Séminaire Lotharingien de Combinatoire **80B** (2018)
Article #95, 12 pp.

Proceedings of the 30th Conference on Formal Power Series and Algebraic Combinatorics (Hanover)

Bijection from multiline queues to rhombic tableaux for the inhomogeneous 2-TASEP

Olya Mandelshtam*[1]

[1] *Brown University, Providence, Rhode Island*

Abstract. The 2-TASEP is a model describing the dynamics of first and second class particles hopping in one direction on a finite 1D lattice. For the 2-TASEP with periodic boundary conditions, there is a well-known description for the stationary probabilities in terms of *multiline queues* of Ferarri and Martin. On the other hand, for the 2-TASEP with open boundary conditions, there is a rich connection to tableaux combinatorics: its stationary probabilities are described using *rhombic alternative tableaux*. In this article, we unify the two approaches by defining a new object, the *toric rhombic tableaux* and describing a simple bijection between these tableaux and multiline queues for the 2-TASEP with periodic boundary conditions. Furthermore, with a natural modification of both the rhombic alternative tableaux and the toric rhombic tableaux, we obtain a tableaux interpretation for probabilities of the inhomogeneous 2-TASEP both with periodic and open boundary conditions, in which different classes of particles hop with different rates. Through our bijection, our result generalizes a result of Ayyer and Linusson on multiline queues.

Keywords: multispecies TASEP, multiline queues, rhombic tableaux

1 Introduction

The 2-TASEP is a Markov chain describing class 1 and 2 particles hopping in one direction on a finite 1D lattice with n sites, with priority given to the class 2 particles. The 2-TASEP with open boundaries is strongly connected to a rich garden of combinatorial objects, including tableaux (alternative tableaux [13], permutation tableaux [5], staircase tableaux [6, 4]). More specifically, stationary probabilities for states of the 2-TASEP with open boundaries can be expressed as a sum over weighted fillings of certain tableaux [12]. The 2-TASEP with periodic boundary conditions, on the other hand, is a special case of the k-TASEP on $\mathbb{Z}/n\mathbb{Z}$, which has been studied primarily using multiline queues of Ferrari and Martin [7]: stationary probabilities for states of the 2-TASEP with periodic boundary conditions can be expressed by enumerating the corresponding multiline

*olya@math.brown.edu. Olya Mandelshtam was supported by the Postdoctoral Presidential Fellowship and NSF grant DMS-1704874.

queues [2]. Our goal is to draw a parallel between the two methods by describing a bijection between certain tableaux and multiline queues in the 2-TASEP case.

The 2-TASEP of size n has sites numbered 1 through n with each site occupied by 1, 2, or 0, representing the particles of class 1 and 2 and holes, respectively. The possible transitions of the 2-TASEP chain are the following, all occurring with rate 1:

$$X\,2\,0\,Y \xrightarrow{1} X\,0\,2\,Y, \qquad X\,2\,1\,Y \xrightarrow{1} X\,1\,2\,Y, \qquad X\,1\,0\,Y \xrightarrow{1} X\,0\,1\,Y,$$

where X, Y are words in $\{2,1,0\}$. In the open boundary case, the bulk transitions are the same as above, with parameters α and β dictating the rates of boundary transitions:

$$0\,Y \xrightarrow{\alpha} 2\,Y, \qquad Y\,2 \xrightarrow{\beta} Y\,0.$$

In Section 2, we define the multiline queues (MLQ) and rhombic alternative tableaux (RAT), which are the main objects we work with. Our first main result is in Section 3, in which we define the *cylindric rhombic tableaux* (CRT) and give a direct bijection Ψ_{CRT} with MLQs.

Theorem 1.1. *For X a state of the 2-TASEP and $\mathrm{CRT}(X)$ denoting the set of associated cylindric rhombic tableaux, the stationary probability of X is proportional to $|\mathrm{CRT}(X)|$.*

Theorem 1.2. *For X a state of the 2-TASEP and $\mathrm{MLQ}(X)$ denoting the set of associated multiline queues, $\Psi_{\mathrm{CRT}} : \mathrm{MLQ}(X) \to \mathrm{CRT}(X)$ is a bijection.*

The inhomogeneous 2-TASEP with periodic boundary conditions is a generalization with additional parameters x_2, x_1, which are the respective rates of the transitions $21 \to 12$ and $10 \to 01$ (by normalization, $20 \to 02$ still has rate 1). In their paper studying a Markov chain on permutations similar to the inhomogeneous k-TASEP with transition rates x_{ij} between particles of class i and j for $k \geq i > j \geq 0$, Lam and Williams conjectured that stationary probabilities for this process are proportional to polynomials in the x_{ij}'s with positive integer coefficients [8]. This conjecture was solved when $x_1 = 1$ by Ayyer and Linusson for the 2-TASEP [3]. Using our tableaux approach, in Section 4 we present a simple solution incorporating x_2 and x_1 for the 2-TASEP case by putting weights $\mathrm{wt}(T)$ on CRT; our solution recovers the result of [3].

