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Stanley character formula for the spin characters of the symmetric groups

Sho Matsumoto*[1] and Piotr Śniady†[2]

[1]*Graduate School of Science and Engineering, Kagoshima University 1-21-35, Korimoto, Kagoshima, Japan*
[2]*Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warszawa, Poland*

Abstract. We give a new formula for the irreducible spin characters of the symmetric groups. This formula is analogous to Stanley's character formula for the usual (linear) characters of the symmetric groups.

Keywords: spin representations of the symmetric groups, Stanley character formula

A full version of this extended abstract [9] will be published elsewhere.

The *spin symmetric group* $\widetilde{\mathfrak{S}}_n$ is the double cover of the symmetric group $\mathfrak{S}_n$. This group is generated by $t_1, \dots, t_{n-1}, z$ subject to the relations:

$$\begin{aligned} z^2 &= 1, && && \\ z t_i &= t_i z, && t_i^2 = z && \text{for } i \in [n-1], \\ (t_i t_{i+1})^3 &= z && && \text{for } i \in [n-2], \\ t_i t_j &= z t_j t_i && && \text{for } |i-j| \geq 2; \end{aligned}$$

we use the convention that $[k] = \{1, \dots, k\}$. This group was introduced by Schur [12]; it is essential for studying *projective representations* of the usual symmetric group $\mathfrak{S}_n$.

Schur proved that, roughly speaking[1], the conjugacy classes of $\widetilde{\mathfrak{S}}_n$ which are non-trivial from the viewpoint of the character theory are indexed by *odd partitions* of n, i.e. partitions $(\pi_1, \dots, \pi_l)$ of n such that $\pi_1 \geq \cdots \geq \pi_l$ are odd positive integers. The set of such odd partitions of n will be denoted by OP_n. We set $\mathrm{OP} = \bigcup_{n=0}^{\infty} \mathrm{OP}_n$.

The central element $z \in \widetilde{\mathfrak{S}}_n$ acts on each irreducible representation by ± 1. An irreducible representation of $\widetilde{\mathfrak{S}}_n$ is said to be *spin* if z corresponds to -1. Schur also proved

*shom@sci.kagoshima-u.ac.jp. Research of SM was supported by JSPS KAKENHI Grant Number 17K05281.

†psniady@impan.pl. Research of PŚ was supported by *Narodowe Centrum Nauki*, grant number 2017/26/A/ST1/00189.

[1]For an exact statement see [9].

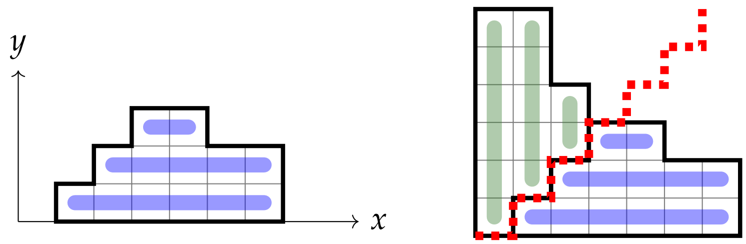

Figure 1: Strict partition $\xi = (6, 5, 2)$ shown as a *shifted Young diagram* and its double $D(\xi) = (7, 7, 5, 3, 2, 2)$.

that the irreducible spin representations of $\widetilde{\mathfrak{S}}_n$, roughly speaking[2], correspond to *strict partitions* of n, i.e. to partitions $(\xi_1, \dots, \xi_l)$ of n which form a *strictly* decreasing sequence $\xi_1 > \cdots > \xi_l$ of positive integers. The set of such strict partitions of n will be denoted by SP_n. We will represent them as *shifted Young diagrams*, cf. Figure 1. We set $\mathrm{SP} = \bigcup_{n=0}^{\infty} \mathrm{SP}_n$.

For an odd partition $\pi \in \mathrm{OP}_n$ (which corresponds to a conjugacy class of $\widetilde{\mathfrak{S}}_n$) and a strict partition $\xi \in \mathrm{SP}_n$ (which corresponds to its irreducible spin representation) we denote by $\widetilde{\phi}^{\xi}(\pi)$ the corresponding *spin character* (for some fine details related to this definition we refer to [5, Section 2] and [9]). **Our goal is to give a closed formula for such spin characters which would be useful for the purposes of the asymptotic representation theory, i.e. which would allow good understanding of the limit $\xi \to \infty$.**

In the following it will be more convenient to pass to quantities

$$X^{\xi}(\pi) := 2^{\frac{\ell(\xi)-\ell(\pi)}{2}} \, \widetilde{\phi}^{\xi}(\pi),$$

where $\ell(\pi)$ denotes the number of parts of a partition π, cf. [5, Proposition 3.3].

