Séminaire Lotharingien de Combinatoire **80B** (2018)
Article #93, 12 pp.

Proceedings of the 30th Conference on Formal Power Series and Algebraic Combinatorics (Hanover)

A Schur–Weyl like construction of the rectangular representation for the double affine Hecke algebra

D. Jordan*[1] and M. Vazirani†[2]

[1] *School of Mathematics, University of Edinburgh, Edinburgh, UK*
[2] *Department of Mathematics, UC Davis, Davis, CA, USA*

Abstract. Let $G = GL_N$ and V be its N-dimensional defining representation. Given a module M for the algebra of quantum differential operators on G, and a positive integer n, we may equip the space $F_n(M)$ of invariant tensors in $V^{\otimes n} \otimes M$, with an action of the double affine Hecke algebra of type GL_n.

In this paper we take M to be the basic module, i.e. the quantized coordinate algebra $M = \mathcal{O}_{\mathsf{q}}(G)$. We describe a weight basis for $F_n(\mathcal{O}_{\mathsf{q}}(G))$ combinatorially in terms of walks in the type A weight lattice; these are equivalent to standard periodic tableaux, and subsequently we identify $F_n(\mathcal{O}_{\mathsf{q}}(G))$ with the irreducible "rectangular representation" of height N of the double affine Hecke algebra.

Keywords: Cherednik algebras, representation theory, quantum differential algebra

1 Introduction

Classic Schur–Weyl duality involves commuting actions of GL_N and the symmetric group $\mathfrak{S}_n$ on $V^{\otimes n}$ where $V = \mathbb{C}^N$ is the defining representation of GL_N. Under this duality, the GL_N-invariants yield the $N \times k$ rectangular representation of $\mathfrak{S}_n$ when $n = kN$. In this paper $\mathfrak{S}_n$ is replaced by the double affine Hecke algebra (DAHA) and GL_N is replaced by the algebra of quantum differential operators $\mathcal{D}_{\mathsf{q}}(GL_N)$. These two algebras have commuting actions on the invariants in $V^{\otimes n} \otimes M$, where M is a $\mathcal{D}_{\mathsf{q}}(GL_N)$-module. We show that in the case M is the "basic" $\mathcal{D}_{\mathsf{q}}(GL_N)$-module that this yields the rectangular representation of the DAHA. This duality is very useful, as the representation theory of the DAHA is well-understood in terms of type A algebraic combinatorics, while the representation theory of $\mathcal{D}_{\mathsf{q}}(GL_N)$ is much less well-understood.

Throughout the paper, $G = GL_N$, $\mathfrak{g} = \mathfrak{gl}_N$, $n = kN$. Associated to the quantum group $U_{\mathsf{q}}(\mathfrak{g})$ is the quantized coordinate algebra $\mathcal{O}_{\mathsf{q}}(G)$ and the algebra of quantum differential operators $\mathcal{D}_{\mathsf{q}}(G)$. $\mathcal{D}_{\mathsf{q}}(G)$ is a q-deformation of the algebra $D(G)$ of differential operators

*djordan@ed.ac.uk. Jordan was partially supported by the European Research Council under the European Union's Horizon 2020 research and innovation programme (grant agreement no. 637618).

†vazirani@math.ucdavis.edu. Vazirani was partially supported by the Simons Foundation.

on G. $M = \mathcal{O}_{\mathfrak{q}}(G)$ is naturally $\mathcal{D}_{\mathfrak{q}}(G)$-module, which we call the basic $\mathcal{D}_{\mathfrak{q}}(G)$-module. We show $F_n(M) := (V^{\otimes n} \otimes M)^{\mathrm{inv}}$ is the rectangular representation of the DAHA. See [7] for more details. We do so by restricting $F_n(M)$ to the commutative subalgebra $\mathcal{Y}$ of the DAHA and analyzing its $\mathcal{Y}$-weights, which has a lovely combinatorial description naturally encoded by standard periodic tableaux. The theory of $\mathcal{Y}$-semisimple DAHA representations then determines the isomorphism type of $F_n(M)$.

2 Combinatorics in type A: lattice walks & skew tableaux

We fix positive integers N, k and $n = kN$ throughout the paper (except for in specific examples), and $G = GL_N, \mathfrak{g} = \mathfrak{gl}_N$. V is the N-dimensional defining representation $V = V_{\epsilon_1}$ of $\mathfrak{g}$ (or more precisely of $U_{\mathfrak{q}}(\mathfrak{g})$).

2.1 The GL_N weight lattice

Consider $\mathbb{R}^N$, with standard basis, $\mathcal{E} = \{\epsilon_i \mid i = 1, \dots, N\}$ and symmetric form $\langle\, , \rangle$ with respect to which $\mathcal{E}$ is an orthonormal basis. The weight lattice of $\mathfrak{g}$ is

$$\Lambda = \bigoplus_{i=1}^{N} \mathbb{Z}\epsilon_i = \mathbb{Z}^N.$$

Elements of Λ are called integral weights. The dominant integral weights are

$$\Lambda^+ = \{m_1\epsilon_1 + \cdots + m_N\epsilon_N \mid m_i \in \mathbb{Z}, m_1 \geq \cdots \geq m_N\}.$$

We remark that $\mathcal{E}$ are the weights of V. Let us denote $\mathbf{d} := \epsilon_1 + \epsilon_2 + \cdots + \epsilon_N$.