Theorem 1.3. *For X a state of the inhomogeneous 2-TASEP with periodic boundary conditions and with transition rates x_2, x_1, the stationary probability of X is proportional to $\sum_{T \in \mathrm{CRT}(X)} \mathrm{wt}(T)$.*

Finally, we are able to naturally extend our results to the inhomogeneous 2-TASEP with open boundaries: in Section 4.1, we use our bijection to define *acyclic multiline queues* which are in bijection with the RAT, and put weights $\mathrm{wt}_M(T)$ on the RAT to obtain an analogous formula for the stationary probabilities.

Theorem 1.4. *For X a state of the inhomogeneous 2-TASEP with open boundaries and with transition rates x_2, x_1, the stationary probability of X is proportional to $\sum_{T \in \mathrm{RAT}(X)} \mathrm{wt}_M(T)$.*

This is an extended abstract of the article [11].

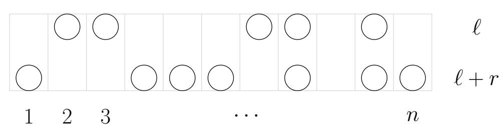
2 Preliminaries

A state of the 2-TASEP is represented by a word $X = X_1 \dots X_n$ where $X_i \in \{2, 1, 0\}$. For the 2-TASEP with periodic boundary conditions, we consider X_1 and X_n to be adjacent. To distinguish between the different 2-TASEPs we study, we will write TASEP(n, r) representing states of the 2-TASEP with open boundaries with n particles and exactly r class 1 particles, and TASEP(k, r, ℓ) representing states of the 2-TASEP with periodic boundary conditions and exactly k class 2 particles, r class 1 particles, and ℓ holes.

We define the two main objects we work with in the 2-TASEP context: multiline queues (MLQs) and rhombic alternative tableaux (RAT).

2.1 Multiline queues

An MLQ of size (k, r, l) with $n := k + r + \ell$ is a choice of ℓ and $\ell + r$ locations in the top and bottom rows, respectively, of a $2 \times n$ array. Locations are labeled from left to right with $[1, n]$. We identify the left and right edges of the array, making it a cylinder; thus location 1 is to the right of and adjacent to location n.



Each MLQ corresponds uniquely to a state of the TASEP(k, r, ℓ), which we call its type. To determine the type of an MLQ, we describe a *ball drop algorithm*, consisting of balls from the top row dropping to occupy balls in the bottom row.

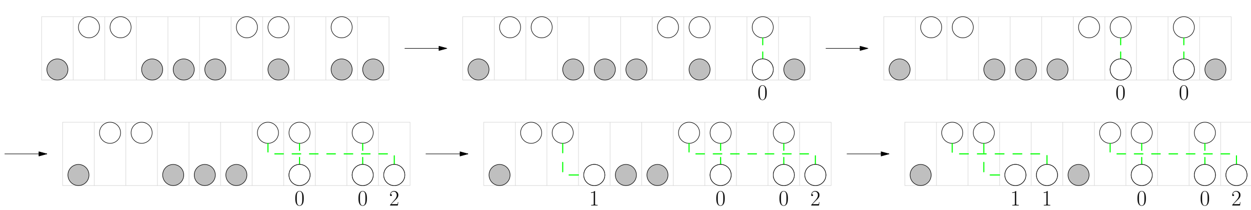


Figure 1: The steps of Algorithm 2.1 for an MLQ of type $X = 12200120200$. The white (occupied) bottom row balls are the 0-balls, and the weight vector is $(1, 1, 0, 0, 2)$.

Algorithm 2.1 (Ball drop)**.** Let $y_1 < \cdots < y_\ell$ be the locations of the top row balls in the MLQ. For $i = \ell, \dots, 1$:

Ball drop of y_i. Drop the top row ball at y_i to the bottom row to occupy the first unoccupied bottom row ball weakly to its right, while marking every unmarked vacancy it hits on its path. The bottom row ball it occupies is marked as a 0-ball. The *hitting weight* of the occupied 0-ball is the number of vacancies that were marked during the drop.

Output. All bottom row balls that are not 0-balls (i.e. were not hit by a dropping top row ball) are marked as 1-balls. A state of the 2-TASEP is read off the bottom row by associating 0-balls, 1-balls, and vacancies to holes, class 1, and class 2 particles, respectively. The output of the algorithm is the type X of the MLQ and a weight vector $(w_1, \dots, w_\ell)$ of hitting weights from left to right.

See Figure 1 for an example of Algorithm 2.1.

Lemma 2.2. *Given the output of the type $X \in \text{TASEP}(k,r,\ell)$ and weight vector $(w_1, \dots, w_\ell)$ of Algorithm 2.1, one can uniquely determine the locations of the top row balls of an MLQ.*

Lemma 2.2 is proved by reversing Algorithm 2.1 with weight vector $(w_1, \dots, w_\ell)$ and "lifting" the 0-balls at locations $x_1 < \cdots < x_\ell$ such that the 0-ball at x_i marks the first w_i unmarked vacancies to its left. Observe that at the end of Algorithm 2.1, each marked vacancy will have been marked by exactly one dropping ball. Consequently, it can be shown that when the lifts are performed from left to right, the original locations of the top row balls of the MLQ are recovered.

