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Enumerating Linear Systems on Graphs

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Abstract. The divisor theory of graphs views a finite connected graph G as a discrete version of a Riemann surface. Divisors on G are formal integral combinations of the vertices of G, and linear equivalence of divisors is determined by the discrete Laplacian operator for G. As in the case of Riemann surfaces, we are interested in the complete linear system $|D|$ of a divisor D—the collection of nonnegative divisors linearly equivalent to D. Unlike the case of Riemann surfaces, the complete linear system of a divisor on a graph is always finite. We compute generating functions encoding the sizes of all complete linear systems on G. We interpret our results in terms of polyhedra associated with divisors and in terms of the invariant theory of the (dual of the) Jacobian group of G. If G is a cycle graph, our results lead to a bijection between complete linear systems and binary necklaces.

Keywords: Chip-firing, divisors on graphs, lattice points in polyhedra, invariant theory, necklaces

1 Introduction.

Divisor theory preliminaries. Let $G = (V, E)$ be a connected, undirected multigraph with finite vertex set V and finite edge multiset E. Loops are allowed but our results are not affected if they are removed.

We recall some of the theory of divisors on graphs, referring readers unfamiliar with this theory to [1] or to Part 1 of the textbook [3]. A *divisor* on G is an element in the free abelian group on the vertices of G,

$$\mathrm{Div}(G) := \mathbb{Z}V = \left\{\textstyle\sum_{v\in V} D(v)v : D(v) \in \mathbb{Z}\right\}.$$

The degree of a divisor D is the sum of its coefficients: $\deg(D) := \sum_{v\in V} D(v)$. For instance, if we consider $v \in V$ as a divisor, then $\deg(v) = 1$. We use the notation $\deg_G(v)$ to refer to the ordinary degree of a vertex—the number of edges incident on v. The set

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of divisors of degree k is denoted by $\mathrm{Div}^k(G)$. The *(discrete) Laplacian operator* of G is the function $L\colon \mathbb{Z}^V \to \mathbb{Z}^V$ given by

$$L(f)(v) = \sum_{vw \in E} (f(v) - f(w))$$

for each $f \in \mathbb{Z}^V$ and $v \in V$. The *divisor of a function* $f\colon V \to \mathbb{Z}$, arising by analogy from the theory of divisors on Riemann surfaces, is then

$$\mathrm{div}(f) := \sum_{v \in V} (L(f)(v))\, v \in \mathrm{Div}(G).$$

The mapping $v \mapsto \chi_v$ which sends each vertex to its corresponding characteristic function determines an isomorphism $\chi\colon \mathrm{Div}(G) \simeq \mathbb{Z}^V$, and we have $\chi \circ \mathrm{div} = L$, which we use to identify div with L.

Divisors of functions are called *principal divisors*, and they form an additive subgroup of $\mathrm{Div}(G)$ denoted $\mathrm{Prin}(G)$. Divisors D and D' are *linearly equivalent* if $D - D' \in \mathrm{Prin}(G)$, in which case we write $D \sim D'$. The *Picard group* of G is then the group of divisors modulo linear equivalence:

$$\mathrm{Pic}(G) := \mathrm{Div}(G) / \mathrm{Prin}(G).$$

Since principal divisors have degree zero, $\mathrm{Pic}(G)$ is graded by degree. Its degree k part is denoted $\mathrm{Pic}^k(G)$. The degree-zero part of the Picard group is a subgroup called the *Jacobian group* of G:

$$\mathrm{Jac}(G) := \mathrm{Pic}^0(G) = \mathrm{Div}^0(G) / \mathrm{Prin}(G) \subseteq \mathrm{Pic}(G).$$

We write $[D]$ for the class of a divisor D modulo $\mathrm{Prin}(G)$. Fixing any vertex $q \in V$, there is an isomorphism

$$\begin{aligned} \mathrm{Pic}(G) &\to \mathbb{Z} \oplus \mathrm{Jac}(G) \\ [D] &\mapsto (\deg(D), [D - \deg(D)q]) . \end{aligned} \tag{1.1}$$