We introduce a special weight ρ given by

$$\rho = \frac{1}{2}((N-1)\epsilon_1 + (N-3)\epsilon_2 + (N-5)\epsilon_3 + \cdots + (1-N)\epsilon_N).$$

Observe $2\rho \in \Lambda^+$, although ρ might not be depending on the parity of N.

Definition 2.1. Given a dominant integral weight $\lambda = \sum_i m_i\epsilon_i \in \Lambda^+$ we denote by $\mathrm{YD}(\lambda)$ the diagram (or integer partition) with fewer than N parts,

$$\mathrm{YD}(\lambda) = (m_1 - m_N, m_2 - m_N, \dots, m_{N-1} - m_N, 0).$$

We will call the diagonal through the upper left box of $\mathrm{YD}(\lambda)$ the *principal diagonal*, and we decree that this diagonal is labelled with m_N. The other diagonals are labelled consecutively, so that the next diagonal to the right is labelled $m_N + 1$, etc. Equivalently, we can say that the upper left box is in row 1 and column $m_N + 1$, and then the diagonal is the column number minus the row number.

Note the diagram $\mathrm{YD}(\lambda + r\mathbf{d})$ is that of $\mathrm{YD}(\lambda)$ shifted r units right, and so its diagonal labels are incremented $+r$. Hence, although we draw the same diagram for λ as well as $\lambda + r\mathbf{d}$, they are distinguished by their diagonal labels.

Given $\lambda = \sum_i m_i \epsilon_i$, its dual weight is $\lambda^* := \sum_i -m_i \epsilon_{N+1-i}$. Observe therefore that if one takes $\mathrm{YD}(\lambda^*)$ and rotates it 180 degrees, then it is the complement to $\mathrm{YD}(\lambda)$ in a $N \times (m_1 - m_N)$ rectangle. See Figure 1.

Let us describe the diagonal labels in terms of the inner product on Λ. Consider λ as compared to $\lambda + \epsilon_i$. The diagram has one extra box and we claim the diagonal of that box is labeled $\langle \lambda, \epsilon_i \rangle + 1 - i = \langle \lambda + \epsilon_i, \epsilon_i \rangle - i$. The new box in the ith row. Note that $m_i = \langle \lambda, \epsilon_i \rangle$. The ith row of $\mathrm{YD}(\lambda)$ has length $m_i - m_N$, which is to say it ends $m_i - m_N$ units to the right of the leftmost column, so the new box is in column $m_i + 1$, and thus the $m_i + 1 - i$ diagonal. The diagonal label of the new box is thus

$$m_i + 1 - i = \langle \lambda, \epsilon_i \rangle + \langle \rho, \epsilon_i \rangle - \langle \rho, \epsilon_1 \rangle. \tag{2.1}$$

2.2 Walks on the weight lattice

Definition 2.2. A *walk in Λ^+ of length n*, from weight λ to weight μ is a finite sequence,

$$\underline{u} = (\lambda = u_0, u_1, \ldots, u_n = \mu),$$

where each $u_i \in \Lambda^+$, and each difference $u_i - u_{i-1}$ lies in $\mathcal{E}$. We denote by $\delta_i(\underline{u})$ the index of $u_i - u_{i-1} \in \mathcal{E}$, so that $u_i - u_{i-1} = \epsilon_{\delta_i(\underline{u})}$.

Definition 2.3. A walk in Λ^+ of length n which begins at λ and ends at $\lambda + k\mathbf{d}$ is called a *looped walk* at λ. We denote by $\mathcal{W}_\lambda^{N,k}$ the set of all looped walks at λ of length $n = kN$.

Note that the multiset $\{\epsilon_{\delta_i(\underline{u})} \mid 1 \leq i \leq n\}$ of steps taken on any looped walk $\underline{u}$ consists of $\mathcal{E}$ with multiplicity $k = n/N$. See Figure 2 for an example of a looped walk.

2.3 Skew tableaux

We shall now recall an alternative combinatorial description of $\mathcal{W}_\lambda^{N,k}$ in terms of skew tableaux. We first associate to a weight $\lambda \in \Lambda^+$ a skew diagram

$$\mathrm{D}_\lambda^{N,k} = (\mathrm{YD}(\lambda) + (k^N)) / \mathrm{YD}(\lambda).$$

Equivalently we may obtain $\mathrm{D}_\lambda^{N,k}$ by removing $\mathrm{YD}(\lambda)$ from the upper left, and $\mathrm{YD}(\lambda^*)$, rotated 180 degrees, from the lower right, of the $N \times (k + m_1 - m_N)$ rectangular diagram. See Figure 1. The skew diagram $\mathrm{D}_\lambda^{N,k}$ inherits diagonal labels from $\mathrm{YD}(\lambda)$ as well as choice of principal diagonal.

Recall that a standard tableau on a (skew) diagram with n boxes is a filling of its boxes with $\{1, 2, \ldots, n\}$ such that entries increase across rows and down columns.

