Séminaire Lotharingien de Combinatoire **82B** (2019)
Article #11, 12 pp.

Proceedings of the 31st Conference on Formal Power Series and Algebraic Combinatorics (Ljubljana)

The CDE property for skew vexillary permutations

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Abstract. We prove a conjecture of Reiner, Tenner, and Yong which says that the initial weak order intervals corresponding to certain vexillary permutations have the coincidental down-degree expectations (CDE) property. Actually our theorem applies more generally to certain "skew vexillary" permutations (a notion we introduce), and shows that these posets are in fact "toggle CDE." As a corollary we obtain a homomesy result for rowmotion acting on semidistributive lattices in the sense of Barnard and of Thomas and Williams.

Keywords: Weak order, coincidental down-degree expectations (CDE), homomesy

1 Introduction

This extended abstract summarizes the results of [6], which contains all the proofs.

Let $w \in \mathfrak{S}_n$ be a permutation. Consider the following two probability distributions on the set of permutations $u \in \mathfrak{S}_n$ which are less than or equal to w in weak order $(\mathfrak{S}_n, \leq)$. For the first distribution: select u uniformly at random among all permutations $u \leq w$. For the second distribution: choose a reduced word $w = s_{i_1} s_{i_2} \cdots s_{i_{\ell(w)}}$ of w uniformly at random; then choose $k \in \{0, 1, \ldots, \ell(w)\}$ uniformly at random; and finally define $u := s_{i_1} s_{i_2} \cdots s_{i_k}$. In general these two distributions will be quite different. Our main result is that for a large family of w ("skew vexillary permutations of balanced shape"), although these two distributions are indeed different, the expected number of descents of the random permutation u is nevertheless the same for both.

This is an instance of the "coincidental down-degree expectations" phenomenon introduced by Reiner, Tenner, and Yong [9].

Definition 1 (See [9, Definition 2.1]). *Let P be a finite poset. Let* uni_P *denote the uniform probability distribution on P. Let* $\mathrm{maxchain}_P$ *denote the probability distribution where each $p \in P$ occurs with probability proportional to the number of maximal chains containing p. Let* $\mathrm{ddeg}\colon P \to \mathbb{N}$ *denote the* down-degree *statistic:* $\mathrm{ddeg}(p)$ *is the number of elements of P which $p \in P$ covers. If μ is a discrete probability distribution on a finite set X and $f\colon X \to \mathbb{R}$ is some statistic on X, we use the notation $\mathbb{E}(\mu; f)$ to denote the expectation of f with respect to μ. Finally, we say that P has the* coincidental down-degree expectations (CDE) property *if we have* $\mathbb{E}(\mathrm{uni}_P; \mathrm{ddeg}) = \mathbb{E}(\mathrm{maxchain}_P; \mathrm{ddeg})$. *In this case we also say that P* is CDE.

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The coincidence of the expected number of descents for the two distributions on permutations described in the first paragraph of this introduction can be recast, in the language of Definition 1, as saying that the weak order interval $[e, w]$ between the identity permutation e and our chosen permutation w is CDE. This is because the maximal chains in this weak order interval naturally correspond to the reduced words of w, and similarly the down-degree of a permutation in weak order is its number of descents.

Note that $\mathbb{E}(\mathrm{uni}_P; \mathrm{ddeg})$ is the *edge density* of P, i.e., the number of edges of the Hasse diagram of P divided by the number of elements of P. As part of our main result, we will not only establish that $\mathbb{E}(\mathrm{uni}_{[e,w]}; \mathrm{ddeg}) = \mathbb{E}(\mathrm{maxchain}_{[e,w]}; \mathrm{ddeg})$ for the aforementioned family of permutations w, but we will also give a simple formula for the edge density of these posets $[e, w]$. There is no a priori reason to expect a simple formula for the edge density of a poset, so our result says that these posets $[e, w]$ have a very special combinatorial structure.

Let us now briefly review the history of the study of CDE posets and explain how our result fits into this history.

The first instance of a poset being shown to be CDE occurred in the context of the algebraic geometry of curves. Chan, López Martín, Pflueger, and Teixidor i Bigas [5] showed that the interval $[\varnothing, b^a]$ in Young's lattice of partitions between the empty shape $\varnothing$ and the $a \times b$ rectangle $b^a := (\overbrace{b, b, \cdots, b}^{a})$ is CDE with edge density $ab/(a+b)$. This was the key combinatorial result these authors needed to reprove a product formula for the genus of Brill-Noether loci of dimension one.