1 Normalized characters

The usual way of viewing the linear characters $\chi^{\lambda}(\pi)$ of the symmetric group $\mathfrak{S}_n$ is to fix the irreducible representation λ and to consider the character as a function of the conjugacy class π. The *dual approach*, initiated by Kerov and Olshanski [6], suggests to do the opposite: *fix the conjugacy class π and to view the character as a function of the irreducible representation λ*. In order for this approach to be successful one has to choose the most convenient normalization constants which we review in the following.

For a fixed integer partition π the corresponding *normalized linear character on the*

[2]For an exact statement see [9].

conjugacy class π (cf. [6]) is the function on the set of *all* Young diagrams given by

$$\mathrm{Ch}_\pi(\lambda) := \begin{cases} n^{\downarrow k} \, \frac{\chi^\lambda(\pi \cup 1^{n-k})}{\chi^\lambda(1^n)} & \text{if } n \geq k, \\ 0 & \text{otherwise,} \end{cases}$$

where $n = |\lambda|$ and $k = |\pi|$ and $n^{\downarrow k} = n(n-1)\cdots(n-k+1)$ denotes the falling power. Above, for partitions $\lambda, \sigma \vdash n$ we denote by $\chi^\lambda(\sigma)$ the irreducible linear character of the symmetric group which corresponds to the Young diagram λ, evaluated on any permutation with the cycle decomposition given by σ.

Following Ivanov [5], for a fixed odd partition $\pi \in \mathrm{OP}$ the corresponding *normalized spin character* is a function on the set of *all* strict partitions given by

$$\mathrm{Ch}^{\mathrm{spin}}_\pi(\xi) := \begin{cases} n^{\downarrow k} \, \frac{X^\xi(\pi \cup 1^{n-k})}{X^\xi(1^n)} = n^{\downarrow k} \, 2^{\frac{k-\ell(\pi)}{2}} \, \frac{\widetilde{\phi}^\xi(\pi \cup 1^{n-k})}{\widetilde{\phi}^\xi(1^n)} & \text{if } n \geq k, \\ 0 & \text{otherwise,} \end{cases} \tag{1.1}$$

where $n = |\xi|$, $k = |\pi|$, and $\ell(\pi)$ denotes the number of parts of π. **We will find a closed formula for such spin characters $\mathrm{Ch}^{\mathrm{spin}}_\pi$. We will achieve it by finding a link between the families $(\mathrm{Ch}^{\mathrm{spin}}_\pi)$ and (Ch_π) of spin and linear characters.**

2 Stanley character formulas

Let $\sigma_1, \sigma_2 \in \mathfrak{S}_k$ be permutations and let λ be a Young diagram. Following [4], we say that (f_1, f_2) *is a coloring of* (σ_1, σ_2) *which is compatible with* λ if:

- $f_i \colon C(\sigma_i) \to \mathbb{Z}_+$ is a function on the set of cycles of σ_i for each $i \in \{1, 2\}$; we view the values of f_1 as columns of λ and the values of f_2 as rows;
- whenever $c_1 \in C(\sigma_1)$ and $c_2 \in C(\sigma_2)$ are cycles which are not disjoint, the box with the Cartesian coordinates $\big(f_1(c_1), f_2(c_2)\big)$ belongs to λ.

We denote by $N_{\sigma_1,\sigma_2}(\lambda)$ the number of colorings of (σ_1, σ_2) which are compatible with λ.

Example 2.1. Let

$$\sigma_1 = \underbrace{(1,5,4,2)}_{V}\underbrace{(3)}_{W}, \qquad \sigma_2 = \underbrace{(2,3,5)}_{\Pi}\underbrace{(1,4)}_{\Sigma}. \tag{2.1}$$

There are three pairs of cycles $(\sigma_1, \sigma_2) \in C(\sigma_1) \times C(\sigma_2)$ with the property that σ_1 and σ_2 are not disjoint, namely $(V, \Pi), (V, \Sigma), (W, \Pi)$. It is now easy to check graphically (cf. Figure 2) that (f_1, f_2) is indeed a coloring compatible with $\lambda = (3, 1)$ for

$$f_1(V) = 1, \quad f_1(W) = 3, \quad f_2(\Pi) = 1, \quad f_2(\Sigma) = 2. \tag{2.2}$$

By considering four possible choices for the values of f_2 and counting the choices for the values of f_1 one can verify that $N_{\sigma_1,\sigma_2}(\lambda) = 3^2 + 3 + 1 + 1 = 14$ for $\lambda = (3, 1)$.