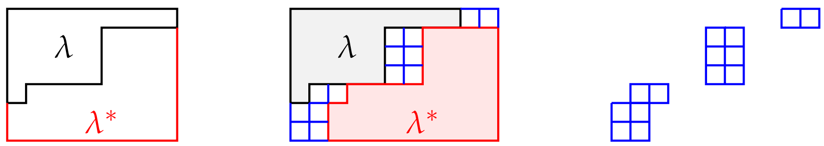

Figure 1: The skew diagram $D_\lambda^{7,2}$ in the case $N = 7$, $k = 2$, $n = 14$.

Definition 2.4. Given a weight $\lambda \in \Lambda^+$, we denote by $\mathcal{SK}_\lambda^{N,k}$ the set of all standard tableaux on the diagonal-labeled skew shape $D_\lambda^{N,k}$.

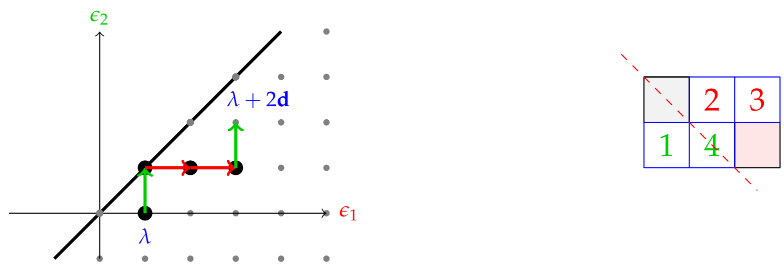

Figure 2: A looped walk $\underline{u}$ at $\lambda = \epsilon_1$ of length 4 and the skew tableau $\mathcal{T}ab(\underline{u})$ with the dashed line indicating the principal diagonal, which here is labeled 0.

Definition 2.5. Define the map $\mathcal{T}ab : \mathcal{W}_\lambda^{N,k} \to \mathcal{SK}_\lambda^{N,k}$ from length $n = kN$ looped walks at $\lambda \in \Lambda^+$ to standard skew tableaux of shape $D_\lambda^{N,k}$ as follows: for each $i = 1, \ldots, n$ fill the leftmost vacant box in the $\delta_i(\underline{u})$-th row of $D_\lambda^{N,k}$ with the symbol i.

Proposition 2.6 ([9])**.** *The map* $\mathcal{T}ab : \mathcal{W}_\lambda^{N,k} \xrightarrow{\sim} \mathcal{SK}_\lambda^{N,k}$ *is a bijection.*

Example 2.7. The looped walk in Figure 2 is $\underline{u} = (\lambda, \lambda + \epsilon_2, \lambda + \epsilon_2 + \epsilon_1, \lambda + \epsilon_2 + \epsilon_1 + \epsilon_1, \lambda + \epsilon_2 + \epsilon_1 + \epsilon_1 + \epsilon_2 = \lambda)$, and so the sequence $(\delta_1(\underline{u}), \delta_2(\underline{u}), \delta_3(\underline{u}), \delta_4(\underline{u})) = (2, 1, 1, 2)$. Compare this to the skew tableau $\mathcal{T} = \mathcal{T}ab(\underline{u})$ which places 2 and 3 in the first row, 1 and 4 in the second row.

2.4 Periodic tableaux

For the rectangular shape $\mu = (k^N)$, we extend it to a "periodic diagram" $\cup_{r \in \mathbb{Z}} \mu[r]$ which coincides with the $N \times \infty$ strip as in Figure 3. In terms of coordinates $\mu[r] = \mu + r(0, k)$.

We always consider the fundamental domain $\mu[0]$ to be anchored on the 0-diagonal, and so extend our diagonal labeling.

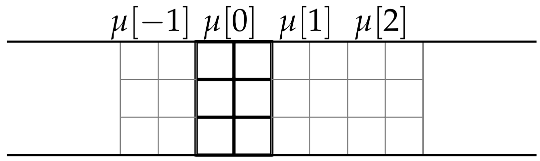

Figure 3: The diagram $\mu = (2^3)$ is made periodic by shifting horizontally.

Definition 2.8. Let $n = kN$. An n-periodic standard tableaux of shape $\mu = (k^N)$ is a bijection $R : \mathbb{Z} \to \{\text{boxes of } N \times \infty \text{ strip}\}$ such that:

- fillings increase across rows and down columns,
- the fillings of $\mu[0]$ are distinct mod n,
- the fillings of $\mu[r]$ are those of $\mu[0]$ $+nr$.

We will denote the set of all such tableaux $\mathrm{P}_n\mathrm{SYT}(k^N)$.

An $R \in \mathrm{P}_n\mathrm{SYT}(k^N)$ is completely determined by the fillings of $\mu[0]$, see Figure 4. (However it may happen that the filling of $\mu[0]$ is row- and column-increasing, but its periodization is not standard.) Observe that $\mathrm{D}_\lambda^{N,k}$ is also a fundamental domain of the periodization of $\mu = (k^N)$. Similarly, "periodizing" a standard skew tableau in $\mathcal{T} \in \mathcal{SK}_\lambda^{N,k}$ (i.e., filling in the rest of the entries according to the periodicity constraint) yields a well-defined standard periodic tableau in $\mathrm{P}_n\mathrm{SYT}(k^N)$, as soon as we specify the compatibility with the diagonal labelling. In other words, since the filling of $\mathcal{T}$ is $\{1, \ldots, n\}$ it is easy to see its periodization is standard. This shows the map $\mathcal{P}er$ below is well-defined.