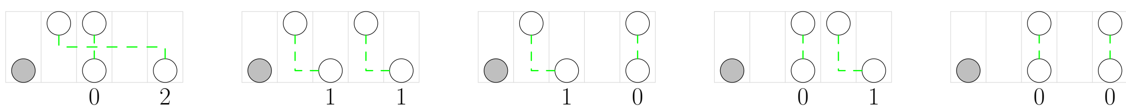


Figure 2: $\text{MLQ}(X)$ for $X = 12020$ with weight vectors $(0,2), (1,1), (1,0), (0,1), (0,0)$.

Since the ℓ balls in the top row and the $r+\ell$ balls in the bottom row are chosen independently, there is a total of $\binom{n}{\ell}\binom{n}{k}$ MLQs of size (k,r,ℓ). The following theorem of Ferrari and Martin gives an elegant expression for probabilities of the 2-TASEP with periodic boundary conditions. We remark that this theorem also holds for the k-TASEP with a more general definition of MLQs. Let $\text{MLQ}(X)$ be the set of MLQs of type X.

Theorem 2.3 ([7])**.** *Let $X \in \text{TASEP}(k,r,\ell)$ for $n := k+r+\ell$ be a state of the 2-TASEP with periodic boundary conditions. Then the stationary probability of X is* $\Pr(X) = \frac{1}{\binom{n}{k}\binom{n}{\ell}}|\text{MLQ}(X)|$.

Example 2.4. From Figure 2, for $X = 12020$ we obtain $\Pr(X) = \frac{5}{\binom{5}{2}\binom{5}{2}} = \frac{1}{20}$.

2.2 Rhombic alternative tableaux

The RAT were defined by the author and Viennot in [12] as a solution for the more general 2-ASEP model with open boundaries, in which the parameter q dictates the rate at which particles can hop in the opposing direction (setting $q = 0$ recovers the 2-TASEP). In this section, we give a definition of the subset of RAT corresponding to the $q = 0$ case.

The RAT are fillings with up-arrows and left-arrows of a tiling of a closed shape whose boundary is composed of south, southwest, and west edges on a triangular lattice.

The tiles are three types of rhombic tiles which we call 20-tiles, 10-tiles, and 21-tiles. Each tile can contain an arrow that points either towards its left vertical edge or the top horizontal edge with the condition that any tile that is "pointed to" by an arrow must be empty; we call them *left-arrows* and *up-arrows* respectively. We give a precise definition below.

Definition 2.5. Let $X \in \text{TASEP}(n,r)$ such that X has k 2's and ℓ 0's with $k+r+\ell = n$. Define a lattice path $P(X)$ as follows: reading X from left to right, draw a south edge for a 2, a southwest edge for a 1, and a west edge for a 0. From the northeast corner of $P(X)$, draw ℓ west edges followed by r southwest edges, followed by k south edges, thus obtaining a closed region $\Gamma(X)$, whose southeast boundary is $P(X)$. We call $\Gamma(X)$ a *rhombic diagram*.

Call a rhombus with south and west edges a *20-tile*, a rhombus with west and southwest edges a *10-tile*, and a rhombus with south and southwest edges a *21-tile*. Choose a tiling $\mathcal{T}$ with the 20-tiles, 21-tiles, and 10-tiles on $\Gamma(X)$.

Definition 2.6. A *north-strip* is a connected strip composed of adjacent 20- and 10-tiles. A *west-strip* is a connected strip composed of adjacent 20- and 21-tiles. The 20-tile and the 21-tile can contain a *left-arrow*, while the 20-tile and the 10-tile can contain an *up-arrow*. See Figure 3.

Definition 2.7. A tile is *pointed at* by an arrow if it is in the same west-strip to the left of a left-arrow or if it is in the same north-strip above an up-arrow. Conversely, a tile is *free* if it is not pointed at by any arrow (a tile containing an arrow is considered free).

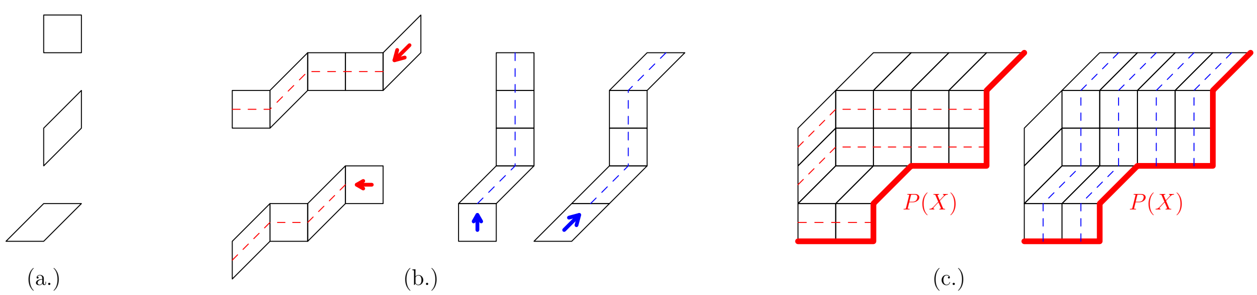


Figure 3: (a.) 20-, 21-, and 10-tiles; (b.) left-arrows and up-arrows and the tiles they point to; (c.) all west-strips and north-strips in $\Gamma(X)$ for $X = 122001200$.

