Séminaire Lotharingien de Combinatoire **82B** (2019)
Article #14, 12 pp.

Proceedings of the 31st Conference on Formal Power Series and Algebraic Combinatorics (Ljubljana)

Combinatorics of generalized exponents

Cédric Lecouvey*[1] and Cristian Lenart†[2]

[1]*Faculté des Sciences et Techniques, Université de Tours Tours, France*
[2]*Department of Mathematics and Statistics, State University of New York at Albany, 1400 Washington Avenue, Albany, NY 12222, U.S.A.*

Abstract. We give a purely combinatorial proof of the positivity of the stabilized forms of the generalized exponents associated to each classical root system. In finite type C_n, we obtain a combinatorial description of the generalized exponents based on symplectic King tableaux. We also present three applications of our combinatorial formula. Our methods are expected to extend to the orthogonal types.

Résumé. Pour chaque système de racines de type classique, nous donnons une preuve combinatoire de la positivité de la forme stabilisée des exposants généralisés. En type C_n, nous obtenons une description combinatoire des exposants généralisés en termes de tableaux de King et en donnons trois applications. Nos méthodes doivent s'étendre aux types orthogonaux.

Keywords: Weyl characters, Lusztig q-analogues, representation theory, crystals, branching rules.

1 Introduction

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of rank n and G its corresponding Lie group. The group G acts on the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$, and it was proved by Kostant [8] that $S(\mathfrak{g})$ factors as $S(\mathfrak{g}) = H(\mathfrak{g}) \otimes S(\mathfrak{g})^G$, where $H(\mathfrak{g})$ is the harmonic part of $S(\mathfrak{g})$. The generalized exponents of $\mathfrak{g}$, as defined by Kostant [8], are the polynomials appearing as the coefficients in the expansion of the graded character of $H(\mathfrak{g})$ in the basis of the Weyl characters. It was shown by Hesselink [2] that these polynomials coincide, in fact, with the Lusztig t-analogues [17] of zero weight multiplicities in the irreducible finite-dimensional representations of $\mathfrak{g}$. In particular, they have non-negative integer coefficients, because they are affine Kazhdan–Lusztig polynomials.

For $\mathfrak{g} = \mathfrak{sl}_n$, the generalized exponents admit a nice combinatorial description in terms of the Lascoux–Schützenberger charge statistic on semistandard tableaux of zero weight [12]. This statistic is defined via the cyclage operation on tableaux, which is based

*cedric.lecouvey@lmpt.univ-tours.fr

†clenart@albany.edu. The second author gratefully acknowledges the partial support from the NSF grant DMS–1362627.

on the Schensted insertion scheme. This combinatorial description extends, in fact, to any Lusztig t-analogue of type A_{n-1}, that is, possibly associated to a nonzero weight (also called Kostka polynomials). Another interpretation of the charge statistic in terms of crystals [3, 6] of type A_{n-1} was given later by Lascoux, Leclerc and Thibon in [11].

Despite many efforts during the last three decades, no general combinatorial proof of the positivity of the Lusztig t-analogues is known beyond type A. Nevertheless such proofs have been obtained in some particular cases [4], [5], [13] and [14].

In this extended abstract, we give a combinatorial description of the stabilized version of the generalized exponents and a proof of their positivity by using the combinatorics of type $A_{+\infty}$ crystal graphs. This can be regarded as a generalization of results in [11] for the weight zero, and in fact we were able to rederive the latter without any reference to the charge statistic or the combinatorics of semistandard tableaux. Our description is in terms of the so-called distinguished vertices in crystals of type $A_{+\infty}$, but we show that these vertices are in natural bijection with some generalizations of symplectic King tableaux, which makes the link with stable Lusztig t-analogue more natural. Next, we provide a complete combinatorial proof of the positivity of the generalized exponents in the non-stable C_n case. Observe there that the non-stable case is much more involved than the stable one, essentially because we need a combinatorial description of the non-Levi branching from $\mathfrak{gl}_{2n}$ to $\mathfrak{sp}_{2n}$, which is complicated in general. Here one needs in a crucial way recent duality results by Kwon [9, 10] giving a crystal interpretation of the previous branching and a combinatorial model relevant to its study. We strongly expect that our approach extends to the orthogonal types, once all the results of [10] are available for the non-Levi orthogonal branchings. We refer the reader to [15] for complete proofs and more results.

Section 2 recalls the definition of the generalized exponents. Section 3 is devoted to the combinatorial description of the stabilized form (in classical type) of the generalized exponents in terms of distinguished tableaux, which we define and study here. In Section 4, we give the promised combinatorial description of the generalized exponents in type C_n by using King tableaux [7]. In Section 5, we give three applications of the description in Section 4.