Definition 2.9. The *periodization* map,

$$\mathcal{P}er : \bigsqcup_{\lambda \in \Lambda^+} \mathcal{SK}_\lambda^{N,k} \to \mathrm{P}_n\mathrm{SYT}(k^N) \tag{2.2}$$

sends $\mathcal{T}$ to the unique periodic tableau in $\mathrm{P}_n\mathrm{SYT}(k^N)$ agreeing with $\mathcal{T}$ in the fundamental domain of shape $\mathrm{D}_\lambda^{N,k}$ located along the $N \times \infty$ strip so that diagonal labels coincide.

See Figure 4. In that example, note the skew tableaux are only differentiated by their diagonal labels and likewise for the periodic tableaux.

Proposition 2.10. *The map $\mathcal{P}er$ is a bijection.*

Definition 2.11. Given $R \in \mathrm{P}_n\mathrm{SYT}(k^N)$, let $\mathrm{diag}_R : \mathbb{Z} \to \mathbb{Z}$ be the map such that $\mathrm{diag}_R(i)$ is the label of the diagonal on which $\boxed{\;i\;}$ lies.

Note $\mathrm{diag}_R(i+n) = \mathrm{diag}_R(i+kN) = \mathrm{diag}_R(i) + k$.

Definition 2.12. The *weight* $\mathrm{wt}(R) \in (\mathcal{K}^\times)^n$ of $R \in \mathrm{SYT}(k^N)$ or of $R \in \mathrm{P}_n\mathrm{SYT}(k^N)$, is the tuple,

$$\mathrm{wt}(R) = \left(t^{2\mathrm{diag}_R(1)}, t^{2\mathrm{diag}_R(2)}, \ldots, t^{2\mathrm{diag}_R(n)}\right) =: t^{(2\mathrm{diag}_R(1), 2\mathrm{diag}_R(2), \ldots, 2\mathrm{diag}_R(n))}.$$

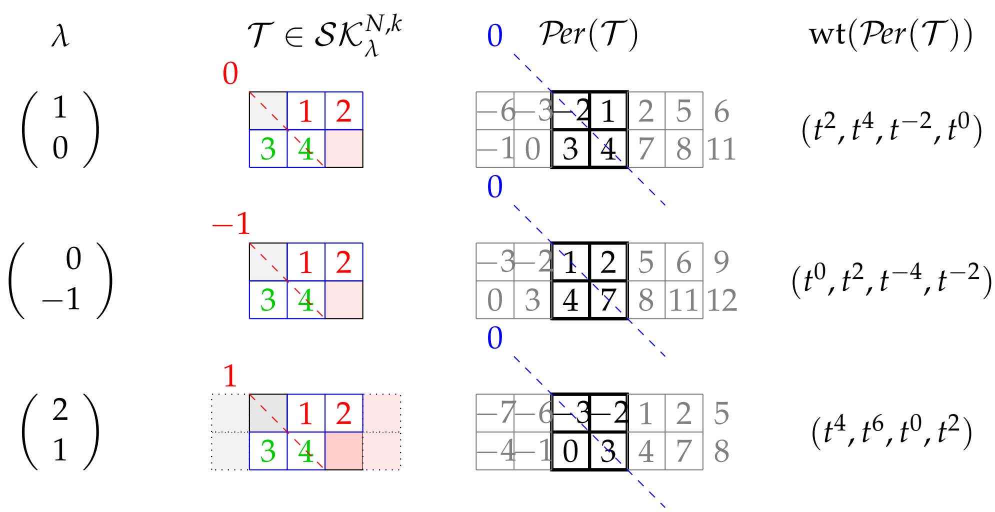

Figure 4: Here $N = 2, k = 2$. The principal diagonal is marked red. The fundamental rectangle of $\mathcal{P}er(\mathcal{T})$ is chosen so that the 0th diagonal matches that of $\mathcal{T} \in \mathcal{SK}_\lambda^{N,k}$.

3 The rectangular representation of the DAHA

The key result of this section is Theorem 3.7.

Let $\mathcal{K}$ denote a field of characteristic zero, and let $q, t \in \mathcal{K}^\times$, and assume neither q nor t is a root of unity. Typical instances are $\mathcal{K} = \mathbb{C}$, $\mathbb{C}(t)$, or $\mathbb{C}(q,t)$.