Definition 2.8. A *rhombic alternative tableau* of type $X \in \text{TASEP}(n,r)$ is a rhombic diagram $\Gamma(X)$ with some tiling $\mathcal{T}$ that is filled with up-arrows and left-arrows such that:

(i.) a tile must be empty if it is pointed at by an up-arrow or a left-arrow, and (ii.) a free tile must contain an up-arrow or a left-arrow.

Definition 2.9. The *weight* of a RAT R of size (n,r) with ufree(R) north-strips free of up-arrows and lfree(R) west-strips free of left-arrows is $\text{wt}(R) = \alpha^{n-r-\text{ufree}(R)}\beta^{n-r-\text{lfree}(R)}$.

For $X \in \text{TASEP}(n,r)$, we define $\text{RAT}_{\mathcal{T}}(X)$ to be the set of fillings of $\Gamma(X)$ with some fixed tiling $\mathcal{T}$. Because it turns out $\sum_{R\in\text{RAT}_{\mathcal{T}}(X)} \text{wt}(R)$ is independent of the tiling (see Prop 2.8 of [12]), we define $\text{weight}(X) = \sum_{R\in\text{RAT}_{\mathcal{T}}(X)} \text{wt}(R)$ for arbitrary tiling $\mathcal{T}$.

The following result is Theorem 3.1 in [12], and is proved with the canonical Matrix Ansatz technique.

Theorem 2.10 ([12])**.** *Let $X \in \text{TASEP}(n,r)$ be a state of the 2-TASEP with open boundary conditions. The stationary probability of X is $\Pr(X) = \frac{1}{\mathcal{Z}_{n,r}}\text{weight}(X)$, where $\mathcal{Z}_{n,r} = \sum_{X\in\text{TASEP}(n,r)} \text{weight}(X)$.*

3 Cylindric rhombic tableaux

We introduce tableaux that we call *cylindric rhombic tableaux* (CRT), related to the RAT, for an analogous formula for the stationary probabilities of the 2-TASEP with periodic boundary conditions.

Figure 4: For $X = 1120200120$, (a.) X-strip and (b.) cylindric diagram $\mathcal{H}(X)$.

Let $X = X_1 \dots X_n \in \text{TASEP}(k,r,\ell)$ with $n := k + r + \ell$. Define an *X-strip* to be a connected strip of 20-tiles and 21-tiles, which is built from right to left by reading X from left to right and appending a 20-tile for every 0 and a 21-tile for every 1. We define the *cylindric diagram* $\mathcal{H}(X)$ to be a stack of k X-strips on top of each other with $P(X)$ as in Definition 2.5 superimposed (see Figure 4).

Identify the vertical edges on the left boundary of $\mathcal{H}(X)$ with the corresponding vertical edges belonging to the same west-strip on the right boundary, making $\mathcal{H}(X)$ a cylinder. We carry over all definitions of tiles, strips, and arrows from the RAT. Set each west-strip to start at the tile directly west of $P(X)$ and end at the tile directly east of $P(X)$. We break symmetry with north-strips, however, by setting each north-strip (which are by construction composed of strips of k 20-tiles) to run from bottom to top of $\mathcal{H}(X)$, disregarding the position of $P(X)$.

Definition 3.1. A CRT of type X is a filling of the tiles of $\mathcal{H}(X)$ with left-arrows and up-arrows according to same rules as the RAT in Definition 2.8. We denote the set of CRT by $\mathrm{CRT}(X)$.

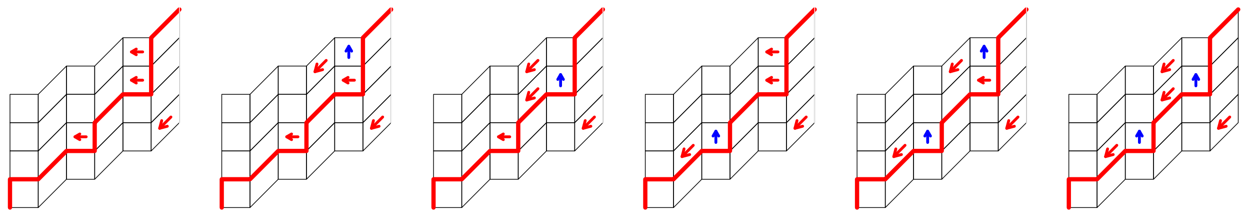


Figure 5: The set $\mathrm{CRT}(X)$ for $X = 1220120102$.

Our main result is the following, illustrated by Figure 5.

Theorem 3.2. *Let $X \in \mathrm{TASEP}(k, r, \ell)$ be a state of the 2-TASEP with periodic boundary conditions, with $n := k + r + \ell$. Then* $\Pr(X) = \frac{1}{\binom{n}{k}\binom{n}{\ell}} |\mathrm{CRT}(X)|$.