Throughout this work, we fix an ordering $v_1, \dots, v_n$ of V, which determines a basis for $\mathrm{Div}(G)$ and a corresponding dual basis $\chi_{v_1}, \dots, \chi_{v_n}$ for $\mathbb{Z}^V$, allowing us to identify both spaces with $\mathbb{Z}^n$. Thus, $D \in \mathrm{Div}(G)$ is identified with $(D(v_1), \dots, D(v_n))$, and for any $v \in V$, we may refer to the v-th coordinate of a vector in $\mathbb{Z}^n$. With respect to the chosen bases, div and L are represented by the $n \times n$ *Laplacian matrix*, which we also denote by L. This matrix is given by

$$L = \mathrm{Deg}(G) - A$$

where

$$\mathrm{Deg}(G) = \mathrm{diag}(\deg_G(v_1), \dots, \deg_G(v_n))$$

and A is the adjacency matrix for G. The i,j-th entry of A is the number of edges connecting v_i to v_j. It is symmetric since G is undirected. We then have the isomorphism

$$\begin{aligned}\operatorname{Pic}(G) &\simeq \operatorname{cok}(L) = \mathbb{Z}^n / \operatorname{im}_{\mathbb{Z}}(L)\\ [\textstyle\sum_{i=1}^n a_i v_i] &\mapsto (a_1, \ldots, a_n) + \operatorname{im}_{\mathbb{Z}}(L).\end{aligned}$$

Fixing any vertex $q \in V$, define the *reduced Laplacian* matrix for G with respect to q as the $(n-1) \times (n-1)$ matrix $\tilde{L}$ formed by removing the row and column corresponding to q from L. There is an isomorphism

$$\begin{aligned}\operatorname{Jac}(G) &\simeq \mathbb{Z}^{n-1} / \operatorname{im}_{\mathbb{Z}}(\tilde{L})\\ [D] &\to D|_{q=0}\end{aligned} \tag{1.2}$$

where $D|_{q=0} := \sum_{v \in V \setminus \{q\}} D(v)v$. The inverse sends the class of the v-th standard basis vector in $\mathbb{Z}^{n-1} / \operatorname{im}_{\mathbb{Z}}(\tilde{L})$ to $[v-q]$ for each $v \neq q$. Isomorphisms (1.1) and (1.2) combine to say that for $D, D' \in \operatorname{Div}(G)$,

$$D \sim D' \quad \iff \quad \left(\deg(D) = \deg(D') \quad \text{and} \quad D|_{q=0} = D'|_{q=0} \bmod \operatorname{im}_{\mathbb{Z}}(\tilde{L})\right).$$

The kernel of the Laplacian matrix is the set of constant vectors, and the reduced Laplacian has full rank $n-1$. By the matrix-tree theorem, the number of spanning trees of G is $\det(\tilde{L})$, and thus by (1.2), it is also the order of $\operatorname{Jac}(G)$.

Partitioning effective divisors. A divisor E is *effective* if $E(v) \geq 0$ for all $v \in V$, in which case we write $E \geq 0$. The *complete linear system* of a divisor D is its set of linearly equivalent effective divisors:

$$|D| := \{E \in \operatorname{Div}(G) : E \geq 0 \text{ and } E \sim D\}.$$

Note that $|D|$ depends only on the divisor class of D. Also, since linearly equivalent divisors have the same degree, $|D|$ is finite.

Fix $q \in V$, and for each $[D] \in \operatorname{Jac}(G)$, define

$$\mathbb{E}_{[D]} := \cup_{k \geq 0} |D + kq| = \{E \in \operatorname{Div}(G) : E \geq 0 \text{ and } E - \deg(E)q \sim D\}.$$

The $\mathbb{E}_{[D]}$ partition the set of effective divisors as D runs over a set of representatives for $\operatorname{Jac}(G)$. The collection $\mathbb{E}_{[0]}$ is a semigroup, and each $\mathbb{E}_{[D]}$ is a $\mathbb{E}_{[0]}$ semi-module. Note that $\mathbb{E}_{[D]}$ depends on q.[1]

[1] For $q' \in V$, writing $D + kq = D + kq' + k(q - q')$ shows the dependence is "periodic" with period equal to the order of $[q - q'] \in \operatorname{Jac}(G)$.