Subsequently, Chan, Haddadan, Hopkins and Moci [4] extended the combinatorial result of [5] to many more shapes beyond rectangles. They showed that if $\sigma = \lambda/\nu$ is a "balanced" skew shape of height a and width b, then the interval $[\nu, \lambda]$ in Young's lattice is also CDE with edge density $ab/(a+b)$. Rectangles b^a are balanced, as are *staircases* $\delta_d := (d-1, d-2, \ldots, 1)$. Furthermore, if σ is a balanced shape then the shape $\sigma \circ b^a$ obtained from σ by replacing every box with an $a \times b$ rectangle is also balanced. So for instance the *rectangular staircases* $\delta_d \circ b^a$ are also balanced shapes.

In fact, Chan–Haddadan–Hopkins–Moci [4] showed these distributive lattices are "toggle CDE," a stronger notion than CDE whose definition is based on "toggling" [11].

The intervals of Young's lattice discussed above are all distributive lattices. Reiner–Tenner–Yong found some interesting examples of CDE posets which are not distributive lattices by considering intervals of weak order. Specifically, Reiner–Tenner–Yong [9, Theorem 1.1] proved that if $\lambda = \delta_d \circ b^a$ is a rectangular staircase, and $w \in \mathfrak{S}_n$ is a dominant permutation of shape λ, then $[e, w]$ is CDE with edge density $(d-1)ab/(a+b)$. To do this they employed tableaux and the theory of Schur polynomials, Schubert polynomials, Grothendieck polynomials, et cetera.

Reiner–Tenner–Yong also conjectured a significant generalization of their result:

Conjecture 2 ([9, Conjecture 1.2])**.** *Let $\lambda = \delta_d \circ b^a$ be a rectangular staircase and $w \in \mathfrak{S}_n$ a vexillary permutation of shape λ. Then $[e, w]$ is CDE with edge density $(d-1)ab/(a+b)$.*

We remind the reader that there is essentially one dominant permutation of shape λ, but there are in general many vexillary permutations of shape λ

Our main result establishes Conjecture 2. In fact, we show that there is nothing particularly special about rectangular staircases in this conjecture: the important thing is that the shape is balanced. And our result will apply to skew shapes as well. Thus as part of our result we introduce the notion of "skew vexillary" permutations.

Our approach to Conjecture 2 is very different from that of Reiner–Tenner–Yong: we do not use tableaux or symmetric functions at all. Rather, we prove our main result by adopting the "toggle perspective" from [4]. In [4], toggling was considered only for distributive lattices; but building on the recent work of Reading [8], Barnard [1], and Thomas and Williams [12], we successfully extend the "toggle perspective" to the *semi*distributive setting (which includes intervals of weak order).

Rowmotion is an invertible operator acting on any finite distributive lattice, studied by many authors [2, 3, 11]. Thanks to an observation of Striker [10], Chan–Haddadan–Hopkins–Moci [4] were able to deduce from their "toggle CDE" result that for a balanced shape $\sigma = \lambda/\nu$, the average down-degree is the same along every rowmotion orbit of $[\nu, \lambda]$. This is a *homomesy* result [7]. A semidistributive generalization of rowmotion was recently considered by Barnard [1] and Thomas-Williams [12]. We deduce a similar homomesy corollary in this setting: for w a skew vexillary permutation of balanced shape, the average down-degree is the same along every rowmotion orbit of $[e, w]$.

2 Skew vexillary permutations

We always consider (integer) partitions ν, λ partially ordered according to *Young's lattice*: we have $\nu \le \lambda$ if and only if the Young diagram of ν is contained in that of λ. We always use $[\nu, \lambda]$ to mean the interval of Young's lattice.

We always consider permutations $u, w \in \mathfrak{S}_n$ partially ordered according to *weak order*: we have $u \le w$ if and only if $w = us_{i_1}s_{i_2}\cdots s_{i_k}$ for some sequence of simple transpositions $s_{i_1}, \ldots, s_{i_k}$ with $\ell(us_{i_1}\cdots s_{i_j}) = \ell(u) + j$ for all $1 \le j \le k$ (where $\ell(u)$ is the length of u). Equivalently, $u \le w$ if and only if $\mathrm{Inv}(u^{-1}) \subseteq \mathrm{Inv}(w^{-1})$ (where $\mathrm{Inv}(u)$ denotes the set of inversions of u). We always use $[u, w]$ to mean the interval of weak order.