Definition 3.1. The *extended affine symmetric group* is[1]

$$\widehat{\mathfrak{S}}_n = \left\langle \pi, s_i,\ i \in \mathbb{Z}/n\mathbb{Z} \;\middle|\; \begin{array}{ll} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{for } i \in \mathbb{Z}/n\mathbb{Z}, \\ s_i s_j = s_j s_i & \text{for } j \not\equiv i \pm 1 \bmod n, \\ \pi s_i = s_{i+1}\pi & \text{for } i \in \mathbb{Z}/n\mathbb{Z}, \\ s_i^2 = 1 & \text{for } i \in \mathbb{Z}/n\mathbb{Z} \end{array} \right\rangle.$$

We recall that $\widehat{\mathfrak{S}}_n$ acts on $\mathbb{Z}$ by n-periodic permutations, i.e. bijections $\sigma : \mathbb{Z} \to \mathbb{Z}$ such that $\sigma(i+n) = \sigma(i) + n$. It also acts on the set $(\mathcal{K}^\times)^n$ via:

$$\begin{aligned} s_i \cdot (a_1, \ldots, a_i, a_{i+1}, \ldots a_n) &= (a_1, \ldots, a_{i+1}, a_i, \ldots, a_n) \\ s_0 \cdot (a_1, a_2, \ldots, a_{n-1}, a_n) &= (q a_n, a_2, \ldots, a_{n-1}, q^{-1} a_1) \\ \pi \cdot (a_1, \ldots, a_n) &= (q a_n, a_1, a_2, \ldots, a_{n-1}). \end{aligned} \tag{3.1}$$

The $\widehat{\mathfrak{S}}_n$ action on $\mathbb{Z}$ descends to an action on periodic tableaux, as follows. We set $\sigma \cdot R$ to be the tableau where $\boxed{i}$ is replaced with $\boxed{\sigma(i)}$.

[1] We drop the first relation when $n = 2$.

The function diag_R is compatible with the $\widehat{\mathfrak{S}}_n$ action: $\mathrm{diag}_{\sigma\cdot R}(\sigma(i)) = \mathrm{diag}_R(i)$ for any $\sigma \in \widehat{\mathfrak{S}}_n$. Furthermore the action intertwines the action (3.1) of $\widehat{\mathfrak{S}}_n$ on $(\mathcal{K}^\times)^n$: we have $\mathrm{wt}(\sigma \cdot R) = \sigma \cdot \mathrm{wt}(R)$. (We observe that $\sigma \cdot R$ need not be standard, even if R is.) Any domain for the n-periodicity in Definition 2.8 is also a domain for the π^n-action. Note that π^n shifts the $N \times \infty$ strip k steps horizontally.

Definition 3.2. The *GL_n double affine Hecke algebra* $\mathbb{H}_{q,t} = \mathbb{H}_{q,t}(GL_n)$ is the $\mathcal{K}$-algebra presented by generators:

$$T_0, T_1, \dots T_{n-1}, \pi^{\pm 1}, Y_1^{\pm 1}, \dots, Y_n^{\pm 1},$$

subject to relations[2]:

$$\begin{array}{ll}
(T_i - t)(T_i + t^{-1}) = 0 \quad (i = 0, \dots, n-1), & \\
T_i T_j T_i = T_j T_i T_j \quad (j \equiv i \pm 1 \bmod n), & T_i T_j = T_j T_i \quad (\text{otherwise}), \\
\pi T_i \pi^{-1} = T_{i+1} \quad (i = 0, \dots, n-2), & \pi T_{n-1} \pi^{-1} = T_0, \\
T_i Y_i T_i = Y_{i+1} \quad (i = 1, \dots, n-1), & T_0 Y_n T_0 = q^{-1} Y_1 \\
T_i Y_j = Y_j T_i \quad (j \not\equiv i, i+1 \bmod n), & \\
\pi Y_i \pi^{-1} = Y_{i+1} \quad (i = 1, \dots, n-1), & \pi Y_n \pi^{-1} = q^{-1} Y_1.
\end{array}$$

We often refer to the double affine Hecke algebra as the DAHA.

We set $\mathcal{Y}$ to be the commutative subalgebra of $\mathbb{H}_{q,t}$ generated by the $Y_i^{\pm 1}$, $1 \le i \le n$ and $H(\mathcal{Y})$ to be the subalgebra generated by $\mathcal{Y}$ and the T_i, $1 \le i < n$.

3.1 The rectangular representations

In his hallmark paper [3], Cherednik gave a complete classification of irreducible $\mathcal{Y}$-semisimple representations, i.e. those $\mathbb{H}_{q,t}$-modules for which the $\mathcal{Y}$-action can be diagonalized. His classification builds on the parallel story for the affine Hecke algebra [5, 4, 2], [10]. Subsequently, the paper [11] built on Cherednik's classification via periodic skew diagrams combinatorially, connecting standard tableaux on the diagrams to $\mathcal{Y}$-weights. In this section we detail a very special case of Cherednik's construction, when the Young diagram indexing the irreducible module is an $N \times k$ rectangle and the periodicity is purely horizontal, so that the shape is not actually skew but an $N \times \infty$ strip.

3.1.1 $\mathcal{Y}$-semisimple representations

A tuple $\underline{z} = (z_1, \dots, z_n) \in (\mathcal{K}^\times)^n$ is called a *$\mathcal{Y}$-weight*. Let M be an $\mathbb{H}_{q,t}$-module. We define its support to be

$$\mathrm{supp}(M) = \{\underline{z} \mid M[\underline{z}] \neq 0\} \quad \text{where} \quad M[\underline{z}] = \{v \in M \mid Y_i v = z_i v,\ 1 \le i \le n\}$$

[2]As with $\widehat{\mathfrak{S}}_n$, we drop the relations on the second line when $n = 2$.

is its $\underline{z}$-weight space. A non-zero $v \in M[\underline{z}]$ is called a weight vector, or $\underline{z}$-weight vector.