2 Generalized exponents

2.1 Background

Let $\mathfrak{g}_n$ be a simple Lie algebra over $\mathbb{C}$ of rank n with triangular decomposition $\mathfrak{g}_n = \oplus_{\alpha\in R_+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \oplus_{\alpha\in R_+}\mathfrak{g}_{-\alpha}$,so that $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}_n$ and R_+ its set of positive roots. The root system $R = R_+ \sqcup (-R_+)$ of $\mathfrak{g}_n$ is realized in a real Euclidean space E with inner product $(\cdot,\cdot)$. For any $\alpha \in R$, we write $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ for its coroot. Let $S \subset R_+$ be the subset of simple roots and Q_+ the $\mathbb{Z}_+$-cone generated by S. The set

P of integral weights for $\mathfrak{g}_n$ satisfies $(\beta, \alpha^\vee) \in \mathbb{Z}$ for any $\beta \in P$ and $\alpha \in R$. We write $P_+ = \{\beta \in P \mid (\beta, \alpha^\vee) \geq 0 \text{ for any } \alpha \in S\}$ for the cone of dominant weights of $\mathfrak{g}_n$, and denote by $\omega_1, \ldots, \omega_n$ its fundamental weights. Let W be the Weyl group of $\mathfrak{g}_n$ generated by the reflections s_α with $\alpha \in S$, and write ℓ for the corresponding length function.

By a classical theorem due to Kostant, the graded character of the harmonic part of the symmetric algebra $S(\mathfrak{g}_n)$ satisfies

$$\mathrm{char}_t(H(\mathfrak{g}_n)) = \frac{\prod_{i=1}^n (1 - t^{d_i})}{(1-t)^n} \prod_{\alpha \in R} \frac{1}{1 - te^\alpha} = \prod_{i=1}^n (1 - t^{d_i}) \mathrm{char}_t(S(\mathfrak{g}_n)),$$

where we have $d_i = m_i + 1$, for $i = 1, \ldots, n$, and $m_1, \ldots, m_n$ are the (classical) exponents of $\mathfrak{g}_n$. In type A_n we have $m_i = i$, in types B_n and C_n $m_i = 2i - 1$ and in type D_n $m_i = 2i - 1$ for $i = 1, \ldots, n-1$ with $m_n = n - 1$. On the other hand, it is known (see [2]) that $\mathrm{char}_t(H(\mathfrak{g}_n))$ coincides with the Hall-Littlewood polynomial Q'_0, namely we have

$$\mathrm{char}_t(H(\mathfrak{g}_n)) = Q'_0 = \sum_{\lambda \in P_+} K^{\mathfrak{g}_n}_{\lambda,0}(t) s^{\mathfrak{g}_n}_\lambda,$$

where $s^{\mathfrak{g}_n}_\lambda$ is the Weyl character associated to the finite-dimensional irreducible representation $V(\lambda)$ of $\mathfrak{g}_n$ with highest weight λ. The polynomials $K^{\mathfrak{g}_n}_{\lambda,0}(t)$ are the generalized exponents of $\mathfrak{g}_n$, and they coincide with the Lusztig t-analogues [17] associated to the zero weight subspaces in the representations $V(\lambda)$. The classical exponents $m_1, \ldots, m_n$ correspond to the adjoint representation of $\mathfrak{g}_n$, namely we have $K^{\mathfrak{g}_n}_{\widetilde{\alpha},0}(t) = \sum_{i=1}^n t^{m_i}$,where $\widetilde{\alpha}$ is the highest root in R_+.

2.2 Classical types

In classical types, $\mathrm{char}_t(S(\mathfrak{g}_n))$ is easy to compute. Let $\mathcal{P}_n$ be the set of partitions with at most n parts, and $\mathcal{P}$ the set of all partitions. The rank of the partition γ is defined as the sum of its parts, and is denoted by $|\gamma|$.

In type A_{n-1}, we start from the Cauchy identity

$$\prod_{1 \leq i,j \leq n} \frac{1}{1 - tx_i y_j} = \sum_{\gamma \in \mathcal{P}_n} t^{|\gamma|} s_\gamma(x) s_\gamma(y).$$

Here $s_\nu(x)$ stands for the ordinary Schur function in the variables $x_1, \ldots, x_n$. By setting $y_i = \frac{1}{x_i}$ for any $i = 1, \ldots, n$, and by considering the images of the symmetric polynomials in $R^{A_{n-1}} = \mathrm{Sym}[x_1, \ldots, x_n]/(x_1 \cdots x_n - 1)$, we get

$$\begin{aligned} \mathrm{char}_t(S(\mathfrak{sl}_n)) &= (1-t) \sum_{\gamma \in \mathcal{P}_n} t^{|\gamma|} s_\gamma(x) s_\gamma(x^{-1}) = (1-t) \sum_{\gamma \in \mathcal{P}_n} t^{|\gamma|} s_\gamma s_{\gamma^*} = \\ &= (1-t) \sum_{\gamma \in \mathcal{P}_n} t^{|\gamma|} \sum_{\lambda \in \mathcal{P}_{n-1}} c^\lambda_{\gamma,\gamma^*} s_\lambda(x). \end{aligned} \tag{2.1}$$

Here $\gamma^* = -w_\circ(\gamma)$, where $w_\circ$ is the permutation of maximal length in S_n and the $c^\lambda_{\gamma,\gamma^*}$'s are the Littlewood-Richardson coefficients. We also use the same notation for a symmetric polynomial and its image in $R^{A_{n-1}}$.