The proof for Theorem 3.2 is a straightforward Matrix Ansatz argument nearly identical to that for Theorem 3.1 in [12]. We proceed directly to show $\mathrm{CRT}(X)$ is in bijection with $\mathrm{MLQ}(X)$.

3.1 Bijections from multiline queues to CRT

Our bijection is based on a technical condition on the weight vector $(w_1, \ldots, w_\ell)$ associated to an MLQ via the ball drop of Algorithm 2.1.

Lemma 3.3. *Let $X = X_1 \ldots X_n \in \mathrm{TASEP}(k, r, \ell)$ with $X_{x_1} = \cdots = X_{x_\ell} = 0$. For $i \in [1, \ell]$, define $b_i = \max\{j < x_i : X_j = 1\}$ to be the locations of the nearest 1's to the left of each 0 in X. Let $(w_1, \ldots, w_\ell)$ be the weight vector of $M \in \mathrm{MLQ}(X)$. The conditions on $(w_1, \ldots, w_\ell)$ are:*

$$\sum_{j:\ b_i < x_j \le x_i} w_j + 1 \le x_i - b_i.$$

for each i. In other words, there are enough vacancies left of each x_i so that it can have weight w_i.

We call a list $(w_1, \ldots, w_\ell)$ that satisfies the condition in Lemma 3.3 an *X-consistent list*. Lemma 2.2 states that each X-consistent list corresponds uniquely to an MLQ in $\mathrm{MLQ}(X)$. By the same principles, it turns out that an X-consistent list also corresponds uniquely to a CRT in $\mathrm{CRT}(X)$.

Lemma 3.4. *Let $R \in \mathrm{CRT}(X)$ with a_i the number of left-arrows in the i'th north-strip of R from right to left. Then $(a_1, \ldots, a_\ell)$ is an X-consistent list. Moreover, for any X-consistent list $(a_1, \ldots, a_\ell)$, there exists a unique filling of $\mathrm{CRT}(X)$ with this property.*

The second statement of Lemma 3.4 is proved by construction: when filling the north-strips from right to left, the left-arrows must be placed in consecutive free tiles from bottom to top (see Prop 3.1 in [9]).

Definition 3.5. Let $M \in \mathrm{MLQ}(X)$ have weight vector $(w_1, \ldots, w_\ell)$. Construct $\Psi_{\mathrm{CRT}}(M) \in \mathrm{CRT}(X)$ so the i'th north-strip of $\Psi_{\mathrm{CRT}}(M)$ from right to left has w_i left-arrows for each i.

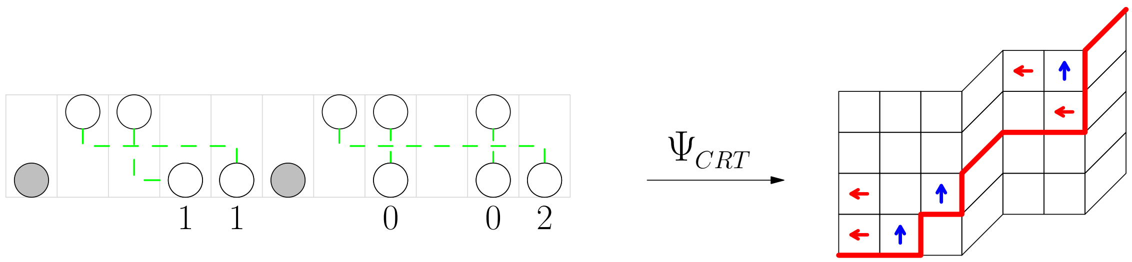


Figure 6: The weights of the MLQ M are $(1,1,0,0,2)$, which is also the number of left-arrows in each north-strip from right to left in the CRT $\Psi_{\mathrm{CRT}}(M)$.

See Figure 6 for an example of the map Ψ_{CRT}. It is similarly easy to define the inverse of Ψ_{CRT} by Lemmas 3.4 and 2.2 and reversing Algorithm 2.1, thus establishing that $\Psi_{\mathrm{CRT}} : \mathrm{MLQ}(X) \to \mathrm{CRT}(X)$ is indeed a bijection.

Remark 3.6. Using nested lattice paths, one obtains a bijection from MLQs to CRT that is different from Ψ_{CRT}. A multiline queue naturally has an interpretation in terms of weighted lattice paths, where each row of the MLQ is mapped to a path on a triangular lattice. At $q = 0$, fillings of RAT are in bijection with weighted nested lattice paths [10], and indeed one can show this property is preserved in the case of the CRT using the canonical lattice path bijection of Catalan paths and Catalan tableaux (see [9, 14]).

4 Inhomogeneous 2-TASEP

We define the transitions on the inhomogeneous 2-TASEP with periodic boundary conditions as follows: for arbitrary words $X, Y \in \{2, 1, 0\}$,

$$X20Y \xrightarrow{1} X02Y, \qquad X21Y \xrightarrow{x_2} X12Y, \qquad X10Y \xrightarrow{x_1} X01Y$$

where $0 \leq x_2, x_1 \leq 1$ are parameters describing the hopping rates. When $x_2 = x_1 = 1$, we recover the usual 2-TASEP. When $x_1 = 1$, we recover the inhomogeneous 2-TASEP in [3] (our solution specializes to the latter after some manipulation).