Definition 3.3. We call M *$\mathcal{Y}$-semisimple* if we have an isomorphism $M \cong \bigoplus_{\underline{z}} M[\underline{z}]$, as $\mathcal{Y}$-modules.

Such M are called calibrated in [10]. Note that M is $\mathcal{Y}$-semisimple if and only if it has a *weight basis*: a basis consisting of $\mathcal{Y}$-weight vectors. Further, in this case $\mathrm{Res}_{\mathcal{Y}}^{\mathbb{H}_{q,t}}(M)$ is semisimple as a $\mathcal{Y}$-module.

The structure of $\mathcal{Y}$-semisimple modules is extremely rigid. Given a $\mathcal{Y}$-semisimple module, one can read its composition factors directly from its support. If M is both simple and $\mathcal{Y}$-semisimple, its nonzero weight spaces are all one-dimensional. Further one need only determine a single $\underline{z} \in \mathrm{supp}(M)$ in order to determine all of $\mathrm{supp}(M)$, and hence the isomorphism type of M.

The support of a simple $\mathcal{Y}$-semisimple modules has a lovely combinatorial structure. It is easy to show that if M is simple, then its support is contained in a single $\widehat{\mathfrak{S}}_n$-orbit. If additionally M is $\mathcal{Y}$-semisimple, we can say *exactly* what subset of the $\widehat{\mathfrak{S}}_n$-orbit we get, i.e. we can completely determine $\mathrm{supp}(M)$. More precisely, given $\underline{z} \in \mathrm{supp}(M)$, one can determine the set $S \subset \widehat{\mathfrak{S}}_n$ such that $\mathrm{supp}(M) = \{w \cdot \underline{z} \mid w \in S\}$. The following theorem uniquely characterizes the set S, which depends on choice of $\underline{z}$:

Theorem 3.4 ([5, 4, 2, 3], [10])**.** *Let M be a simple and $\mathcal{Y}$-semisimple $\mathbb{H}_{q,t}$-module. Let $\underline{z} \in \mathrm{supp}(M)$. We have:*

1. *For $1 \leq i < n$, we have $M[s_i \cdot \underline{z}] = 0$ if and only if $\frac{z_i}{z_{i+1}} \in \{t^2, t^{-2}\}$. Further*
$$T_i M[\underline{z}] \subset M[\underline{z}] \oplus M[s_i \cdot \underline{z}].$$
2. *$M[s_0 \cdot \underline{z}] = 0$ if and only if $\frac{qz_n}{z_1} \in \{t^2, t^{-2}\}$. Further $T_0 M[\underline{z}] \subset M[\underline{z}] \oplus M[s_0 \cdot \underline{z}]$.*
3. *We have $M[\pi \cdot \underline{z}] \neq 0$, and $\pi M[\underline{z}] = M[\pi \cdot \underline{z}]$.*

Note that Theorem 3.4 allows us to precisely describe the action of the $\mathbb{H}_{q,t}$-generators on a weight basis, once we have chosen a sensible normalization or scaling. The proof of this theorem uses the theory of "intertwiners" [3], for which the reader may also consult [11].

3.1.2 Induction of the rectangular representation to the DAHA

Associated to the partition $\mu = (k^N)$ is a finite dimensional irreducible representation of the finite Hecke algebra $\langle T_i, 1 \leq i < n \rangle$. A basis for this representation is indexed by the set of standard Young tableaux of shape (k^N). We denote by $\mathrm{Rect}(N,k)$ the $H(\mathcal{Y})$-module obtained by inflating this module via the homomorphism sending $T_i \mapsto T_i$, $Y_1 \mapsto t^0 = 1$. It is well-known for generic t (i.e. away from small roots of unity) that

$\mathrm{Rect}(N,k)$ is $\mathcal{Y}$-semisimple. Identifying the $N \times k$ rectangular diagram (k^N) with $\mathrm{D}_0^{N,k}$, we assign diagonal labels as in Section 2. Below we denote standard tableaux of shape (k^N) by $\mathrm{SYT}(k^N)$.

Proposition 3.5. *The irreducible $H(\mathcal{Y})$-module $\mathrm{Rect}(N,k)$ has a $\mathcal{Y}$-weight basis,*

$$\{v_R \mid R \in \mathrm{SYT}(k^N)\} \quad \textit{with} \quad Y_i v_R = t^{2\mathrm{diag}_R(i)} v_R$$

when t is generic. In particular each v_R has weight $\mathrm{wt}(R)$.

Let

$$M(k^N) = \mathrm{Ind}_{H(\mathcal{Y})}^{\mathbb{H}_{q,t}} \mathrm{Rect}(N,k)$$

denote the induced module $\mathbb{H}_{q,t} \otimes_{H(\mathcal{Y})} \mathrm{Rect}(N,k)$. This module has basis

$$\{T_\sigma \otimes v_R \mid \sigma \in \widehat{\mathfrak{S}}_n/\mathfrak{S}_n, R \in \mathrm{SYT}(k^N)\}$$

which can be ordered so that, with respect to this basis, $\mathcal{Y}$ acts triangularly. $M(k^N)$ thus has support $\mathrm{supp}\left(M(k^N)\right) = \{\sigma \cdot \mathrm{wt}(R) \mid \sigma \in \widehat{\mathfrak{S}}_n/\mathfrak{S}_n, R \in \mathrm{SYT}(k^N)\}$.