For any positive integer m, define $\mathcal{P}_m^{(2)}$ as the set of partitions of the form 2κ with $\kappa \in \mathcal{P}_m$, and $\mathcal{P}_m^{(1,1)}$ as the subset of $\mathcal{P}_m$ containing the partitions of the form $(2\kappa)'$ with $\kappa \in \mathcal{P}$. Moreover, we denote by $s_\lambda^{\mathfrak{so}_{2n+1}}$, $s_\lambda^{\mathfrak{sp}_{2n}}$, and $s_\lambda^{\mathfrak{so}_{2n}}$ the irreducible characters corresponding to the highest weight λ, for the Lie algebras of types B_n, C_n, and D_n, respectively.

In type B_n, we start from the Littlewood identity [16]

$$\prod_{1\le i<j\le 2n+1} \frac{1}{1-ty_iy_j} = \sum_{\nu\in\mathcal{P}_{2n+1}^{(1,1)}} t^{|\nu|/2} s_\nu(y)\,,$$

and we specialize $y_{2n+1}=1, y_{2i-1}=x_i$, and $y_{2i}=\frac{1}{x_i}$, for any $i=1,\ldots,n$. This gives

$$\mathrm{char}_t(S(\mathfrak{so}_{2n+1})) = \sum_{\nu\in\mathcal{P}_{2n+1}^{(1,1)}} t^{|\nu|/2} \sum_{\lambda\in\mathcal{P}_n} c_\nu^\lambda(\mathfrak{so}_{2n+1}) s_\lambda^{\mathfrak{so}_{2n+1}}\,,$$

where $c_\nu^\lambda(\mathfrak{so}_{2n+1})$ is the branching coefficient corresponding to the restriction from $\mathfrak{gl}_{2n+1}$ to $\mathfrak{so}_{2n+1}$. Similarly, we get:

$$\mathrm{char}_t(S(\mathfrak{sp}_{2n})) = \sum_{\nu\in\mathcal{P}_{2n}^{(2)}} t^{|\nu|/2} \sum_{\lambda\in\mathcal{P}_n} c_\nu^\lambda(\mathfrak{sp}_{2n}) s_\lambda^{\mathfrak{sp}_{2n}} \quad \text{and}$$
$$\mathrm{char}_t(S(\mathfrak{so}_{2n})) = \sum_{\nu\in\mathcal{P}_{2n}^{(1,1)}} t^{|\nu|/2} \sum_{\lambda\in\mathcal{P}_n} c_\nu^\lambda(O_{2n}) s_\lambda^{O_{2n}}\,.$$

Note that here we considered the character $s_\lambda^{O_{2n}}$ of the $O(2n)$-module $V^{O(2n)}(\lambda)$ parametrized by the partition λ.

Proposition 2.1. *We have the following identities.*

1. *In type A_{n-1}, for any $\lambda \in \mathcal{P}_{n-1}$, we have* $\frac{K_{\lambda,0}^{\mathfrak{sl}_n}(t)}{\prod_{i=1}^n(1-t^i)} = \sum_{\gamma\in\mathcal{P}_n} t^{|\gamma|} c^\lambda_{\gamma,\gamma^*}\,.$
2. *In type B_n, for any $\lambda \in \mathcal{P}_n$, we have* $\frac{K_{\lambda,0}^{\mathfrak{so}_{2n+1}}(t)}{\prod_{i=1}^n(1-t^{2i})} = \sum_{\nu\in\mathcal{P}_{2n+1}^{(1,1)}} t^{|\nu|/2} c_\nu^\lambda(\mathfrak{so}_{2n+1})\,.$
3. *In type C_n, for any $\lambda \in \mathcal{P}_n$, we have* $\frac{K_{\lambda,0}^{\mathfrak{sp}_{2n}}(t)}{\prod_{i=1}^n(1-t^{2i})} = \sum_{\nu\in\mathcal{P}_{2n}^{(2)}} t^{|\nu|/2} c_\nu^\lambda(\mathfrak{sp}_{2n})\,.$
4. *In type D_n, for any $\lambda \in \mathcal{P}_n$, we have* $\frac{K_{\lambda,0}^{O(2n)}(t)}{(1-t^n)\prod_{i=1}^{n-1}(1-t^{2i})} = \sum_{\nu\in\mathcal{P}_{2n}^{(1,1)}} t^{|\nu|/2} c_\nu^\lambda(O_{2n})\,.$

Observe that in the previous Assertion 1, the factor $(1-t)$ in (2.1) gives the missing "$d_i = 1$" in type A_{n-1} and in Assertion 2, the partition λ can have an odd rank.

For type D_n, the dominant weights are not necessarily partitions, whereas this is the case in Assertion 4 of the previous proposition. So here we have in fact to write

$$K_{\lambda,0}^{O(2n)}(t) = K_{\omega(\lambda),0}^{\mathfrak{so}_{2n}}(t) = K_{\iota(\omega(\lambda)),0}^{\mathfrak{so}_{2n}}(t)$$

for any partition $\lambda \in \mathcal{P}_n \setminus \mathcal{P}_{n-1}$ and $\omega(\lambda)$.

We have then by a theorem of Lascoux and Schützenberger [12]

$$K_{\lambda,0}^{\mathfrak{sl}_n}(t) = \sum_{T \in SST(\lambda)_0} t^{\mathrm{ch}_n(T)},$$

where $SST(\lambda)_0$ is the set of semistandard tableaux labeled by letters of $\{1 < \cdots < n\}$ of weight $\mu = (a, \ldots, a) = 0$ (i.e. each letter i appear a times in T) where $a = |\lambda|/n$, and $\mathrm{ch}_n(T)$ is the charge statistic evaluated on T. Recall that this charge statistic is defined by rather involved combinatorial operation such as cyclage on tableaux.