We introduce a weight on the CRT, which is a monomial in x_2, x_1 to give a formula for the probabilities of the inhomogeneous 2-TASEP with periodic boundary conditions.

Definition 4.1. Let $X \in \text{TASEP}(k, r, \ell)$ and $R \in \text{CRT}(X)$. The *weight* $\text{wt}(R)$ is defined as

$$\text{wt}(R) = x_2^{k-\text{Left}(R)} x_1^{\ell-\text{Up}(R)},$$

where $\text{Left}(R)$ is the number of 20-tiles in R containing a left-arrow and $\text{Up}(R)$ is the number of 20-tiles in R containing an up-arrow. Set $\text{weight}(X) = \sum_{R \in \text{CRT}(X)} \text{wt}(R)$.

Example 4.2. The CRTs in Figure 5 have respective weights $x_2x_1^3$, $x_2^2x_1^2$, $x_2^2x_1^2$, $x_2^3x_1$, $x_2^4x_1$.

A straightforward Matrix Ansatz argument gives the following result.

Theorem 4.3. *Let* $X \in \text{TASEP}(k, r, \ell)$ *be a state of the inhomogeneous 2-TASEP with periodic boundary conditions. Then* $\Pr(X) = \frac{1}{\mathcal{Z}_{k,r,\ell}} \text{weight}(X)$*, where* $\mathcal{Z}_{k,r,\ell} = \sum_{X \in \text{TASEP}(k,r,\ell)} \text{weight}(X)$.

Using Ψ_{CRT}, one can define statistics umv and rest on MLQs that correspond to Left and Up respectively on CRT, the former being a natural statistic, but the latter slightly cumbersome.

Definition 4.4. For $M \in \text{MLQ}$, let umv denote the number of unmarked vacancies in the bottom row of M after Algorithm 2.1. We call a *restricted 0-ball* a 0-ball such that at the time it is hit during Algorithm 2.1, between it and the nearest 1-ball to its left, there are zero unmarked vacancies. Let $\text{rest}(M)$ denote the number of such restricted 0-balls.

Example 4.5. In Figure 1, the 2nd and 5th 0-balls from the left are restricted, since after they are hit, there are no unmarked vacancies to their left with no 1-ball in between. At termination there are no unmarked vacancies, so $\text{umv}(M) = 0$ and $\text{rest}(M) = 2$.

Lemma 4.6. *For* $M \in \text{MLQ}$, $\text{umv}(M) = \text{Left}(\Psi_{\text{CRT}}(M))$ *and* $\text{rest}(M) = \text{Up}(\Psi_{\text{CRT}}(M))$.

Thus by applying Lemma 4.6 to Theorem 4.3, we obtain the following corollary, which is equivalent to the solution of [3] for the inhomogeneous 2-TASEP at $x_1 = 1$.

Corollary 4.7 ([3])**.** *Let* $X \in \text{TASEP}(k, r, \ell)$ *be a state of the inhomogeneous 2-TASEP with periodic boundary conditions. Then* $\Pr(X)$ *is proportional to* $\sum_{M \in \text{MLQ}(X)} x_2^{-\text{umv}(M)}$.

4.1 Inhomogeneous 2-TASEP with open boundaries and acyclic multiline queues

A nice consequence of our CRT-MLQ bijection is that we can apply the same methods to obtain analogous results for the inhomogeneous 2-TASEP with open boundaries. In this generalization, α and β dictate the rates of entry and exit of class 2 particles at the left and right boundaries of the lattice, while $1, x_2, x_1$ dictate the rates at which the respective transitions $20 \to 02$, $21 \to 12$, and $10 \to 01$ occur.

In the expected way, to obtain probabilities for the inhomogeneous process, we modify the weight of a RAT by counting its tiles of different types that contain arrows.

Definition 4.8. For $R \in \text{RAT}(n,r)$, set

$$\text{wt}_M(R) = x_2^{-\text{Left}(R)} x_1^{-\text{Up}(R)} \alpha^{n-r-\text{ufree}(R)} \beta^{n-r-\text{lfree}(R)}$$

where $\text{Left}(R)$ and $\text{Up}(R)$ are statistics carried over from the CRT.

Using a Matrix Ansatz argument, we obtain that the stationary probability of a state $X \in \text{TASEP}(n,r)$ of the inhomogeneous 2-TASEP is proportional to $\sum_{R \in \text{RAT}_{\mathcal{T}}(X)} \text{wt}_M(R)$.

Observe that we can obtain a simple bijection $c : \text{CRT}(1X1) \to \text{RAT}_{\mathcal{T}_X}(X)$ by simply taking the restriction of $T \in \text{CRT}(1X1)$ to the region corresponding to $\Gamma(X)$ above the path $P(1X1)$ and appending the necessary 10-tiles and up-arrows in a deterministic way, as in Figure 8. (Here $\mathcal{T}_X$ the restriction of the canonical tiling of $\mathcal{H}(X)$ to $\Gamma(X)$.) Consequently, in the next section we use Ψ_{CRT} to introduce a new object, the *acyclic multiline queue*, which is derived from the usual MLQ, and is in bijection with the RAT.