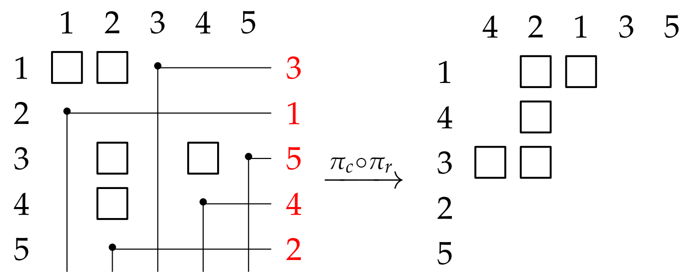
Our principal objects of interest in this paper are the initial weak order intervals $[e, w]$ where w is a "skew vexillary" permutation, a notion we now introduce.

Recall that for a permutation $w \in \mathfrak{S}_n$, the *Rothe diagram* of w is the diagram which has boxes $(i, w(j))$ for all $(i, j) \in \mathrm{Inv}(w)$.

Definition 3. *Let $\sigma = \lambda/\nu$ be a skew shape. We say that $w \in \mathfrak{S}_n$ is* skew vexillary of shape σ

if its Rothe diagram can be transformed to σ via some permutation of rows and columns. We say that w is skew vexillary *if it is skew vexillary of some shape.*

Example 4. *Consider $w = 31542$:*



As shown above, by applying the permutation $\pi_r = 14325$ to the rows and $\pi_c = 42135$ to the columns of this Rothe diagram, we transform it to the shape $(3,2,2)/(1,1)$. Hence w is skew vexillary of shape $(3,2,2)/(1,1)$.

The skew vexillary permutations include many important sub-families:

- the *Grassmannian* permutations, i.e., those with at most one descent;
- the *inverse Grassmannian* permutations, i.e., those whose inverse is Grassmannian;
- the *dominant* permutations, i.e., those whose Rothe diagram is a straight shape (equivalently, the 132-avoiding permutations);
- the *vexillary* permutations, i.e., those whose Rothe diagram can be transformed to a straight shape via permutation of rows and columns (equivalently, the 2143-avoiding permutations);
- the *fully commutative* permutations, i.e., those whose reduced words are all connected by commutation relations (equivalently, the 321-avoiding permutations).

Proposition 5. *Let $\sigma = \lambda/\nu$ be a skew shape and $\mathcal{P}$ the set of isomorphism classes of posets of the form $[e,w]$ for w a skew vexillary permutation of shape σ. Then:*

- *if $\sigma = \sigma_1 \sqcup \sigma_2$ is disconnected, then for each $[e,w] \in \mathcal{P}$ we have $[e,w] \simeq [e,w_1] \times [e,w_2]$ where w_i is skew vexillary of shape σ_i, $i = 1,2$;*
- *if σ is connected of height a and width b then for each $[e,w] \in \mathcal{P}$ we have $[e,w] \simeq [e,w']$ where $w' \in \mathfrak{S}_{a+b}$ is skew vexillary of shape σ (hence $\mathcal{P}$ is finite);*
- *$[\nu,\lambda] \in \mathcal{P}$, and if $\sigma = b^a$ is a rectangle then $[\varnothing, b^a]$ is the only element of $\mathcal{P}$, but otherwise there are some other elements which are not distributive lattices;*
- *$\mathcal{P}$ is closed under duality.*

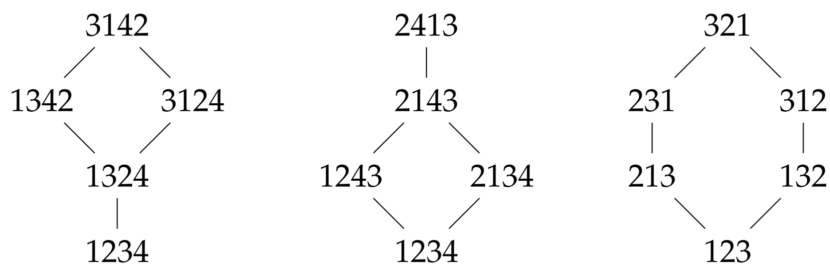


Figure 1: Example 6: the initial weak order intervals corresponding to vexillary permutations of shape $(2,1)$.