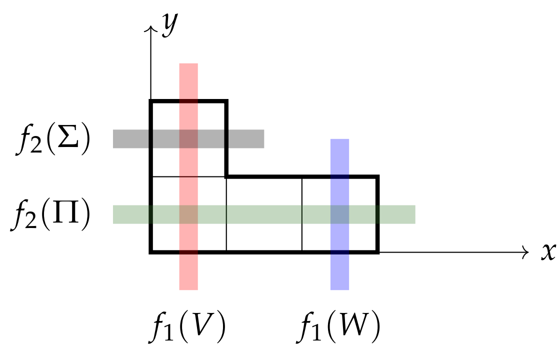

Figure 2: Graphical representation of the coloring (2.2) of the permutations (2.1) which is compatible with the Young diagram $\lambda = (3, 1)$.

2.1 Linear Stanley character formula

Stanley [15] conjectured a certain closed formula for the linear characters of the symmetric groups. One of its proofs [4] was obtained by rewriting it in an equivalent form which we will recall in the following.

We will identify a given integer partition $\pi = (\pi_1, \dots, \pi_\ell) \vdash k$ with an arbitrary permutation $\pi \in \mathfrak{S}_k$ with the corresponding cycle structure.

Theorem 2.2 ([4])**.** *For any partition $\pi \vdash k$ and any Young diagram λ*

$$\mathrm{Ch}_\pi(\lambda) = \sum_{\substack{\sigma_1, \sigma_2 \in \mathfrak{S}_k \\ \sigma_1 \sigma_2 = \pi}} (-1)^{\sigma_1} N_{\sigma_1, \sigma_2}(\lambda), \tag{2.3}$$

where $(-1)^{\sigma_1} \in \{-1, 1\}$ denotes the sign of the permutation σ_1.

This formula is closely related to *Kerov polynomials* [2] which are expressions of the characters Ch_π in terms of *free cumulants* of Young diagrams. Recently, the first author [8] found spin counterparts for Kerov polynomials. The current paper was initiated by attempts to understand the underlying structures behind this result.

2.2 The main result: spin Stanley character formula

For a strict partition $\xi \in \mathrm{SP}_n$ we consider its *double* $D(\xi)$ which is an integer partition of $2n$. Graphically, $D(\xi)$ corresponds to a Young diagram obtained by arranging the *shifted Young diagram* ξ and its 'transpose' so that they nicely fit along the 'diagonal', cf. Figure 1, see also [7, page 9].

For $\sigma_1, \sigma_2 \in \mathfrak{S}_k$ we denote by $|\sigma_1 \vee \sigma_2|$ the number of orbits in the set $[k] = \{1, \dots, k\}$ under the action of the group $\langle \sigma_1, \sigma_2 \rangle$ generated by σ_1 and σ_2. As before, we identify an integer partition $\pi \vdash k$ with an arbitrary permutation $\pi \in \mathfrak{S}_k$ with the corresponding cycle structure.

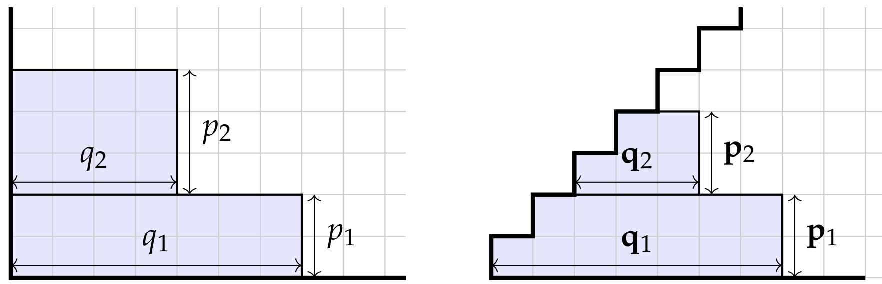

Figure 3: Multirectangular Young diagram $P \times Q$ and multirectangular shifted Young diagram $\mathbf{P} \rtimes \mathbf{Q}$.

Theorem 2.3 (The main result)**.** *For any odd partition $\pi \in \mathrm{OP}_k$ and $\xi \in \mathrm{SP}$*

$$\mathrm{Ch}^{\mathrm{spin}}_{\pi}(\xi) = \sum_{\substack{\sigma_1, \sigma_2 \in \mathfrak{S}_k \\ \sigma_1 \sigma_2 = \pi}} \frac{1}{2^{|\sigma_1 \vee \sigma_2|}} (-1)^{\sigma_1} N_{\sigma_1, \sigma_2}\big(D(\xi)\big). \tag{2.4}$$

The remaining sections of this paper (Sections 3 to 5) are devoted to a sketch of the proof of this result. In the following we will discuss some of its applications.

