P)$,*

$$ch(\chi_I) = \omega(F_I(\mathbf{x})).$$

We obtain the following theorem as a corollary.

Theorem 4.7. *Let P be a graded poset with a Whitney labeling λ. Then*

$$F_p(x) = ch(\chi_P) = ch(\chi_{Q_\lambda(P)}) = \omega(F_{Q_\lambda}(x)).$$

Idea of the proof. Note that $F_{Q_\lambda(P)}(\mathbf{x}) = \sum_I (F_I(\mathbf{x}))$ and that $\chi_P = \chi_{Q_\lambda(P)} = \sum_I \chi_I$, where the sums are over maximal intervals I of $Q_\lambda(P)$. $\square$

We finish with some structural results concerning the maximal intervals of $Q_\lambda(P)$. A poset P is called *bowtie-free* if there does not exist distinct $a, b, c, d \in P$ with $c \lessdot a$, $d \lessdot a$, $c \lessdot b$ and $d \lessdot b$. In [5], McNamara showed that a bowtie-free poset P with a $\hat{0}$ and a $\hat{1}$ has a local $H_n(0)$-action with the property that the characteristic of this action is $\omega(F_p(x))$ if and only if P is snellable. Additionally, he showed that if P is a lattice, then P is supersolvable. Proposition 4.6 then implies the following corollary.

Corollary 4.8. *Let P be a graded poset with a Whitney labeling λ. If I is a maximal interval of $Q_\lambda(P)$ and is bowtie free, then I is snellable. Moreover, if I is a lattice, then I is supersolvable.*

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