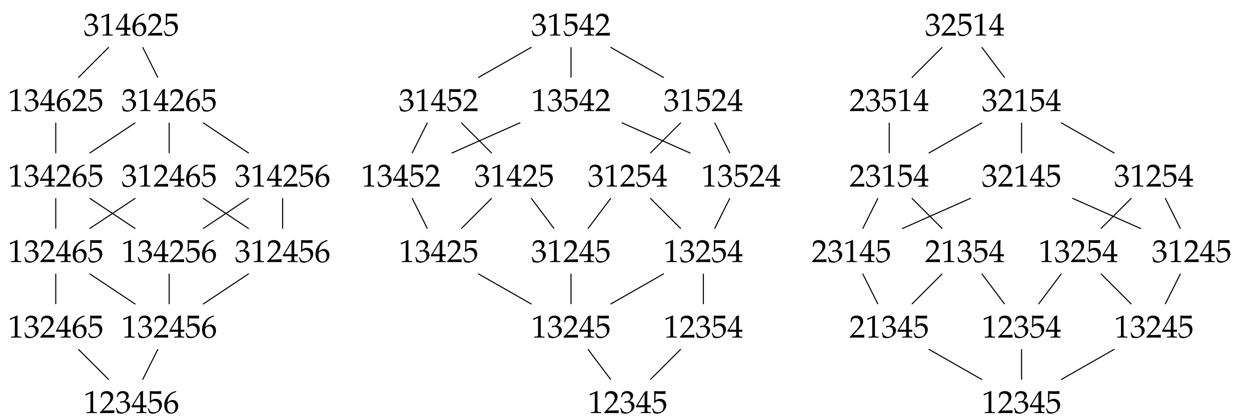


Figure 2: Example 7: the initial weak order intervals corresponding to skew vexillary permutations of shape $(3,2,2)/(1,1)$.

Let's see some examples of these families of posets.

Example 6. *Let $\lambda = (2,1)$. Up to isomorphism there are three posets of the form $[e,w]$ for w a vexillary permutation of shape λ: these are $[1234, 3142] \simeq [\varnothing, (2,1)]$ for the inverse Grassmannian permutation 3142; $[1234, 2413] \simeq [\varnothing, (2,1)]^*$ for the Grassmannian permutation 2413; and the self-dual poset $[123, 321]$ for the dominant permutation 321. These are depicted in Figure 1. One can check that all these posets are CDE with edge density 1.*

Example 7. *Let $\sigma = (3,2,2)/(1,1)$. Up to isomorphism there are three posets of the form $[e,w]$ for w a skew vexillary permutation of shape σ: these are the self-dual poset $[123456, 314625] \simeq [(1,1),(3,2,2)]$ for the fully commutative permutation 314625; and the posets $[12345, 31542]$ and $[12345, 32514]$, which are dual to one another. These are depicted in Figure 2. One can check that all these posets are CDE with edge density $3/2$.*

Examples 6 and 7 both concern "balanced" shapes so our main result will explain why all the posets in these examples are CDE.

3 Toggling for weak order intervals

Let L be a finite *semidistributive lattice* (this means that L is a finite lattice for which a "canonical join representation" and "canonical meet representation" exist for each $x \in L$; see [1] for a precise definition). For instance, any interval of weak order $[e, w]$ is a semidistributive lattice. We now explain a canonical labeling of the cover relations of L due to Barnard [1], building on work of Reading [8] in the case of weak order on $\mathfrak{S}_n$.

For any cover relation $x \lessdot y \in L$, we define the *canonical edge labeling*

$$\gamma(x \lessdot y) := \min\{z \in L \colon x \vee z = y\}.$$

The semidistributivity of L guarantees that for any $x \lessdot y$ the set $\{z \in L \colon x \vee z = y\}$ has a unique minimal element (see [1, Proposition 3.4]), and thus that this label $\gamma(x \lessdot y) \in L$ always exists. Moreover, it is easy to see that we always have $\gamma(x \lessdot y) \in \mathrm{Irr}(L)$, where $\mathrm{Irr}(L)$ is the set of *join irreducible* elements of L.

For example, consider the case where L is a *distributive lattice*. Hence L is isomorphic to $\mathcal{J}(P)$, the set of order ideals of some finite poset P ordered by containment (recall that $I \subseteq P$ is an *order ideal* if $q \in I, p \leq q \Rightarrow p \in I$). Then $\mathrm{Irr}(\mathcal{J}(P)) = P$, and for $I, J \in \mathcal{J}(P)$ with $I \lessdot J$ we have $\gamma(I \lessdot J) = p$ where $p \in P$ is such that $J = I \cup \{p\}$. This fundamental example explains the "toggling" terminology: the toggles we define below will toggle the status of p in I (when possible).

Following Reading [8], let us also explain the canonical γ-labeling for weak order on $\mathfrak{S}_n$. First of all $\mathrm{Irr}(\mathfrak{S}_n)$ consists of the non-identity Grassmannian permutations. And if $u \lessdot w$ with $u = ws_k$, then

$$\gamma(u \lessdot w) = \begin{array}{c} \text{the Grassmannian permutation } g \in \mathfrak{S}_n \text{ whose descent is } (w_k, w_{k+1}) \text{ and} \\ \text{with } (i, w_{k+1}) \in \mathrm{Inv}(g^{-1}) \Leftrightarrow (i, w_{k+1}) \in \mathrm{Inv}(u^{-1}) \text{ for } w_{k+1} < i < w_k \end{array}$$

For example, Figure 3 shows the γ-labeling for $[12345, 35142]$.