Theorem 3.6. *Let $q = t^{-2k}$. Then $M(k^N)$ has unique simple quotient.*

It is not hard to show that for any $R \in \mathrm{SYT}(k^N)$ the $\mathrm{wt}(R)$ weight space of $M(k^N)$ is one-dimensional, and thus it follows that it has unique simple quotient.

Using the next theorem one may explicitly construct this unique simple quotient and thereby show it is $\mathcal{Y}$-semisimple.

Theorem 3.7 ([3], [11])**.** *When $q = t^{-2k}$ there exists a unique irreducible representation $L(k^N)$ of $\mathbb{H}_{q,t}$ that is $\mathcal{Y}$-semisimple with support $\{\mathrm{wt}(R) \mid R \in \mathrm{P}_n\mathrm{SYT}(k^N)\}$. In particular we may realize $L(k^N)$ as the linear span over $\mathcal{K}$ of*

$$\{v_R \mid R \in \mathrm{P}_n\mathrm{SYT}(k^N)\},$$

such that each v_R is a $\mathcal{Y}$-weight vector of weight $\mathrm{wt}(R)$, i.e., $Y_i v_R = t^{2\mathrm{diag}_R(i)} v_R, \quad 1 \le i \le n$.

Corollary 3.8. *When $q = t^{-2k}$, the unique simple quotient of $M(k^N)$ is $L(k^N)$; in particular it is $\mathcal{Y}$-semisimple and its support is given in Theorem 3.7.*

4 The functor and the isomorphism

4.1 Quantum algebras: $U_{\mathrm{q}}(\mathfrak{g})$, $\mathcal{O}_{\mathrm{q}}(G)$ and $\mathcal{D}_{\mathrm{q}}(G)$

Recall $G = GL_N$, $\mathfrak{g} = \mathfrak{gl}_N$.

We refer to [8] and [7] for detailed definitions, in particular the Serre presentation of the quantum group $U_{\mathsf{q}}(\mathfrak{g})$, the formulas for R-matrices, and the Peter–Weyl theorem.

The algebra of quantum differential operators on G, which we denote by $\mathcal{D}_{\mathsf{q}}(G)$, was studied in many different settings. The presentation as a twisted tensor product

$$\mathcal{D}_{\mathsf{q}}(G) = \mathcal{O}_{\mathsf{q}}(G)\widetilde{\otimes}\mathcal{O}_{\mathsf{q}}(G), \tag{4.1}$$

is adapted from the paper [12] (see also [1]).

We denote by ℓ and $\partial_{\triangleleft}$ the inclusions into the first and second tensor factor of (4.1), so that $\ell \otimes \partial_{\triangleleft} : \mathcal{O}_{\mathsf{q}}(G) \otimes \mathcal{O}_{\mathsf{q}}(G) \to \mathcal{D}_{\mathsf{q}}(G)$ is the tautological isomorphism of $U_{\mathsf{q}}(\mathfrak{g})$-modules (however it is not an algebra homomorphism). This is a q-deformation of the tensor decomposition $\mathcal{D}(G) \cong \mathcal{O}(G) \otimes U(\mathfrak{g})$ into functions on G, and the vector fields of left-translation, hence the notation.

For the purposes of this paper, all we need is

Theorem 4.1 (Peter–Weyl decomposition)**.** *As a module for $U_{\mathsf{q}}(\mathfrak{g})$ we have an isomorphism: $\mathcal{O}_{\mathsf{q}}(G) \cong \bigoplus_{\lambda\in\Lambda^+} V_\lambda \otimes V_\lambda^*$.*

4.2 The functor F_n

Recall $V = V_{\epsilon_1}$ is the N-dimensional defining representation of $U_{\mathsf{q}}(\mathfrak{g})$. We denote by $\det_{\mathsf{q}}$ the 1-dimensional representation of $U_{\mathsf{q}}(\mathfrak{g})$ that has weight $\mathbf{d}$. Let M be a $\mathcal{D}_{\mathsf{q}}(G)$-module. Given $n \in \mathbb{N}$, we define the functor F_n via setting $F_n(M)$ to be the following space of $\det_{\mathsf{q}}^k$-variants:

$$F_n(M) = \left(\det_{\mathsf{q}}^{-k}(V) \otimes \underset{n}{V} \otimes \cdots \otimes \underset{1}{V} \otimes M\right)^{U_{\mathsf{q}}(\mathfrak{g})}.$$

In [6], an action of the double affine Hecke algebra was constructed on the space $F_n^{SL_N}$. Let us summarize its GL-modification.