2.3 Application: bounds on spin characters

The following character bound is a spin version of an analogous result for the linear characters of the symmetric group [4]. It is a direct application of Theorem 2.3 and its proof follows the same line as its linear counterpart [4].

Corollary 2.4. *There exists a universal constant $a > 0$ with the property that for any integer $n \geq 1$, any strict partition $\xi \in \mathrm{SP}_n$, and any odd partition $\pi \in \mathrm{OP}_n$*

$$2^{\frac{n-\ell(\pi)}{2}} \left| \frac{\widetilde{\phi}^{\xi}(\pi)}{\widetilde{\phi}^{\xi}(1^n)} \right| = \left| \frac{X^{\xi}(\pi)}{X^{\xi}(1^n)} \right| < \left[a \max\left(\frac{\xi_1}{n}, \frac{n-\ell(\pi)}{n} \right) \right]^{n-\ell(\pi)}.$$

Several asymptotic results about (random) Young diagrams and tableaux which use the inequality from [4] can be generalized in a rather straightforward way to (random) *shifted* Young diagrams and *shifted* tableaux thanks to Corollary 2.4. A good example is provided by the results about the asymptotics of the number of skew standard Young tableaux of prescribed shape [3] which can be generalized in this way to asymptotics of the number of skew *shifted* standard Young tableaux.

2.4 Application: characters on multirectangular Young diagrams

Following Stanley [14], for tuples of integers $P = (p_1, \ldots, p_l)$, $Q = (q_1, \ldots, q_l)$ which fulfill some obvious inequalities we consider the corresponding *multirectangular Young dia-*

gram $P \times Q$, cf. Figure 3. Stanley [14, 15] initiated investigation of the characters $\mathrm{Ch}_\pi(P \times Q)$ viewed as polynomials in the multirectangular coordinates $p_1, \dots, p_l, q_1, \dots, q_l$; these polynomials now are referred to as *(linear) Stanley character polynomials*.

The number of colorings $N_{\sigma_1,\sigma_2}(P \times Q) \in \mathbb{Z}[p_1, \dots, p_l, q_1, \dots, q_l]$ is given by a very explicit, convenient polynomial. In this way the linear Stanley formula (Theorem 2.2) gives an explicit expression for the linear Stanley polynomials.

De Stavola [1] adapted these concepts to shifted multirectangular Young diagrams $\mathbf{P} \rtimes \mathbf{Q}$ cf. Figure 3 and initiated investigation of *spin Stanley polynomials* $\mathrm{Ch}^{\mathrm{spin}}_\pi(\mathbf{P} \rtimes \mathbf{Q})$. Thanks to Theorem 2.3, by expressing the multirectangular coordinates P, Q of the double $P \times Q = D(\mathbf{P} \rtimes \mathbf{Q})$ in terms of the shifted multirectangular coordinates $\mathbf{P}, \mathbf{Q}$ one can obtain a rather straightforward expression for the spin Stanley polynomial $\mathrm{Ch}^{\mathrm{spin}}_\pi(\mathbf{P} \rtimes \mathbf{Q})$. Applications of this result to *spin Kerov polynomials* will be discussed in a forthcoming paper.

2.5 Towards irreducible representations of spin groups

The proof of the linear Stanley formula (2.3) presented in [4] was found in the following way. We attempted to reverse-engineer the right-hand side of (2.3) and to find

- some natural vector space V with the basis indexed by combinatorial objects; the space V should be a representation of the symmetric group $\mathfrak{S}_n$ with $n := |\lambda|$, and
- a projection $\Pi \colon V \to V$ such that Π commutes with the action of $\mathfrak{S}_n$ and such that its image ΠV is an irreducible representation of $\mathfrak{S}_n$ which corresponds to the specified Young diagram λ

in such a way that the corresponding character of ΠV would coincide with the right-hand side of (2.3).

Our attempt was successful: one could consider a vector space V with the basis indexed by fillings of the boxes of λ with the numbers from $[n]$. The action of $\mathfrak{S}_n$ on this basis was given by pointwise relabelling of the values in the boxes. The projection Π turned out to be the Young symmetrizer with the action given by shuffling of the boxes in the rows and columns of λ. The resulting representation ΠV clearly coincides with the Specht module, which concluded the proof.