Definition 1.1. The *λ-sequence* for $[D] \in \mathrm{Jac}(G)$ is the sequence with k-th term

$$\lambda_{[D]}(k) := \#|D + kq|.$$

(It does not depend on the choice of representative of the class $[D]$.) The *λ-sequence generating function* is

$$\Lambda_{[D]}(z) := \sum_{k \geq 0} \lambda_{[D]}(k) z^k.$$

Goal. The study of complete linear systems is fundamental to the divisor theory of both Riemann surfaces and graphs. For instance, the Riemann-Roch Theorem[2] in either area can be interpreted as a statement about the existence of effective divisors. However, unlike the case of Riemann surfaces, the complete linear system of a divisor on a graph is always finite. Thus, it is natural to wonder about its cardinality. We know of no prior work focused on this question. Moreover, there are special representatives for the elements $\mathrm{Jac}(G)$ relative to the choice of a vertex q variously called *G-parking functions* or *q-reduced divisors* (for example, see [3]). The structures we introduce here provide new invariants to attach to these objects.

Our goal for this extended abstract is to provide a means of computing the λ-sequence generating function $\Lambda_{[D]}$ for any graph G and divisor $[D]$. In Section 2, the computation is achieved by introducing systems of *primary* and *secondary divisors*. In Section 3, the primary and secondary divisors are related to standard theory for counting lattice points in polyhedra. In Section 4, primary and secondary divisors are interpreted as primary and secondary invariants for a certain representation of the dual group $\mathrm{Jac}^*(G)$. Finally, in Section 5, we relate complete linear systems on cycle graphs to binary necklaces. This final section also serves as a set of concrete examples for the results in earlier sections.

2 Primary and secondary divisors

We first compute $\Lambda_{[D]}$ using *primary* and *secondary divisors*, defined as part of the following theorem.

Theorem 2.1. *Fix $q \in V$, and for each $v \in V$, let $\mathrm{ord}_q(v)$ be the order of $[v - q]$ in the finite group $\mathrm{Jac}(G)$.*

(1) (Existence) *There exists a finite subset $\mathcal{P} \subset \mathbb{E}_{[0]}$ and for each $[D] \in \mathrm{Jac}(G)$, a finite subset $\mathcal{S}_{[D]} \subset \mathbb{E}_{[D]}$ such that each $E \in \mathbb{E}_{[D]}$ can be written uniquely as*

$$E = F + \sum_{P \in \mathcal{P}} a_P P$$

[2] See [1] for the Riemann-Roch Theorem for graphs.

with $F \in \mathcal{S}_{[D]}$ and $a_P \in \mathbb{Z}_{\geq 0}$ for all $P \in \mathcal{P}$. The set $\mathcal{P}$ is called a set of primary divisors *for G, and $\mathcal{S}_{[D]}$ is called the set of $[D]$-*secondary *divisors with respect to $\mathcal{P}$.*

(2) (Uniqueness) *Sets $\mathcal{P}$ and $\{\mathcal{S}_{[D]}\}_{[D]\in \mathrm{Jac}(G)}$ satisfy part (1) if and only if*

$$\mathcal{P} = \{\ell_v v : v \in V\} \quad \text{and} \quad \mathcal{S}_{[D]} = \{E \in \mathbb{E}_{[D]} : E(v) < \ell_v \text{ for all } v \in V\},$$

where ℓ_v is any positive multiple of $\mathrm{ord}_q(v)$ for all $v \in V$.

(3) *With $\mathcal{P}$ and $\mathcal{S}_{[D]}$ as above,*

$$|\mathcal{S}_{[D]}||\,\mathrm{Jac}(G)| = \prod_{v\in V} \ell_v.$$

(4) *Choose $\ell_i := \ell_{v_i}$ for $j = 1, \ldots, n$ in accordance with part (2), and let $q = v_n$. Let A be a set of standard representatives for the image of $D|_{q=0} + \mathrm{im}_{\mathbb{Z}}(\tilde{L})$ under the natural projection $\mathbb{Z}^{n-1} \to \prod_{i=1}^{n-1} \mathbb{Z}/\ell_i\mathbb{Z}$. Then the corresponding secondary divisors are*

$$\mathcal{S}_{[D]} = \{(a_1, \ldots, a_{n-1}, a_n) : (a_1, \ldots, a_{n-1}) \in A, 0 \leq a_n < \ell_n\}.$$

A standard argument then expresses the λ-sequence generating function as a rational function in terms of primary and secondary divisors.