2.3 Stable versions

When the ranks of the classical root systems considered go to infinity, the previous relations simplify. In particular, for n sufficiently large, we have

$$c_\nu^\lambda(\mathfrak{so}_{2n+1}) = \sum_{\delta \in \mathcal{P}} c_{\lambda,2\delta}^\nu, \quad c_\nu^\lambda(\mathfrak{sp}_{2n}) = \sum_{\delta \in \mathcal{P}} c_{\lambda,(2\delta)'}^\nu, \quad \text{and} \quad c_\nu^\lambda(\mathfrak{so}_{2n}) = \sum_{\delta \in \mathcal{P}} c_{\lambda,2\delta}^\nu.$$

Observe that, for $\mathfrak{g} = \mathfrak{so}_{2n+1}$, this implies in particular that $c_\nu^\lambda(\mathfrak{so}_{2n+1}) = 0$ when the ranks of λ and ν do not have the same parity, which is false in general. Thus when $|\lambda|$ is even, we get the relations

$$\frac{K_{\lambda,0}^{B_\infty}(t)}{\prod_{i=1}^{\infty}(1-t^{2i})} = \frac{K_{\lambda,0}^{D_\infty}(t)}{\prod_{i=1}^{\infty}(1-t^{2i})} = \sum_{\nu \in \mathcal{P}^{(1,1)}} \sum_{\delta \in \mathcal{P}^{(2)}} t^{|\nu|/2} c_{\lambda,\delta}^\nu \text{ in type } B_\infty, D_\infty,$$
$$\frac{K_{\lambda,0}^{C_\infty}(t)}{\prod_{i=1}^{\infty}(1-t^{2i})} = \sum_{\nu \in \mathcal{P}^{(2)}} \sum_{\delta \in \mathcal{P}^{(1,1)}} t^{|\nu|/2} c_{\lambda,\delta}^\nu \quad \text{in type } C_\infty.$$

In particular, this gives

$$K_{\lambda,0}^{B_\infty}(t) = K_{\lambda,0}^{D_\infty}(t) \quad \text{and} \quad K_{\lambda,0}^{B_\infty}(t) = K_{\lambda',0}^{C_\infty}(t). \tag{2.2}$$

All these stabilized forms are in fact formal power series in t equal to zero when the rank of λ is odd.

3 Stabilized generalized exponents and crystal graphs of type $A_{+\infty}$

3.1 Crystal of type $A_{+\infty}$

We refer to [3, 6] for complements on Kashiwara's crystals, including the standard notation. Recall that crystals of type $A_{+\infty}$ are those associated to the infinite Dynkin diagram

$$\overset{1}{\circ} - \overset{2}{\circ} - \overset{3}{\circ} \cdots$$

The partitions label the dominant weights of $\mathfrak{sl}_{+\infty}$. If we denote by $(\omega_i)_{\geq 1}$ the sequence of fundamental weights of $\mathfrak{sl}_{+\infty}$, we have for any partition $\lambda \in \mathcal{P}$, $\lambda = \sum_i a_i \omega_i$, where a_i is the number of columns with height i in the Young diagram of λ.

To each partition λ corresponds the crystal $B(\lambda)$ of the irreducible infinite-dimensional representation of $\mathfrak{sl}_{+\infty}$ parametrized by λ. A classical model for $B(\lambda)$ is that of semistandard tableaux of shape λ on the infinite alphabet $\mathbb{Z}_{>0} = \{1 < 2 < 3 < \cdots\}$. Given $b \in B(\lambda)$, we define $\boldsymbol{\varepsilon}(b) = \sum_{i=1}^{+\infty} \varepsilon_i(b)\omega_i$ and $\boldsymbol{\varphi}(b) = \sum_{i=1}^{+\infty} \varphi_i(b)\omega_i$, where both sums are in fact finite. The weight of $b \in B(\lambda)$ then verifies $\mathrm{wt}(b) = \boldsymbol{\varphi}(b) - \boldsymbol{\varepsilon}(b)$.

3.2 Combinatorial preliminaries

In the sequel we consider the order $\leq$ on $\mathcal{P}$ such that $\lambda \leq \mu$ if and only if $\mu - \lambda \in P_+^\infty$, that is, $\mu - \lambda$ decomposes in the basis of the ω_i's with non-negative integer coefficients.

The partitions in $\mathcal{P}^{(2)}$ (resp. in $\mathcal{P}^{(1,1)}$) are those which can be tiled with horizontal (resp. vertical) dominoes. Equivalently, a partition κ belongs to $\mathcal{P}^{(2)}$ (resp. $\mathcal{P}^{(1,1)}$) if and only if the number of columns (resp. rows) of fixed height (resp. length) is even. So

$$\kappa \in \mathcal{P}^{(2)} \iff \kappa = \sum_i 2a_i\omega_i \quad \text{and} \quad \kappa \in \mathcal{P}^{(1,1)} \iff \kappa = \sum_i a_i\omega_{2i}\,.$$