Theorem 4.2. *Let M be a module for $\mathcal{D}_{\mathsf{q}}(G)$, k be a positive integer, and $n = kN$. When $q = \mathsf{q}^{-2k}$ and $t = \mathsf{q}$, we have an exact functor, $F_n : \mathcal{D}_{\mathsf{q}}(G)$-mod $\to \mathbb{H}_{q,t}(GL_n)$-mod such that the generators of $\mathbb{H}_{q,t}(GL_n)$ act on $F_n(M)$ as follows.*

1. *T_i $(i = 1, \ldots, n-1)$ acts by the braiding $\sigma_{V,V}$ on the $\underset{i+1}{V} \otimes \underset{i}{V}$ factors.*
2. *The double braiding on $\det_{\mathsf{q}}^{-k}(V) \otimes \underset{n}{V}$ acts as the scalar q^{-2k}.*
3. *Y_1 acts only in the rightmost two tensor factors $\underset{1}{V} \otimes M$ via*

$$Y_1 = \sigma_{M_\triangleleft,V} \circ \sigma_{V,M_\triangleleft}, \text{ (the double-braiding of } V \text{ and } M\text{, using } \partial_\triangleleft).$$

4. *X_1 acts only in the rightmost two tensor factors $\underset{1}{V} \otimes M$ via*

$$X_1 = V \otimes M \xrightarrow{\Delta_V \otimes \mathrm{Id}_M} V \otimes \mathcal{O}_{\mathrm{q}}(G) \otimes M \xrightarrow{\mathrm{Id}_V \otimes \mathrm{act}_M} V \otimes M,$$

where in the second arrow, $\mathcal{O}_{\mathrm{q}}(G)$ acts on M via the homomorphism $\ell : \mathcal{O}_{\mathrm{q}}(G) \to \mathcal{D}_{\mathrm{q}}(G)$.

In the special case $M = \mathcal{O}_{\mathrm{q}}(G)$, the above action is compatible with Peter-Weyl decomposition, in the following sense. Let

$$W^n_\lambda = \left(\det{}_{\mathrm{q}}^{-k}(V) \otimes V^{\otimes n} \otimes V_\lambda \otimes V^*_\lambda\right)^{U_{\mathrm{q}}(\mathfrak{g})} \cong \mathrm{Hom}_{U_{\mathrm{q}}(\mathfrak{gl}_N)}(\det{}^k_{\mathrm{q}}(V) \otimes V_\lambda, V^{\otimes n} \otimes V_\lambda). \quad (4.2)$$

Then as vector spaces

$$F_n(\mathcal{O}_{\mathrm{q}}(G)) \cong \bigoplus_{\lambda \in \Lambda^+} W^n_\lambda.$$

The Pieri rule applied to (4.2) gives $W^n_\lambda \cong \bigoplus_{\underline{u} \in \mathcal{W}^{N,k}_\lambda} L_{\underline{u}}$, a decomposition into one-dimensional subspaces. Relating the double braiding to the action of the Casimir operator then yields the following theorem.

Theorem 4.3. *Let $q = \mathrm{q}^{-2k}$ and $t = \mathrm{q}$. For any $v \in L_{\underline{u}}$ we have*

$$Y_i v = t^{\langle u_i + 2\rho, u_i\rangle - \langle u_{i-1} + 2\rho, u_{i-1}\rangle - \langle \epsilon_1 + 2\rho, \epsilon_1\rangle} v. \quad (4.3)$$

4.3 The isomorphism type of $F_n(\mathcal{O}_{\mathrm{q}}(G))$

We are finally ready to prove our main theorem:

Theorem 4.4. *Let $\lambda \in \Lambda^+$, $\underline{u}$ be a looped walk in $\mathcal{W}^{N,k}_\lambda$, and $R = \mathcal{P}er(\mathcal{T}ab(\underline{u}))$.*

1. *Each subspace $L_{\underline{u}}$ is a $\mathcal{Y}$-weight space of weight $\mathrm{wt}(R)$; for any $v \in L_{\underline{u}}$, we have*

$$Y_i v = t^{2\mathrm{diag}_R(i)} v.$$

2. *We have isomorphisms of $\mathbb{H}_{q,t}(GL_n)$-modules $F_n(\mathcal{O}_q(GL_N)) \cong L(k^N)$.*

Proof. All that remains is to simplify the exponent of t appearing in Theorem 4.3.

$$\begin{aligned}
&\langle u_i + 2\rho, u_i\rangle - \langle u_{i-1} + 2\rho, u_{i-1}\rangle - \langle \epsilon_1 + 2\rho, \epsilon_1\rangle \\
&\quad = 2\left(\langle u_{i-1}, \epsilon_{\delta_i(\underline{u})}\rangle + \langle \rho, \epsilon_{\delta_i(\underline{u})}\rangle - \langle \rho, \epsilon_1\rangle\right) \\
&\quad = 2(\text{the label of the diagonal on which } \boxed{i} \text{ lies in } \mathcal{T}ab(\underline{u})), \text{ by (2.1)} \\
&\quad = 2\,\mathrm{diag}_{\mathcal{P}er(\mathcal{T}ab(\underline{u}))}(i).
\end{aligned}$$

Hence we may re-write the scalar (4.3) by which Y_i acts as $t^{2\mathrm{diag}_R(i)}$. Having matched their support, the isomorphism $F_n(\mathcal{O}_{\mathrm{q}}(G)) \cong L(k^N)$ then follows from Theorem 3.7.

□

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