Corollary 2.2. *Fix primary and secondary divisors as in Theorem 2.1. For each $[D] \in \mathrm{Jac}(G)$,*

$$\Lambda_{[D]}(z) = \frac{\sum_{F\in\mathcal{S}_{[D]}} z^{\deg(F)}}{\prod_{v\in V}\left(1 - z^{\ell_v}\right)}.$$

Theorem 2.1 has a direct proof, but in the next two sections we show how the theorem is a consequence of standard results in the theory of lattice points in polyhedra and in invariant theory, respectively.

3 Polyhedra

We now interpret the results of Section 2 in terms of lattice points in polyhedra naturally associated with divisors. We first recall some relevant theory, using [2] as our reference. An *affine n-cone in $\mathbb{R}^n$*, or simply an *n-cone*, is a set of the form

$$\mathcal{K} = \{\, p + \lambda_1\omega_1 + \cdots + \lambda_m\omega_m : \lambda_1, \ldots, \lambda_m \geq 0\}$$

where $\omega_1, \ldots, \omega_m, p \in \mathbb{R}^n$ and the span of the ω_i has dimension n. The ω_i are called *generators* of the cone. Any generator that is not a convex combination of the remaining generators is called an *extreme ray*. The cone is *pointed* if it contains no line, and in that case p is called its *apex*. We say $\mathcal{K}$ is *rational* if $p, \omega_1, \ldots, \omega_m \in \mathbb{Q}^n$, and then, by rescaling, we may assume the ω_i have integer coordinates. An n-cone is *simplicial* if it may be written using n generators. Simplicial cones are necessarily pointed.

Equivalently, we may define a rational pointed n-cone in $\mathbb{R}^n$ to be an n-dimensional intersection of finitely many half-planes of the form

$$\{x \in \mathbb{R}^n : a_1x_1 + \cdots + a_nx_n \geq \beta\}$$

where $a_1, \ldots, a_n, \beta \in \mathbb{Z}$ and such that the hyperplanes

$$\{x \in \mathbb{R}^n : a_1x_1 + \cdots + a_nx_n = \beta\}$$

meet in a single point. In that case, we may express the cone as $\{x \in \mathbb{R}^n : Ax \geq b\}$ where A is an integral $m \times n$ matrix of rank n and $b \in \mathbb{Z}^m$.

If $\mathcal{K}$ is a simplicial n-cone in $\mathbb{R}^n$ with an integral generating set $\Omega = \{\omega_1, \ldots, \omega_n\}$ and apex p, define the *fundamental parallelepiped* for $\mathcal{K}$ with respect to Ω to be

$$\Pi := \{\, p + \textstyle\sum_{i=1}^n \lambda_i\omega_i : 0 \leq \lambda_1, \lambda_2, \ldots, \lambda_n < 1\} \,.$$

Every point $\alpha \in \mathcal{K} \cap \mathbb{Z}^n$ has a unique expression as

$$\alpha = p + \pi + m_1\omega_1 + \cdots + m_n\omega_n$$

with $\pi \in \Pi$ and $m_1, \ldots, m_n \in \mathbb{N}$.

Define the *integer-point transform* of a set $S \subset \mathbb{R}^n$ by

$$\sigma_S(\vec{z}\,) = \sigma_S(z_1, \ldots, z_n) := \sum_{\alpha \in S \cap \mathbb{Z}^n} \vec{z}^{\,\alpha}$$

where $\vec{z}^{\,\alpha} := \prod_{i=1}^n z_i^{\alpha_i}$.

When

$$\mathcal{K} = \{\, p + \lambda_1\omega_1 + \cdots + \lambda_n\omega_n : \lambda_1, \ldots, \lambda_n \geq 0\}$$

is a simplicial n-cone in $\mathbb{R}^n$ with $\omega_1, \ldots, \omega_n \in \mathbb{Z}^n$ and $p \in \mathbb{R}^n$, then

$$\sigma_{\mathcal{K}}(\vec{z}\,) = \frac{\sigma_\Pi(\vec{z}\,)}{\prod_{i=1}^n (1 - \vec{z}^{\,\omega_i})},$$

where Π is the fundamental parallelepiped of $\mathcal{K}$ with respect to the ω_i.

Linear systems and polyhedra. Throughout this section, we fix the embedding

$$\mathbb{R}^{n-1} = \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n.$$

Thus, if $D \in \mathrm{Div}(G) = \mathbb{Z}^n$, then we may regard $D|_{q=0}$ as an element of either $\mathbb{Z}^{n-1}$ or $\mathbb{Z}^n$. Similarly, given $f \in \mathbb{R}^{n-1}$, we write Lf in place of $L\binom{f}{0}$.