It follows from work of Barnard (see [1, Lemma 3.3]) that for any $y \in L$, among edges incident to y each join irreducible element appears as a γ-label at most once. Barnard's results allow us to define a notion of "toggling" in this semidistributive context (see also Thomas-Williams [12]). For each join irreducible element $p \in \mathrm{Irr}(L)$ we define *toggling at p* to be the involution $\tau_p \colon L \to L$ defined by

$$\tau_p(y) := \begin{cases} \mathrm{x} & \text{if } \gamma(x \lessdot y) = p; \\ \mathrm{z} & \text{if } \gamma(y \lessdot z) = p; \\ \mathrm{y} & \text{otherwise.} \end{cases}$$

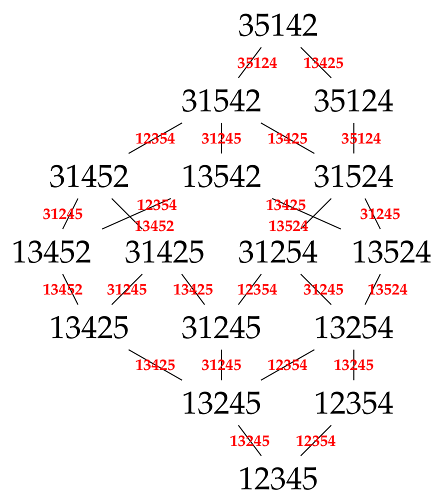


Figure 3: The canonical γ-labeling for a weak order interval.

Note that $\tau_p(y)$ is well-defined precisely because at most one edge incident to y has γ-label p. This notion of toggle generalizes the toggles studied by Striker and Williams [11] in the distributive lattice setting.

For $p \in \mathrm{Irr}(L)$ we define the toggleability statistics $\mathcal{T}_p^+, \mathcal{T}_p^-, \mathcal{T}_p \colon L \to \mathbb{Z}$ by

$$\mathcal{T}_p^+(y) := \begin{cases} 1 & \text{if } y \lessdot \tau_p(y), \\ 0 & \text{otherwise;} \end{cases} \quad \mathcal{T}_p^-(y) := \begin{cases} 1 & \text{if } \tau_p(y) \lessdot y, \\ 0 & \text{otherwise;} \end{cases} \quad \mathcal{T}_p(y) := \mathcal{T}_p^+(y) - \mathcal{T}_p^-(y).$$

Definition 8. *Let μ be a probability distribution on L. We say that μ is* toggle-symmetric *if* $\mathbb{E}(\mu; \mathcal{T}_p) = 0$ *for all* $p \in \mathrm{Irr}(L)$.

This notion of toggle-symmetric distribution generalizes the notion for distributive lattices defined in [4].

Proposition 9. *For any semidistributive lattice L, the distribution* uni_L *is toggle-symmetric.*

Proof. For each $p \in \mathrm{Irr}(L)$ and $y \in L$, we have $\mathcal{T}_p(y) = -\mathcal{T}_p(\tau_p(y))$; but y and $\tau_p(y)$ are equally probable in the uniform distribution. □

Definition 10. *We say that the semidistributive lattice L is* toggle CDE (tCDE) *if* $\mathbb{E}(\mu; \mathrm{ddeg}) = \mathbb{E}(\mathrm{uni}_L; \mathrm{ddeg})$ *for every toggle-symmetric distribution μ on L.*

In [4, Corollary 2.20] it was shown that for a distributive lattice L the distribution $\mathrm{maxchain}_L$ is toggle-symmetric, and hence that L being tCDE implies that it is CDE. This is actually **false** for general semidistributive lattices, but we show that it is true in our case of interest, namely for $L = [e, w]$ an initial interval of weak order:

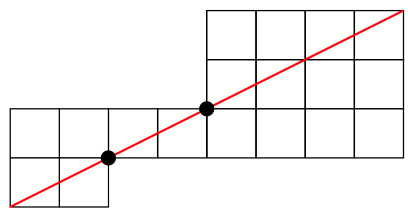
Lemma 11. *For any $w \in \mathfrak{S}_n$, the distribution* $\mathrm{maxchain}_{[e,w]}$ *is toggle-symmetric.*

4 Skew vexillary permutations of balanced shape are CDE

In this section we state our main result. So now we recall the notion of balanced shapes first defined by Chan–Haddadan–Hopkins–Moci [4].