4.2 Acyclic multiline queues

Definition 4.9. An *acyclic MLQ* (AMLQ) of type $X \in \text{TASEP}(n,r)$ is an MLQ of size (k,r,ℓ) for any $0 \leq k \leq n-r$ that is *not* on a cylinder, with the following restriction: for each $1 \leq i \leq \ell$, the i'th top row ball (from the left) has at least i bottom row balls weakly to its right. We denote the set of AMLQs of type X by $\text{AMLQ}(X)$, and by $\text{AMLQ}(n,r)$ the set of acyclic MLQs of size (n,r).

In other words, an AMLQ is an MLQ configuration in which every top row ball occupies a bottom row ball to its right without wrapping. See Figure 7 for an example.

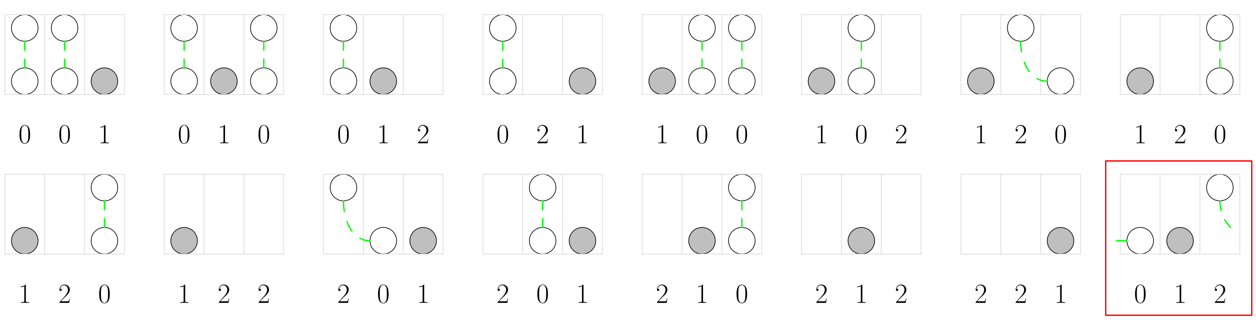


Figure 7: The acyclic MLQs of size $(3,1)$ are shown. The final marked configuration is not an AMLQ since the top row ball must wrap around to occupy the bottom row.

Let $A \in \text{AMLQ}(X)$. Construct $e(A)$ by appending a column containing a vacancy in the top row and a 1-ball in the bottom row to the left and right of A. The leftmost column of $e(A)$ trivially contains a 1-ball, and since every top row ball in A occupies some ball weakly to its right without wrapping around, the bottom row ball at the rightmost location of the $e(A)$ must be a 1-ball; thus $e(A) \in \mathrm{MLQ}(1X1)$ and is moreover a bijection. Now set $\Psi_{\mathrm{RAT}} = c \circ \Psi_{\mathrm{CRT}} \circ e$. See Figure 8.

Definition 4.10. The *weight* of an AMLQ $A \in AMLQ(n,r)$ is

$$\mathrm{wt}_M(A) = \alpha^{n-r-\mathrm{ufree}(A)}\beta^{n-r-\mathrm{lfree}(A)}x_2^{\mathrm{lfree}(A)-\mathrm{umv}(A)}x_1^{\mathrm{ufree}(A)-\mathrm{rest}(A)}$$

where $\mathrm{ufree}(A)$ is the number of restricted 0-balls to the left of the leftmost 1-ball in A, and $\mathrm{lfree}(A)$ is the number of unmarked vacancies to the right of the rightmost 1-ball.

Lemma 4.11. *The maps* $\tilde{\Psi}_{\mathrm{CRT}} : \mathrm{AMLQ}(X) \to \mathrm{RAT}(X)$ *is a weight-preserving bijection with the following properties: (i.)* $\mathrm{ufree}(A) = \mathrm{ufree}(\Psi_{\mathrm{RAT}}(A))$*, (ii.)* $\mathrm{lfree}(A) = \mathrm{lfree}(\Psi_{\mathrm{RAT}}(A))$*, (iii.)* $\mathrm{umv}(A) = \mathrm{Left}(\Psi_{\mathrm{RAT}}(A))$*, (iv.)* $\mathrm{rest}(A) = \mathrm{Up}(\Psi_{\mathrm{RAT}}(A))$*, and finally (v.)* $\mathrm{wt}_M(A) = \mathrm{wt}_m(\Psi_{\mathrm{RAT}}(A))$.

Combining Lemma 4.11 and the usual Matrix Ansatz proof, we obtain our final result.