The structure of the right-hand side of (2.4) might be an indication that an analogous reverse-engineering process could be applied to the spin case. The result would be a very explicit construction of the irreducible spin representations which would be an alternative to the somewhat complicated approach of Nazarov [11].

3 Linear characters in terms of spin characters

For $\xi \in \mathrm{SP}$ and $\pi \in \mathrm{OP}$ we denote

$$\widetilde{\mathrm{Ch}}_\pi(\xi) := \frac{1}{2}\,\mathrm{Ch}_\pi\big(D(\xi)\big).$$

The following result is an intermediate step in the proof of Theorem 2.3 but it might be of independent interest. In particular, in a forthcoming paper [10] we shall discuss its applications in the study of random strict partitions as well as random shifted standard Young tableaux.

Theorem 3.1. *For any odd integers $k_1, k_2, \ldots \geq 1$ the following equalities between functions on the set* SP *of strict partitions hold true:*

$$\begin{aligned} \widetilde{\mathrm{Ch}}_{k_1} &= \mathrm{Ch}^{\mathrm{spin}}_{k_1}, \qquad (3.1)\\ \widetilde{\mathrm{Ch}}_{k_1,k_2} &= \mathrm{Ch}^{\mathrm{spin}}_{k_1,k_2} + \mathrm{Ch}^{\mathrm{spin}}_{k_1}\,\mathrm{Ch}^{\mathrm{spin}}_{k_2},\\ \widetilde{\mathrm{Ch}}_{k_1,k_2,k_3} &= \mathrm{Ch}^{\mathrm{spin}}_{k_1,k_2,k_3} + \mathrm{Ch}^{\mathrm{spin}}_{k_1,k_2}\,\mathrm{Ch}^{\mathrm{spin}}_{k_3} + \mathrm{Ch}^{\mathrm{spin}}_{k_1,k_3}\,\mathrm{Ch}^{\mathrm{spin}}_{k_2} + \mathrm{Ch}^{\mathrm{spin}}_{k_2,k_3}\,\mathrm{Ch}^{\mathrm{spin}}_{k_1},\\ &\;\;\vdots\\ \widetilde{\mathrm{Ch}}_{k_1,\ldots,k_l} &= \sum_{\substack{I:\\ |I|\leq 2}} \prod_{b\in I} \mathrm{Ch}^{\mathrm{spin}}_{(k_i:i\in b)}, \qquad (3.2) \end{aligned}$$

where the sum in (3.2) *runs over all set-partitions of* $[l]$ *into at most two blocks.*

Proof. For an integer partition π we consider the standard numerical factor

$$z_\pi = \prod_{j\geq 1} j^{m_j(\pi)} m_j(\pi)!,$$

where $m_j(\pi)$ denotes the *multiplicity* of j in the partition π. We denote by $f^\lambda = \chi^\lambda\left(1^{|\lambda|}\right)$ the number of standard tableaux of shape λ. For a strict partition ξ we denote

$$g^\xi = X^\xi\left(1^{|\xi|}\right)$$

which also happens to be the number of *shifted* standard tableaux with the shape given by the shifted Young diagram ξ, see [7, III–8, Ex. 12].

Recall the symmetric function algebra $\Lambda = \mathbb{C}[p_1, p_2, p_3, \ldots]$ and its subalgebra, *the algebra of supersymmetric functions* $\Gamma = \mathbb{C}[p_1, p_3, p_5, \ldots]$, where the p_r are Newton's power-sums. Define the algebra homomorphism $\varphi : \Lambda \to \Gamma$ by

$$\varphi(p_r) = \begin{cases} 2p_r & \text{if } r \text{ is odd,}\\ 0 & \text{if } r \text{ is even.}\end{cases}$$

Then [7, III–8, Ex. 10] implies that for any strict partition ξ we have

$$\varphi(s_{D(\xi)}) = 2^{-\ell(\xi)} (Q_\xi)^2, \tag{3.3}$$

where $Q_\xi = Q_\xi(x;-1)$ denotes Schur's Q-function [7, pp. III–8].