Two divisors D and D' on G are linearly equivalent exactly when there is a function $f \in \mathbb{Z}^V$ such that $D' = D + \operatorname{div}(f)$. In this context f is referred to as a *firing script*, and we express the complete linear system for D as

$$|D| = \{E \in \operatorname{Div}(G) : E = D + Lf \geq 0 \text{ for some firing script } f\}.$$

The set of firing scripts appearing above for the complete linear system for D form the polyhedron

$$Q_D := \{f \in \mathbb{R}^n : Lf \geq -D\} \subset \mathbb{R}^n.$$

However, the integer points of Q_D are not in bijection with elements of $|D|$ since L has a non-trivial kernel. The kernel is generated by the all-ones vector $\vec{1}$; so modulo $\ker(L)$, each firing vector $f = (f_1, \ldots, f_n)$ has the unique representative $f - f_n \cdot \vec{1}$ with last coordinate 0, leading us to define

$$P_D := Q_D \cap \{f \in \mathbb{R}^n : f_n = 0\} \subset \mathbb{R}^{n-1}$$

so that $Q_D = P_D + \mathbb{R}\vec{1} \subset \mathbb{R}^n$. It is straightforward to see that the integer points $P_D \cap \mathbb{Z}^{n-1}$ are in bijection with $|D|$:

$$f \in P_D \cap \mathbb{Z}^{n-1} \quad \longleftrightarrow \quad D + Lf \in |D|. \tag{3.1}$$

Since $|D|$ is finite, it follows that the polyhedron P_D is bounded, and hence is a *polytope*. (For a direct proof of boundedness, see [3, Proposition 2.20].)

If $D \sim D'$ with $D' = D + Lf$, then the polyhedra associated with these divisors differ by a translation: $Q_D = Lf + Q_{D'}$, and as discussed above, we may assume $f_n = 0$ to write $P_D = \tilde{L}f + P_{D'}$.

We extend the ideas presented above to interpret the results of Section 2 in terms of counting lattice points in polyhedra.

Definition 3.1. The *cone for a divisor* $D \in \operatorname{Div}^0(G)$ with respect to q is the set

$$\mathcal{K}_D := \{(f, t) \in \mathbb{R}^n \times \mathbb{R} : Lf + tq \geq -D \text{ and } f_n = 0\} \subset \mathbb{R}^{n-1} \times \mathbb{R}.$$

Theorem 3.2. *Let $D \in \operatorname{Div}^0(G)$. Then $\mathcal{K}_D$ is a rational simplicial n-cone with apex $p := \tilde{L}^{-1}(-D|_{q=0}) \in \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$ and has the following properties:*

(1) *The set of integer points of $\mathcal{K}_D$ is in bijection with $\mathbb{E}_{[D]}$ via the mapping*

$$\begin{aligned} \psi_D \colon \mathcal{K}_D \cap \mathbb{Z}^n &\xrightarrow{\sim} \mathbb{E}_{[D]} \\ (f, k) &\mapsto D + kq + Lf. \end{aligned}$$

(2) *The mapping ψ_0 is a bijection between the generating set of integral extreme rays for $\mathcal{K}_D$ and the set of primary divisors for G. Furthermore, ψ_D is a bijection between integer points of the fundamental parallelepiped and secondary divisors of $[D]$ with respect to $\mathcal{P}$.*

(3) *Fix a generating set Ω for $\mathcal{K}_D$ with fundamental parallelepiped Π and corresponding primary and secondary divisors $\mathcal{P} = \psi_0(\Omega)$ and $\mathcal{S}_{[D]} = \psi_D(\Pi \cap \mathbb{Z}^n)$ as in part (2). Then the λ-sequence generating function for $\mathbb{E}_{[D]}$ is*

$$\Lambda_{[D]}(z) = \sigma_{\mathcal{K}}(1,\ldots,1,z) = \frac{\sigma_\Pi(1,\ldots,1,z)}{\prod_{\omega\in\Omega}(1-z^{\deg(\omega)})},$$

where $\deg(\omega)$ is the sum of the coordinates of ω. The numerator and denominator of the expression on the right are the same as those appearing in Corollary 2.2.