Set $\mathcal{P}^{\boxplus} = \mathcal{P}^{(2)} \cap \mathcal{P}^{(1,1)}$. It follows that

$$\kappa \in \mathcal{P}^{\boxplus} \iff \kappa = \sum_i 2a_i\omega_{2i}\,,$$

that is, λ decomposes in terms of the fundamental weights ω_{2i} with even coefficients. In the general case of a partition $\kappa \in \mathcal{P}$ written as $\kappa = \sum_i a_i\omega_i$ we define

$$\kappa_{\boxplus} = \sum_i (a_{2i} - (a_{2i} \bmod 2))\omega_{2i} \quad \text{and} \quad \kappa^{\boxplus} = \kappa - \kappa_{\boxplus} = \sum_i a_{2i+1}\omega_{2i+1} + \sum_i (a_{2i} \bmod 2)\omega_{2i}\,.$$

So $\kappa_{\boxplus}$ and $\kappa^{\boxplus}$ are partitions and $\kappa_{\boxplus} \in \mathcal{P}^{\boxplus}$.

We denote by $P^{\infty}_{(2)}$ and $P^{\infty}_{(1,1)}$ the sublattices of $P = \oplus_{i\geq 1}\mathbb{Z}_i$ defined by $P^{\infty}_{(2)} = \oplus_{i\geq 1} 2\mathbb{Z}_i$ and $P^{\infty}_{(1,1)} = \oplus_{i\geq 1}\mathbb{Z}_{2i}$. Observe that $P^{\infty}_{(2)} \cap \mathcal{P} = \mathcal{P}_{(2)}$ and $P^{\infty}_{(1,1)} \cap \mathcal{P} = \mathcal{P}_{(1,1)}$. We have also $P^{\infty}_{\boxplus} = P^{\infty}_{(2)} \cap P^{\infty}_{(1,1)} = \oplus_{i\geq 1} 2\mathbb{Z}_{2i}$ and $\mathcal{P}^{\boxplus} = \mathcal{P} \cap P^{\infty}_{(2)} \cap P^{\infty}_{(1,1)}$. We define the order $\leq_{\boxplus}$ on $\mathcal{P}$ by $\lambda \leq_{\boxplus} \mu \iff \mu - \lambda \in \mathcal{P}^{\boxplus}$.

3.3 A combinatorial description of the series $K^{C_\infty}_{\lambda,0}(t)$

Definition 3.1. *Consider a partition μ. A vertex $b \in B(\lambda)$ is called μ-distinguished if there exists $(\nu, \delta) \in \mathcal{P}^{(2)} \times \mathcal{P}^{(1,1)}$ such that $\boldsymbol{\varphi}(b) = \nu - \mu$ and $\boldsymbol{\varepsilon}(b) = \delta - \mu$.*

Definition 3.2. *Let $D(\lambda)$ be the set of all vertices in $B(\lambda)$ which are μ-distinguished for at least a partition μ.*

Clearly, if b is μ-distinguished, then b is $(\mu + \kappa)$-distinguished for any $\kappa \in \mathcal{P}^{\boxplus}$ (change $(\nu, \delta) \in \mathcal{P}^{(2)} \times \mathcal{P}^{(1,1)}$ to $(\nu + \kappa, \delta + \kappa) \in \mathcal{P}^{(2)} \times \mathcal{P}^{(1,1)}$). For any $b \in D(\lambda)$, set

$$S_b = \{\mu \in \mathcal{P} \mid b \text{ is } \mu\text{-distinguished}\}.$$

Lemma 3.3. *The set S_b has the form $S_b = \mu_b + \mathcal{P}^{\boxplus}$ and μ_b is minimal for $\leq_{\boxplus}$ such that b is μ_b-distinguished. Moreover, for any $\mu \in S_b$, we have $\mu_b = \mu^{\boxplus}$.*

The following proposition makes more explicit the structure of the distinguished tableaux.

Proposition 3.4. *Let b be a vertex of $B(\lambda)$ with $\lambda \in \mathcal{P}$. Then b is distinguished if and only if $\varepsilon_i(b) = 0$ for any odd i and $\varphi_i(b)$ is even for any odd i. Moreover, we then have $\mu_b = \sum_i (\varphi_{2i}(b) \bmod 2)\omega_{2i} =: \varphi(b) \bmod 2$.*

Theorem 3.5. *We have $K^{C_\infty}_{\lambda,0}(t) = \sum_{b\in D(\lambda)} t^{|\boldsymbol{\varphi}(b)+\mu_b|/2}$.*

3.4 Distinguished tableaux and zero weight King type tableaux

To see that the distinguished tableaux we introduced previously are in natural bijection with zero weight tableaux very close to King tableaux, consider the sets $T_{C_\infty}(\lambda)$ of semistandard tableaux of shape λ on the infinite ordered alphabet $\{1 < \overline{1} < 2 < \overline{2} < \cdots\}$. There will be no condition on the position of the barred letters here, contrary to the definition of King tableaux. Recall the notation of Section 3.3. For any distinguished vertex b in $D(\lambda)$, set $\boldsymbol{\theta}(b) = \boldsymbol{\varphi}(b) + \mu_b$, and let $\theta_j(b)$ be the coefficient of ω_j in the expansion of $\boldsymbol{\theta}(b)$. Since $\boldsymbol{\theta}(b)$ is a dominant weight for $\mathfrak{sl}_\infty$, it can be regarded as a partition. Recall also that $|\lambda|$ is even, says $|\lambda| = 2\ell$. In the sequel of this section, we shall assume that $B(\lambda)$ is realized as the set of semistandard tableaux on the infinite ordered alphabet $\mathbb{Z}_{>0}$. For any integer $i \geq 1$, a reverse lattice skew tableau on $\{2i-1, 2i\}$ is a semistandard filling of a skew Young diagram with columns of height at most 2 by letters $2i-1$ and $2i$

whose Japanese reading is a lattice word (i.e., in each left factor the number of letters $2i$ is less or equal to that of letters $2i-1$).