Recall the Frobenius formula for Schur functions:

$$s_\mu = \sum_\pi z_\pi^{-1} \, \chi^\mu(\pi) \; p_\pi.$$

Applying the homomorphism φ to this identity with $\mu = D(\xi)$, we obtain

$$\varphi(s_{D(\xi)}) = \sum_{\pi \in \mathrm{OP}_{2n}} 2^{\ell(\pi)} z_\pi^{-1} \, \chi^{D(\xi)}(\pi) \; p_\pi.$$

And, recall the Frobenius formula for Schur Q-functions:

$$Q_\xi = \sum_{\nu \in \mathrm{OP}_n} 2^{\ell(\nu)} z_\nu^{-1} \, X^\xi(\nu) \; p_\nu.$$

Substituting these formulas to (3.3), we have for any $\xi \in \mathrm{SP}_n$

$$\sum_{\pi \in \mathrm{OP}_{2n}} 2^{\ell(\pi)} z_\pi^{-1} \, \chi^{D(\xi)}(\pi) \; p_\pi = 2^{-\ell(\xi)} \left(\sum_{\nu \in \mathrm{OP}_n} 2^{\ell(\nu)} z_\nu^{-1} \, X^\xi(\nu) \; p_\nu \right)^2. \tag{3.4}$$

By comparing the coefficients of $p_{(1^{2n})} = p_{(1^n)} p_{(1^n)}$ in both sides of (3.4), we find

$$\frac{f^{D(\xi)}}{(2n)!} = 2^{-\ell(\xi)} \left(\frac{g^\xi}{n!} \right)^2. \tag{3.5}$$

First we assume that π is an odd partition which does not have parts equal to 1, i.e., $m_1(\pi) = 0$. By comparing the coefficients of $p_{\pi \cup (1^{2n-|\pi|})}$ in both sides of (3.4) we find

$$\frac{\chi^{D(\xi)}\left(\pi \cup (1^{n-|\pi|})\right)}{z_{\pi \cup (1^{n-|\pi|})}} = 2^{-\ell(\xi)} \sum_{\substack{\mu^1, \mu^2 \\ \mu^1 \cup \mu^2 = \pi}} \frac{X^\xi\left(\mu^1 \cup (1^{n-|\mu^1|})\right)}{z_{\mu^1 \cup (1^{n-|\mu^1|})}} \frac{X^\xi\left(\mu^2 \cup (1^{n-|\mu^2|})\right)}{z_{\mu^1 \cup (1^{n-|\mu^2|})}}.$$

By the assumption $m_1(\pi) = 0$, we have $z_{\pi \cup (1^{2n-|\pi|})} = z_\pi \cdot (2n - |\pi|)!$ and $z_{\mu^i \cup (1^{n-|\mu^i|})} = z_{\mu^i} \cdot (n - |\mu^i|)!$. Thus, we obtain

$$\frac{\chi^{D(\xi)}\left(\pi \cup (1^{n-|\pi|})\right)}{z_\pi \cdot (2n - |\pi|)!} = 2^{-\ell(\xi)} \sum_{\substack{\mu^1, \mu^2 \\ \mu^1 \cup \mu^2 = \pi}} \frac{X^\xi\left(\mu^1 \cup (1^{n-|\mu^1|})\right)}{z_{\mu^1} \cdot (n - |\mu^1|)!} \frac{X^\xi\left(\mu^2 \cup (1^{n-|\mu^2|})\right)}{z_{\mu^2} \cdot (n - |\mu^2|)!}.$$

Taking the quotient of this and (3.5), we have

$$\frac{1}{z_\pi}\frac{(2n)!}{(2n-|\pi|)!}\frac{\chi^{D(\xi)}\left(\pi\cup(1^{2n-|\pi|})\right)}{f^{D(\xi)}} =$$
$$\sum_{\substack{\mu^1,\mu^2\\ \mu^1\cup\mu^2=\pi}} \frac{1}{z_{\mu^1}z_{\mu^2}}\frac{n!}{(n-|\mu^1|)!}\frac{X^\xi\left(\mu^1\cup(1^{n-|\mu^1|})\right)}{g^\xi}\frac{n!}{(n-|\mu^2|)!}\frac{X^\xi\left(\mu^2\cup(1^{n-|\mu^2|})\right)}{g^\xi},$$

which is equivalent to

$$\mathrm{Ch}_\pi\big(D(\xi)\big) = \sum_{\substack{\mu^1,\mu^2\\ \mu^1\cup\mu^2=\pi}} \frac{z_\pi}{z_{\mu^1}z_{\mu^2}}\mathrm{Ch}^{\mathrm{spin}}_{\mu^1}(\xi)\ \mathrm{Ch}^{\mathrm{spin}}_{\mu^2}(\xi).$$

It is easy to see that this is equivalent to the desired formula. Thus, we completed the proof of the theorem under the assumption $m_1(\pi)=0$.