Definition 12. *Let $\sigma = \lambda/\nu$ be a connected skew shape of height a and width b. The* main antidiagonal *of σ is the line connecting the northeast and southwest corners of the boundary of the rectangle $a \times b$ containing σ. A* corner *of σ is a point where two line segments which are part of the boundary of σ meet; we say the corner is* outward *if no box of σ intersects both of the line segments (except at the corner point). We say that σ is* balanced *if all its outward corners are exactly on its main antidiagonal.*

Example 13. *Let $\sigma = (8,8,8,2)/(4,4)$. Below we draw σ with its main antidiagonal in red and its two outward corners marked with black circles:*



All the outward corners of σ are on its main antidiagonal, so σ is balanced.

It is easy to see that rectangles b^a and staircases δ_d are balanced. Moreover, it is also easy to see that if σ is balanced then $\sigma \circ b^a$ is balanced for any $a, b \in \mathbb{N}$. So in particular the rectangular staircases $\delta_d \circ b^a$ are balanced.

Chan–Haddadan–Hopkins–Moci introduced the balanced shapes in order to prove the following theorem about them.

Theorem 14 ([4, Theorem 3.4])**.** *Let $\sigma = \lambda/\nu$ be a balanced shape of height a and width b. Then $[\nu, \lambda]$ is tCDE (and hence CDE), with edge density $ab/(a+b)$.*

Our main result is:

Theorem 15. *Let $\sigma = \lambda/\nu$ be a balanced shape of height a and width b, and $w \in \mathfrak{S}_n$ a skew vexillary permutation of shape σ. Then $[e, w]$ is tCDE (and hence CDE by Lemma 11), with edge density $ab/(a+b)$.*

Let's explain how Theorem 14 was proved, as it will be the inspiration for our proof of Theorem 15. Let $\sigma = \lambda/\nu$ be a skew shape and $(i,j) \in \sigma$ a box of σ. The *rook* statistic $R_{(i,j)} \colon [\nu, \lambda] \to \mathbb{Z}$ is the following linear combination of toggleability statistics:

$$R_{(i,j)} := \sum_{\substack{1 \le i' \le i, 1 \le j' \le j, \\ (i',j') \in \sigma}} \mathcal{T}^{+}_{(i',j')} + \sum_{\substack{i \le i' \le a, j \le j' \le b, \\ (i',j') \in \sigma}} \mathcal{T}^{-}_{(i',j')} - \sum_{\substack{1 \le i' < i, 1 \le j' < j, \\ (i',j') \in \sigma}} \mathcal{T}^{-}_{(i',j')} - \sum_{\substack{i < i' \le a, j < j' \le b, \\ (i',j') \in \sigma}} \mathcal{T}^{+}_{(i',j')}.$$

1 -1	1			
1 -1	1			
1	1 1	1	1	1
	1	-1 1	-1 1	-1 1

Figure 4: The "rook" $R_{(3,2)}$ for a 4×5 rectangle.

For example, Figure 4 depicts the rook $R_{(3,2)}$ for σ a 4×5 rectangle. In this picture, the number in the northwest corner of a box (i', j') is the coefficient of $\mathcal{T}^+_{(i',j')}$ in the rook, and the number in the southeast corner is the coefficient of $\mathcal{T}^-_{(i',j')}$. We omit zero coefficients.

The following two lemmas explain the significance of the rooks.

Lemma 16 (See [4, Lemma 3.5])**.** *For any skew shape $\sigma = \lambda/\nu$ and any box $(i, j) \in \sigma$, and any toggle-symmetric distribution μ on $[\nu, \lambda]$, we have*

$$\mathbb{E}(\mu; R_{(i,j)}) = \sum_{(i',j)\in\sigma} \mathbb{E}(\mu; \mathcal{T}^-_{(i',j)}) + \sum_{(i,j')\in\sigma} \mathbb{E}(\mu; \mathcal{T}^-_{(i,j')}).$$

Definition 17. *Let $D \subseteq \mathbb{Z}^2$ be a diagram (i.e., any finite subset of $\mathbb{Z}^2$). Let $(i, j) \in D$ be a box of D. We say that (i, j) is* cross-saturated *in D if whenever $(i, j') \in D$ is a box of D in the same row as (i, j) and $(i', j) \in D$ is a box of D in the same column as (i, j), we have $(i', j') \in D$.*

Lemma 18 (See [4, Lemma 3.6])**.** *For any skew shape $\sigma = \lambda/\nu$ and any cross-saturated box $(i, j) \in \sigma$, we have $R_{(i,j)}(\rho) = 1$ for all $\rho \in [\nu, \lambda]$.*

Lemma 16 explains the name "rook" for these random variables: they "attack" in expectation every box in the same row or same column as the box they are placed on.