Theorem 4.12. *Let X be a state of the inhomogeneous 2-TASEP of size (n,r). Then* $\Pr(X)$ *is proportional to*

$$\sum_{A \in \mathrm{AMLQ}(X)} \mathrm{wt}_M(A) = \sum_{R \in \mathrm{RAT}_{\mathcal{T}}(X)} \mathrm{wt}_M(R).$$

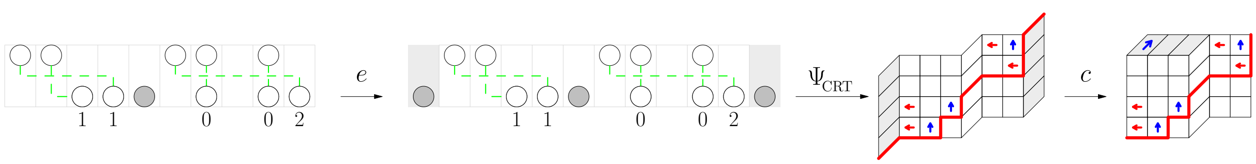


Figure 8: Let $X = 2200120200$. We have: $A \in \mathrm{AMLQ}(X) \longleftrightarrow e(A) \in \mathrm{MLQ}(1X1) \longleftrightarrow \Psi_{\mathrm{CRT}}(e(A)) \in \mathrm{CRT}_{\mathcal{T}_X}(1X1) \longleftrightarrow \Psi_{\mathrm{RAT}}(A) = c(\Psi_{\mathrm{CRT}}(e(A))) \in \mathrm{RAT}(X)$. The highlighted columns in $e(A)$ correspond to the highlighted diagonal strips in $\Psi_{\mathrm{CRT}}(e(A))$.

5 Conclusion

Although tableaux are limited to the 2-species context, their capacity for containing much structure makes the connection with TASEP more transparent than that of the MLQs. For instance the intuitive tableaux solution for the inhomogeneous process is superior to the MLQ solution; it is also much more natural to define a Markov chain that projects to the TASEP on the tableaux rather than on the MLQs. Moreover, in the open boundary case we get solutions for generalizations of the 2-TASEP with larger numbers of parameters (see [12, 4]). Multiline queues, on the other hand, generalize well to a k-TASEP on a ring [1], but are thus far limited to the $q = 0$ case. Thus by linking these two families of objects in the 2-TASEP context, we hope to both generalize the MLQs to obtain solutions for the k-ASEP with additional parameters, as well as to generalize the tableaux method for more species of particles.

References

[1] C. Arita and K. Mallick. "Matrix product solution of an inhomogeneous multi-species TASEP". *J. Phys. A* **46**.8 (2013). DOI: 10.1088/1751-8113/46/8/085002.

[2] A. Ayyer, J.L. Lebowitz, and E.R. Speer. "On the two species asymmetric exclusion process with semi-permeable boundaries". *J. Stat. Phys.* **135**.5–6 (2009), pp. 1009–1037. DOI: 10.1007/s10955-009-9724-2.

[3] A. Ayyer and S. Linusson. "An inhomogeneous multispecies TASEP on a ring". *Adv. in Appl. Math.* **57** (2014), pp. 21–43. DOI: 10.1016/j.aam.2014.02.001.

[4] S. Corteel, O. Mandelshtam, and L. Williams. "Combinatorics of the two-species ASEP and Koornwinder moments". *Adv. Math.* **321** (2017), pp. 160–204. DOI: 10.1016/j.aim.2017.09.034.

[5] S. Corteel and L. Williams. "Tableaux combinatorics for the asymmetric exclusion process". *Adv. in Appl. Math.* **39**.3 (2007), pp. 293–310. DOI: 10.1016/j.aam.2006.08.002.

[6] S. Corteel and L. Williams. "Tableaux combinatorics for the asymmetric exclusion process and Askey-Wilson polynomials". *Duke Math. J.* **159**.3 (2011), pp. 385–415. URL.

[7] P.A. Ferrari and J.B. Martin. "Stationary distributions of multi-type totally asymmetric exclusion processes". *Ann. Probab.* **35**.3 (2007), pp. 807–832. DOI: 10.1214/009117906000000944.

[8] T. Lam and L. Williams. "A Markov chain on the symmetric group which is Schubert positive?" *Experimental Mathematics* **21**.2 (2012), pp. 189–192. DOI: 10.1080/10586458.2011.579020.

[9] O. Mandelshtam. "A determinantal formula for Catalan tableaux and TASEP probabilities". *J. Combin. Theory Ser. A* **132** (2015), pp. 120–141. DOI: 10.1016/j.jcta.2014.12.005.

[10] O. Mandelshtam. "Multi-Catalan Tableaux and the Two-Species TASEP". *Ann. Inst. Henri Poincaré D* **3**.3 (2016), pp. 321–348. DOI: 10.4171/AIHPD/30.

[11] O. Mandelshtam. "Toric tableaux and the inhomogeneous two-species TASEP on a ring". 2017. arXiv: 1707.02663.

[12] O. Mandelshtam and X. Viennot. "Tableaux combinatorics of the two-species PASEP". *J. Combin. Theory Ser. A* **159** (2018), pp. 215–239. DOI: 10.1016/j.jcta.2018.05.006.

[13] X. Viennot. ""Alternative tableaux, permutations and partially asymmetric exclusion process." Slides available at: `http://sms.cam.ac.uk/media/1004`. 2008.

[14] X. Viennot. "Canopy of binary trees, Catalan tableaux and the asymmetric exclusion process". *FPSAC 2007* (2007).