In the general case we write $\pi=\tilde{\pi}\cup(1^r)$ with $m_1(\tilde{\pi})=0$ and $r=m_1(\pi)$. We apply Theorem 3.1 for $\tilde{\pi}$; simple manipulations with the binomial coefficients imply that the claim holds true for π as well. □

4 Spin characters in terms of linear characters

Formulas (3.1)–(3.2) can be viewed as an upper-triangular system of equations with unknowns $(\mathrm{Ch}^{\mathrm{spin}}_\pi)_{\pi\in\mathrm{OP}}$. It can be solved, for example

$$\left.\begin{aligned}
\mathrm{Ch}^{\mathrm{spin}}_{k_1} &= \widetilde{\mathrm{Ch}}_{k_1},\\
\mathrm{Ch}^{\mathrm{spin}}_{k_1,k_2} &= \widetilde{\mathrm{Ch}}_{k_1,k_2} - \widetilde{\mathrm{Ch}}_{k_1}\ \widetilde{\mathrm{Ch}}_{k_2},\\
\mathrm{Ch}^{\mathrm{spin}}_{k_1,k_2,k_3} &= \widetilde{\mathrm{Ch}}_{k_1,k_2,k_3}\\
&\quad - \widetilde{\mathrm{Ch}}_{k_1,k_2}\ \widetilde{\mathrm{Ch}}_{k_3} - \widetilde{\mathrm{Ch}}_{k_1,k_3}\ \widetilde{\mathrm{Ch}}_{k_2} - \widetilde{\mathrm{Ch}}_{k_2,k_3}\ \widetilde{\mathrm{Ch}}_{k_1}\\
&\quad + 3\widetilde{\mathrm{Ch}}_{k_1}\widetilde{\mathrm{Ch}}_{k_2}\widetilde{\mathrm{Ch}}_{k_3},\\
&\ \ \vdots
\end{aligned}\right\} \tag{4.1}$$

The general pattern is given by the following result. In this way several problems involving spin characters are reduced to investigation of their linear counterparts.

Theorem 4.1. *For any $\pi\in\mathrm{OP}$*

$$\mathrm{Ch}^{\mathrm{spin}}_\pi = \sum_I (-1)^{|I|-1}\ (2|I|-3)!!\ \prod_{b\in I}\widetilde{\mathrm{Ch}}_{(\pi_i:i\in b)}, \tag{4.2}$$

where the sum runs over all set-partitions of the set $[\ell(\pi)]$; by definition $(-1)!!=1$.

Proof. The process of solving the upper-triangular system of equations (3.1)–(3.2) can be formalized as follows. By singling out the partition I in (3.2) which consists of exactly one block we may express the spin character $\mathrm{Ch}^{\mathrm{spin}}_{\pi}$ in terms of the linear character $\widetilde{\mathrm{Ch}}_{\pi}$ and spin characters $\mathrm{Ch}^{\mathrm{spin}}_{\pi'}$ which correspond to partitions $\pi' \in \mathrm{OP}$ with $\ell(\pi') < \ell(\pi)$:

$$\mathrm{Ch}^{\mathrm{spin}}_{\pi} = \widetilde{\mathrm{Ch}}_{\pi} - \sum_{\substack{I:\\ |I|=2}} \prod_{b\in I} \mathrm{Ch}^{\mathrm{spin}}_{(\pi_i : i\in b)}. \tag{4.3}$$

By applying this procedure recursively to the spin characters on the right-hand side, we end up with an expression for $\mathrm{Ch}^{\mathrm{spin}}_{\pi}$ as a linear combination (with integer coefficients) of the products of the form

$$\prod_{b\in I} \widetilde{\mathrm{Ch}}_{(\pi_i : i\in b)} \tag{4.4}$$

over set-partitions I of $[\ell(\pi)]$. The remaining difficulty is to determine the exact value of the coefficient of (4.4) in this linear combination.

The above recursive procedure can be encoded by a tree in which each non-leaf vertex has two children and the leaves are labelled by the factors in (4.4) or, equivalently, by the blocks of the set-partition I. Such trees are known under the name of *total binary partitions*; the cardinality of such trees with prescribed leaf labels I is equal to $(2|I|-3)!!$ [13, Example 5.2.6].

Our recursive procedure involves change of the sign; such a change occurs once for each non-leaf vertex. Thus each total binary tree contributes with multiplicity $(-1)^{|I|-1}$ which concludes the proof. □

5 Proof of spin Stanley formula

Proof of Theorem 2.3. We start with Theorem 4.1 and substitute each normalized linear character Ch_{ν} which contributes to the right-hand side of (4.2) by linear Stanley character formula (2.3).