Lemma 16 is immediate from the definition of the rooks; but Lemma 18 is the subtle fact which makes rooks useful. And now we see what makes balanced shapes special: they have cross-saturated boxes in the "right places." That is to say, for any balanced shape it is possible to place rooks on the cross-saturated boxes so that all the boxes of the shape are attacked the same number of times.

So to prove Theorem 15, the main thing we need to do is construct analogous rook statistics for permutations. This is possible, as the following lemma asserts:

Lemma 19. *For any permutation w and any cross-saturated box $(i, j) \in \mathrm{Inv}(w^{-1}) \subseteq \mathbb{Z}^2$, there exists a* rook statistic *$R_{(i,j)} \colon [e, w] \to \mathbb{Z}$ such that:*

- *$R_{(i,j)}$ is a linear combination of the toggleability statistics $\mathcal{T}^+_g$, $\mathcal{T}^-_g$, $g \in \mathrm{Irr}([e, w])$*

- *for any toggle-symmetric distribution μ on $[e,w]$, we have*

$$\mathbb{E}(\mu; R_{(i,j)}) = \sum_{(i',j)\in \mathrm{Inv}(w^{-1})} \sum_{\substack{g\in \mathrm{Irr}([e,w]),\\ (i',j) \text{ a descent of } g}} \mathbb{E}(\mu; \mathcal{T}_g^-) + \sum_{(i,j')\in \mathrm{Inv}(w^{-1})} \sum_{\substack{g\in \mathrm{Irr}([e,w]),\\ (i,j') \text{ a descent of } g}} \mathbb{E}(\mu; \mathcal{T}_g^-).$$

- *$R_{(i,j)}(w') = 1$ for any $w' \in [e,w]$.*

Note that cross-saturation is preserved under permutation of rows and columns of a diagram. Hence, if w is skew vexillary of balanced shape σ, then the fact that σ has many cross-saturated boxes implies that $\mathrm{Inv}(w^{-1})$ will also have many cross-saturated boxes. Thus, Theorem 15 follows easily from Lemma 19.

5 Rowmotion down-degree homomesy

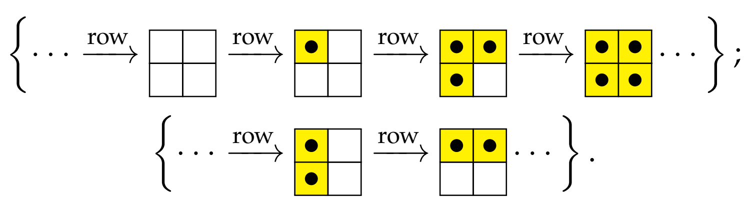
Let $L = \mathcal{J}(P)$ be a distributive lattice. *Rowmotion* on L is the map $\mathrm{row}\colon \mathcal{J}(P) \to \mathcal{J}(P)$ defined by

$$\mathrm{row}(I) := \{p \in P\colon p \le q \text{ for some } q \in \min(P \setminus I)\},$$

where $\min(P \setminus I)$ denotes the minimal elements of P not in I. Rowmotion and its generalizations have been the focus of research of many authors [2, 3, 11]. Rowmotion is in fact invertible; this follows from a description, due to Cameron and Fon-der-Flaass [3], of rowmotion as a composition of toggles: $\mathrm{row} = \tau_{p_1} \circ \tau_{p_2} \circ \cdots \circ \tau_{p_{\#P}}$, where $p_1, p_2, \ldots, p_{\#P}$ is any linear extension of P.

The poset on which the action of rowmotion has been studied the most is the distributive lattice $L = \mathcal{J}([a] \times [b]) = [\varnothing, b^a]$ corresponding to the product of two chains.

Example 20. *Consider rowmotion acting on $\mathcal{J}([2] \times [2]) = [\varnothing, 2^2]$. The orbits of rowmotion are:*



Observe that the order of rowmotion is 4, *and that the average of* ddeg *along each orbit is* 1.

Initially the main interest was in understanding the orbit structure of rowmotion acting on $\mathcal{J}([a] \times [b])$, and in particular in computing its order. For example, Brouwer and Schrijver [2] proved that the order of row acting on $\mathcal{J}([a] \times [b])$ is $a + b$.

More recently, in the context of dynamical algebraic combinatorics, various authors have become interested in other aspects of rowmotion beyond its orbit structure. One particular goal has been to exhibit "homomesies" for rowmotion. So let's review the homomesy paradigm of Propp-Roby [7].