Example 3.6. Assume $i = 2$. Then

$$
\begin{array}{cccccc}
 & & & \boxed{3} & \boxed{3} & \boxed{4} \\
\boxed{3} & \boxed{3} & \boxed{4} & \boxed{4} & &
\end{array}
\qquad
\begin{array}{cccccccc}
 & & & & \boxed{3} & \boxed{3} & \boxed{3} & \boxed{3} \\
\boxed{3} & \boxed{3} & \boxed{4} & \boxed{4} & \boxed{4} & & &
\end{array}
\tag{3.1}
$$

is a reverse lattice skew tableau on $\{3,4\}$.

The following proposition is a reformulation of Proposition 3.4.

Proposition 3.7. *A semistandard tableau T of shape λ is distinguished if and only if for any integer $i \geq 1$, the skew tableau obtained by keeping only the letters $2i-1$ and $2i$ in T is a reverse lattice tableau, and the rows of $\boldsymbol{\theta}(T)$ have even lengths.*

We now explain the correspondence between distinguished tableaux and zero weight King type tableaux. Observe that a tableau T in $T_{C_\infty}(\lambda)$ of weight zero is a juxtaposition of skew tableaux of weight 0 on $\{i,\bar{\imath}\}$ obtained by keeping only the letters i and $\bar{\imath}$. So to obtain a bijection between the set of distinguished tableaux of shape λ and the subset $T^0_{C_\infty}(\lambda) \subset T_{C_\infty}(\lambda)$ of zero weight tableaux, it suffices to describe a bijection between the set of reverse lattice tableaux on $\{2i-1,2i\}$ of given shape and weight in $2\omega_i\mathbb{Z}_{\geq 0}$, and the set of skew tableaux on $\{i,\bar{\imath}\}$ with weight 0. Now recall that we have the structure of a $U_q(\mathfrak{sl}_2)$-crystal on the set of all skew semistandard tableaux of fixed skew shape both on $\{2i-1,2i\}$ and $\{i,\bar{\imath}\}$. By replacing each letter $2i-1$ by i and each letter $2i$ by $\bar{\imath}$, we get a crystal isomorphism f. The distinguished tableaux correspond to the highest weight vertices of weight in $2\omega_i\mathbb{Z}_{\geq 0}$ for the $\{2i-1,2i\}$-structure, whereas the tableaux of weight 0 give the vertices of weight 0 in the $\{i,\bar{\imath}\}$-crystal structure. By observing that only $U_q(\mathfrak{sl}_2)$-crystals with highest weight in $2\omega_i\mathbb{Z}_{\geq 0}$ admit a vertex of weight 0, which is then unique, we obtain that the map $\mathcal{C}$ which associates to each zero weight vertex in the $\{i,\bar{\imath}\}$-crystal structure its highest weight vertex in the $\{2i-1,2i\}$-crystal structure is the bijection we need. More precisely, the map $\mathcal{C}$ (resp. its inverse) is obtained as usual: we start by encoding in the reading of each $\{i,\bar{\imath}\}$-tableau (resp. of each $\{2i-1,2i\}$-tableau) the letters i by $+$ and the letters $\bar{\imath}$ by $-$ (resp. the letters $2i-1$ by $+$ and the letters $2i-1$ by $-$), and next by recursively deleting all the factors $+-$, thus obtaining a reduced word of the form $-^m+^m$ (resp. $+^{2m}$). It then suffices to change the m letters $\bar{\imath}$ corresponding to the m surviving symbols $-$ into i and to apply the isomorphism f^{-1} (resp. change m letters $2i-1$ corresponding to the rightmost m surviving symbols $+$ into $2i$ and apply the isomorphism f).

Example 3.8. The skew tableau of weight 0 on $\{2, \overline{2}\}$ corresponding to (3.1) is

$$\begin{array}{ccccccccc} & & & & \boxed{2} & \boxed{2} & \boxed{2} & \boxed{\overline{2}} \\ \boxed{2} & \boxed{\overline{2}} & \boxed{\overline{2}} & \boxed{\overline{2}} & \boxed{\overline{2}} & & & \end{array}$$

$$\begin{array}{ccccccc} & & & \boxed{2} & \boxed{2} & \boxed{\overline{2}} \\ \boxed{2} & \boxed{2} & \boxed{\overline{2}} & \boxed{\overline{2}} & & \end{array}$$

In the sequel, we shall abuse the notation and identify the two crystal structures corresponding up to the isomorphism f.