We shall discuss in detail the case when $\pi = (\pi_1, \pi_2)$ consists of just two parts. We will view $\mathfrak{S}_{\pi_1}$, $\mathfrak{S}_{\pi_2}$ and $\mathfrak{S}_{\pi_1+\pi_2}$ as the groups of permutations of, respectively, the set $\{1, \dots, \pi_1\}$, $\{\pi_1+1, \dots, \pi_1+\pi_2\}$ and $\{1, \dots, \pi_1+\pi_2\}$. In this way we may identify $\mathfrak{S}_{\pi_1} \times \mathfrak{S}_{\pi_2}$ as a subgroup of $\mathfrak{S}_{\pi_1+\pi_2}$. Thanks to these notations

$$\mathrm{Ch}^{\mathrm{spin}}_{\pi_1,\pi_2}(\xi) = \frac{(-1)!!}{2} \mathrm{Ch}_{\pi_1,\pi_2}\big(D(\xi)\big) - \frac{1!!}{2^2} \mathrm{Ch}_{\pi_1}\big(D(\xi)\big)\, \mathrm{Ch}_{\pi_2}\big(D(\xi)\big) =$$

$$\frac{(-1)!!}{2} \sum_{\substack{\sigma_1,\sigma_2 \in \mathfrak{S}_{\pi_1+\pi_2} \\ \sigma_1\sigma_2 = (\pi_1,\pi_2)}} (-1)^{\sigma_1} N_{\sigma_1,\sigma_2}\big(D(\xi)\big) - \frac{1!!}{2^2} \sum_{\substack{\sigma_1,\sigma_2 \in \mathfrak{S}_{\pi_1}\times\mathfrak{S}_{\pi_2} \\ \sigma_1\sigma_2 = (\pi_1,\pi_2)}} (-1)^{\sigma_1} N_{\sigma_1,\sigma_2}\big(D(\xi)\big), \tag{5.1}$$

where the last equality follows from the observation that a double sum over factorizations of $\pi_1 \in \mathfrak{S}_{\pi_1}$ and over factorizations of $\pi_2 \in \mathfrak{S}_{\pi_2}$ can be combined into a single sum over factorizations of $(\pi_1, \pi_2) \in \mathfrak{S}_{\pi_1} \times \mathfrak{S}_{\pi_2}$.

In general,

$$\mathrm{Ch}^{\mathrm{spin}}_{\pi}(\xi) = \sum_{\substack{\sigma_1,\sigma_2 \in \mathfrak{S}_{|\pi|} \\ \sigma_1\sigma_2=\pi}} c_{\sigma_1,\sigma_2} \, (-1)^{\sigma_1} \, N_{\sigma_1,\sigma_2}\big(D(\xi)\big) \tag{5.2}$$

for some combinatorial factor c_{σ_1,σ_2}. The exact value of this factor is equal to

$$c_{\sigma_1,\sigma_2} = C_m = (-1) \sum_p \left\{ {m \atop p} \right\} \left(-\frac{1}{2}\right)^p (2p-3)!!, \tag{5.3}$$

where m denotes the number of orbits in $[|\pi|]$ under the action of $\langle \sigma_1, \sigma_2 \rangle$, and $\left\{ {m \atop p} \right\}$ denotes Stirling numbers of the second kind. Indeed, the set-partition I (over which we sum in (4.2)) can be identified with a set-partition of the set $C(\pi)$ of the cycles of the permutation $\pi \in \mathfrak{S}_{|\pi|}$. With this in mind we see that to c_{σ_1,σ_2} contribute only these set-partitions I on the right-hand side of (4.2) for which I is bigger than the set-partition given by the orbits of $\langle \sigma_1, \sigma_2 \rangle$. The collection of such set-partitions can be identified with the collection of set-partitions of an m-element set (i.e. the set of orbits of $\langle \sigma_1, \sigma_2 \rangle$).

The exact form of the right-hand side of (5.3) is not important; the key point is that it depends only on m, the number of orbits of $\langle \sigma_1, \sigma_2 \rangle$. In order to evaluate its exact value C_m we shall consider (5.2) in the special case of $\pi = 1^m$. In this case $\sigma_2 = \sigma_1^{-1}$; we denote by $l = |C(\sigma_1)|$ the number of cycles of σ_1. It follows that

$$\mathrm{Ch}^{\mathrm{spin}}_{1^m}(\xi) = n^{\downarrow m} = \sum_l \begin{bmatrix} m \\ l \end{bmatrix} C_l \, (-1)^{m-l} \, (2n)^l,$$

where $n = |\xi|$ and $\left[{m \atop l} \right]$ denotes Stirling number of the first kind. Both sides of the equality are polynomials in the variable n; by comparing the leading coefficients we conclude that

$$C_m = \frac{1}{2^m}.$$

By substituting this value to (5.2) we conclude the proof. □

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