Definition 21. *Let X be a finite set, $\Phi\colon X \to X$ an invertible operator, and $f\colon X \to \mathbb{R}$ some statistic. We say f is* homomesic *with respect to the action of Φ on X if the average of f along every Φ-orbit of X is the same; we say f is c*-mesic *if this same orbit average is $c \in \mathbb{R}$.*

Propp-Roby [7] exhibited homomesies with a number of different statistics for rowmotion acting on $\mathcal{J}([a]\times[b])$. One of their results is:

Theorem 22 ([7, Theorem 27])**.** *The statistic* ddeg *is $ab/(a+b)$-mesic with respect to the action of* row *on $\mathcal{J}([a]\times[b])$.*

Recently, Barnard [1] and Thomas-Williams [12] explained how rowmotion generalizes in a natural way to the semidistributive setting. So now let L be a semidistributive lattice, and γ its canonical edge labeling. Following Thomas-Williams [12], we define for each $y \in L$ the sets $D^\gamma(y), U^\gamma(y) \subseteq \mathrm{Irr}(L)$ of *downwards* and *upwards labels at y* to be

$$D^\gamma(y) := \{\gamma(x \lessdot y)\colon x \in L \text{ with } x \lessdot y\}; \quad U^\gamma(y) := \{\gamma(y \lessdot z)\colon z \in L \text{ with } y \lessdot z\}.$$

Rowmotion is the map $\mathrm{row}\colon L \to L$ defined as follows:

$$\mathrm{row}(y) := \text{ the unique } x \in L \text{ with } D^\gamma(x) = U^\gamma(y).$$

That rowmotion is well-defined and invertible follows from work of Barnard [1].

The study of tCDE posets is related to rowmotion homomesies by the following observation of Striker [10] (which was originally stated only for distributive lattices):

Lemma 23 ([10, Lemma 6.2])**.** *Let $\mathcal{O}$ be an orbit of* row *acting on the semistributive lattice L. Then the distribution μ which is uniform on $\mathcal{O}$ and zero outside of $\mathcal{O}$ is toggle-symmetric.*

Lemma 23 allowed Chan–Haddadan–Hopkins–Moci [4] to deduce the following corollary of Theorem 14 (which generalizes Theorem 22):

Corollary 24 ([4, Corollary 3.11])**.** *Let $\sigma = \lambda/\nu$ be a balanced shape of height a and width b. Then* ddeg *is $ab/(a+b)$-mesic with respect to the action of* row *on $[\nu,\lambda]$.*

Similarly, we deduce the following corollary of Theorem 15:

Corollary 25. *For σ a balanced shape of height a and width b, and $w \in \mathfrak{S}_n$ a skew vexillary permutation of shape σ,* ddeg *is $ab/(a+b)$-mesic with respect to the action of* row *on $[e,w]$.*

Example 26. *Let $w = 35142 \in \mathfrak{S}_5$, a vexillary permutation of balanced shape $\lambda = (3,2,1)$. The weak order interval $[e,w]$ is depicted in Figure 3. The four orbits of* row *acting on $[e,w]$ are:*

$$\{\cdots \xrightarrow{\mathrm{row}} 12345 \xrightarrow{\mathrm{row}} 13254 \xrightarrow{\mathrm{row}} 31524 \xrightarrow{\mathrm{row}} 35142 \cdots\};$$
$$\{\cdots \xrightarrow{\mathrm{row}} 13245 \xrightarrow{\mathrm{row}} 31542 \xrightarrow{\mathrm{row}} 35124 \xrightarrow{\mathrm{row}} 13425 \xrightarrow{\mathrm{row}} 31452 \xrightarrow{\mathrm{row}} 12354 \cdots\};$$
$$\{\cdots \xrightarrow{\mathrm{row}} 13452 \xrightarrow{\mathrm{row}} 31254 \xrightarrow{\mathrm{row}} 13524 \xrightarrow{\mathrm{row}} 31425 \cdots\};$$
$$\{\cdots \xrightarrow{\mathrm{row}} 13542 \xrightarrow{\mathrm{row}} 31245 \cdots\}.$$

In agreement with Corollary 25, we compute that the average down-degrees for these orbits are:

$$\frac{1}{4}(0+2+2+2) = \frac{1}{6}(1+3+1+1+2+1) = \frac{1}{4}(1+2+1+2) = \frac{1}{2}(2+1) = \frac{3}{2}.$$

Acknowledgements

I thank V. Reiner for useful discussions and encouragement during this project.

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