Example 3.9. Assume $\lambda = (1, 1)$. Then we get

$$T^0_{C_\infty}(\lambda) = \left\{ \begin{array}{|c|} \hline k \\ \hline \overline{k} \\ \hline \end{array} \mid k \in \mathbb{Z}_{\geq 1} \right\} \quad \text{and} \quad K^0_{C_\infty}(\lambda) = \left\{ \begin{array}{|c|} \hline k \\ \hline \overline{k} \\ \hline \end{array} \mid k \in \mathbb{Z}_{\geq 2} \right\}.$$

This gives

$$H\left(\begin{array}{|c|} \hline k \\ \hline \overline{k} \\ \hline \end{array} \right) = \begin{array}{|c|} \hline 2k-1 \\ \hline 2k \\ \hline \end{array} \quad \text{and} \quad \boldsymbol{\varphi}\left(\begin{array}{|c|} \hline 2k-1 \\ \hline 2k \\ \hline \end{array} \right) = \omega_{2k} \text{ for any } k \geq 1.$$

Therefore

$$\boldsymbol{\theta}\left(\begin{array}{|c|} \hline 2k-1 \\ \hline 2k \\ \hline \end{array} \right) = 2\omega_{2k} \text{ for any } k \geq 1.$$

Finally $K^{C_\infty}_{\lambda,0}(t) = \sum_{k \geq 1} t^{2k} = \frac{t^2}{1-t^2}$.

4 Type C_n generalized exponents via the Kwon model

In this section, we refine the results in Section 3.4 to the finite type C_n, based on Kwon's model for the corresponding branching coefficients [9, 10]. We also need to use a combinatorial map realizing the conjugation symmetry of Littlewood-Richardson coefficients. It turns out that Kwon's model, the version of the conjugation symmetry map used here, and the distinguished tableaux in Section 3.3 fit together in a beautiful way. This allows us to express the related statistic in terms of a natural combinatorial labeling of the vertices of weight 0 in the corresponding type C_n crystal of highest weight λ, namely the corresponding tableaux due to King [7].

Theorem 4.1. *We have*

$$K^{C_n}_{\lambda,0}(t) = \sum_{T \in K^0_{C_n}(\lambda)} t^{\mathrm{ch}_{C_n}(L(T))},$$

where

$$\mathrm{ch}_{C_n}(L(T)) = \sum_{i=1}^{2n-1} (2n-i) \left\lceil \frac{\varepsilon_i(L(T))}{2} \right\rceil.$$

Remarks 4.2. (1) There does not seem to be a simple way to express the related statistic above directly in terms of T. However, the map $T \mapsto L(T)$ is a simple one.

(2) Theorem 4.1 gives a statistic for computing the Kostka–Foulkes polynomial on King tableaux, rather than on the Kashiwara–Nakashima (KN). A natural question is whether the statistic above can be translated to the KN tableaux and moreover if it is related to the charge statistic constructed in [13] (which conjecturally computes the Kostka–Foulkes polynomials). This question is addressed in [1].

We will now continue Example 3.9.

Example 4.3. Assume $\lambda = (1,1)$ in type C_n. Then we get

$$K^0_{C_n}(\lambda) = \left\{ \begin{array}{|c|}\hline k \\ \hline \bar{k} \\ \hline \end{array} \mid k = 2, \ldots, n \right\}.$$

This gives

$$L\left(\begin{array}{|c|}\hline k \\ \hline \bar{k} \\ \hline \end{array}\right) = \begin{array}{|c|}\hline 2k-1 \\ \hline 2k \\ \hline \end{array} \quad \text{and} \quad \boldsymbol{\varepsilon}^*\left(\begin{array}{|c|}\hline 2k-1 \\ \hline 2k \\ \hline \end{array}\right) = \omega_{2(n-k+1)} \text{ for any } k = 2, \ldots, n.$$

Therefore

$$\boldsymbol{\theta}^*_n\left(\begin{array}{|c|}\hline 2k-1 \\ \hline 2k \\ \hline \end{array}\right) = 2\omega_{2(n-k+1)} \text{ for any } k = 2, \ldots, n.$$

Finally$[K^{C_n}_{\lambda,0}(t) = \sum_{k=1}^{n-1} t^{2k} = \frac{t^2 - t^{2n}}{1-t^2}$.

5 Three applications of Theorem 4.1

5.1 Growth of generalized exponents

First we analyze the growth of the generalized exponents of type C_n with respect to the rank n. The (weight 0) symplectic King tableaux of type C_n embed into those of type C_{n+1} by changing the entries $k, \bar{k}$ to $k+1, \overline{k+1}$, for all k, respectively. Moreover, it is easy to see that this map preserves the statistic in Theorem 4.1. So we obtain the following result, which to our knowledge is new.

Theorem 5.1. *For any integer n and any partition λ with at most n parts, we have $K^{C_{n+1}}_{\lambda,0}(t) - K^{C_n}_{\lambda,0}(t) \in \mathbb{Z}_{\geq 0}[t]$.*

5.2 Reducing a type C generalized exponent to one of type A

We now prove a conjecture of the first author [13]. This conjecture is the first step in the construction of the type C_n charge statistic in [13], and proves the conjecture that this charge computes the corresponding Kostka–Foulkes polynomials in the case of column shapes; see Remarks 4.2 (2). We now label the Dynkin diagram of type C_n such that the special node is n. Consider the fundamental weight ω_{2p}, where $p \in \{1, \dots, \lfloor n/2 \rfloor\}$. All the zero weight vertices in the crystal $B(\omega_{2p})$ belong to the same type A_{n-1} component, which has highest weight $\gamma_p := \varepsilon_1 + \ldots + \varepsilon_p - \varepsilon_{n-p+1} - \ldots - \varepsilon_n$, where ε_i are the coordinate vectors in $\mathbb{R}^n$. In type A_{n-1}, this weight corresponds to the partition $(1^{n-2p}, 2^p)$.

Theorem 5.2. *We have* $K^{C_n}_{\omega_{2p},0}(t) = K^{A_{n-1}}_{\gamma_p,0}(t^2)$.

Remark 5.3. Theorem 5.2 also permits to establish the conjecture of [13] for Lusztig t-analogues associated to any fundamental weight.

5.3 The smallest power of t in $K^{C_n}_{\lambda,0}(t)$

The largest power of t in $K^{C_n}_{\lambda,0}(t)$ is well-known to be $\langle \lambda, \rho^\vee \rangle$, where $\rho^\vee$ is half the sum of the positive coroots. Furthermore, it is also known that the smallest power is greater or equal to $|\lambda|/2$. See [14]. As the third application of our formula for $K^{C_n}_{\lambda,0}(t)$, we will determine this smallest power. Let $\lambda \in \mathcal{P}_n$ be such that $|\lambda|$ is even, and write $\lambda = \sum_{i=1}^n a_i\, \omega_{n+1-i}$. Define

$$s_k := \sum_{i=1}^{k} a_i\,, \qquad b_i := \begin{cases} a_i + 1 & \text{if } a_i \text{ odd and } s_i \text{ odd} \\ a_i - 1 & \text{if } a_i \text{ odd and } s_i \text{ even} \\ a_i & \text{if } a_i \text{ even}\,. \end{cases}$$

Also let $s_0 := 0$ and $S := s_n$.

Theorem 5.4. *The smallest power of t in $K^{C_n}_{\lambda,0}(t)$, for $|\lambda|$ even, is*

$$\frac{1}{2}\sum_{i=1}^{n}(n+1-i)b_i = \frac{|\lambda|}{2} + \frac{1}{2}\sum_{i\,:\,a_i \text{ odd}} (-1)^{s_i - 1}(n+1-i)\,. \tag{5.1}$$

References

[1] M. Dolega, T. Gerber, and J. Torres. “A positive combinatorial formula for symplectic Kostka-Foulkes polynomials I: Rows”. `arXiv:1911.06732`. 2019.

[2] W. H. Hesselink. “Characters of the nullcone”. *Math. Ann.* **252** (1980), pp. 179–182.

[3] J. Hong and S.-J. Kang. *Introduction to quantum groups and crystal bases*. Vol. 42. Graduate Studies in Mathematics. Amer. Math. Soc., 2002.

[4] B. Ion. "Generalized exponents of small representations. I". *Represent. Theory* **13** (2009), pp. 401–426.

[5] B. Ion. "Generalized exponents of small representations. II". *Represent. Theory* **15** (2011), pp. 433–493.

[6] M. Kashiwara. "On crystal bases". Vol. 16. CMS Conf. Proc. Amer. Math. Soc., Providence, RI, 1995, pp. 155–197.

[7] R. C. King. "Weight multiplicities for the classical groups" (1976), 490–499. Lecture Notes in Phys., Vol. 50.

[8] B. Kostant. "Lie group representations on polynomial rings". *Amer. J. Math.* **85** (1963), pp. 327–404.

[9] J.-H. Kwon. "Combinatorial extension of stable branching rules for classical groups". *Trans. Amer. Math. Soc.* **370** (2018), pp. 6125–6152.

[10] J.-H. Kwon. "Lusztig data of Kashiwara-Nakashima tableaux in types B and C". *J. Algebra* **503** (2018), pp. 222–264.

[11] A. Lascoux, B. Leclerc, and J.-Y. Thibon. "Crystal graphs and q-analogues of weight multiplicities for the root system A_n". *Lett. Math. Phys.* **35** (1995), pp. 359–374.

[12] A. Lascoux and M.-P. Schützenberger. "Sur une conjecture de H. O. Foulkes". *C. R. Acad. Sci. Paris Sér. I Math.* **288** (1979), pp. 95–98.

[13] C. Lecouvey. "Kostka-Foulkes Polynomials, Cyclage Graphs and Charge Statistic for the Root System C_n". *J. Algebraic Combin.* **21** (2005), pp. 203–240.

[14] C. Lecouvey. "Combinatorics of crystal graphs and Kostka-Foulkes polynomials for the root systems B_n, C_n and D_n". *Eur. J. Comb.* **27** (2006), pp. 526–557.

[15] C. Lecouvey and C. Lenart. "Combinatorics of generalized exponents". *Int. Math. Res. Not.* (2018). `DOI 10.1093/imrn/rny157`.

[16] D. E. Littlewood. *The theory of group characters and matrix representations of groups*. Reprint of the second (1950) edition. AMS Chelsea Publishing, Providence, RI, 2006.

[17] G. Lusztig. "Singularities, character formulas, and a q-analog of weight multiplicities". *Analysis and topology on singular spaces, II, III (Luminy, 1981)*. Vol. 101. Astérisque. Paris: Soc. Math. France, 1983, pp. 208–229.