4 Invariant theory

The results of Section 2 may also be interpreted in terms of the invariant theory for a representation of the dual group $\mathrm{Jac}^*(G)$. Through this lens, primary and secondary divisors become primary and secondary invariants, and $\Lambda_{[D]}(z)$ is given a substantially different expression as a Molien series.

We recall basic invariant theory for finite groups with [5] as reference. Given a matrix $A \in \mathrm{GL}(\mathbb{C}^n)$ and a polynomial $f \in \mathbb{C}[\mathbf{x}] := \mathbb{C}[x_1,\ldots,x_n]$, define $f \circ A$ by

$$(f\circ A)(x_1,\ldots,x_n) = f(A\vec{x})$$

where $\vec{x}$ is the column vector $[x_1,\ldots,x_n]^t$. Given a finite subgroup Γ of $\mathrm{GL}(\mathbb{C}^n)$ and a character $\chi\colon \Gamma \to \mathbb{C}^\times := \mathbb{C}\setminus\{0\}$, define the *$\chi$-relative invariants of* Γ to be elements of

$$\mathbb{C}[\mathbf{x}]^\Gamma_\chi := \{f \in \mathbb{C}[\mathbf{x}] : f\circ\gamma = \chi(\gamma)f \text{ for all } \gamma\in\Gamma\}.$$

In the case $\chi = \varepsilon$, the trivial character, $\mathbb{C}[\mathbf{x}]^\Gamma := \mathbb{C}[\mathbf{x}]^\Gamma_\varepsilon$ is a subring of $\mathbb{C}[\mathbf{x}]$, graded by degree, called the *invariant subring of* Γ. Its elements are simply called *invariants of* Γ. The invariant subring $\mathbb{C}[\mathbf{x}]^\Gamma$ is generated by the homogeneous polynomials

$$\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} f\circ\gamma$$

as f ranges over all monomials of degree at most $|\Gamma|$. For arbitrary χ, the relative invariants $\mathbb{C}[\mathbf{x}]^\Gamma_\chi$ form a $\mathbb{C}[\mathbf{x}]^\Gamma$-module, generated by homogeneous polynomials of degree at most $|\Gamma|$.

There exist algebraically independent homogeneous invariants $p_1,\ldots,p_n$ such that $\mathbb{C}[\mathbf{x}]^\Gamma$ is a finitely-generated free module over $\mathbb{C}[p_1,\ldots,p_n]$. For any character χ, if $q_1,\ldots,q_t$ are homogeneous polynomials forming a $\mathbb{C}$-basis for $\mathbb{C}[\mathbf{x}]^\Gamma_\chi$ modulo the submodule $\sum_{i=1}^t p_i\mathbb{C}[\mathbf{x}]^\Gamma_\chi$, then

$$\mathbb{C}[\mathbf{x}]^\Gamma_\chi = \bigoplus_{i=1}^t q_i\mathbb{C}[p_1,\ldots,p_n].$$

The p_i are called *primary invariants* and are independent of χ. The q_i are called *secondary (relative) invariants* and depend on χ. The number of secondary invariants, t, also depends on χ in general. Letting t_ε be the number of secondary invariants for the trivial character, we have

$$t_\varepsilon|\Gamma| = \prod_{i=1}^{n} \deg(p_i).$$

The *Hilbert series* for $\mathbb{C}[\mathbf{x}]^\Gamma_\chi$ is

$$\Phi_{\Gamma,\chi}(z) := \sum_{d\geq 0} \dim_{\mathbb{C}}(\mathbb{C}[\mathbf{x}]^\Gamma_{\chi,d})z^d,$$

where $\mathbb{C}[\mathbf{x}]^\Gamma_{\chi,d}$ is the d-th graded piece of $\mathbb{C}[\mathbf{x}]^\Gamma_\chi$. The Hilbert series is also known as the (relative) *Molien series* for Γ due to a theorem of Molien which states that

$$\Phi_{\Gamma,\chi}(z) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} \frac{\overline{\chi(\gamma)}}{\det(I_n - z\gamma)}. \tag{4.1}$$

Linear systems and Molien series. Order the vertices $v_1, \ldots, v_n$ of G, and fix $q = v_n$. To see the relevance of invariant theory to our problem, start with the sequence of projections

$$\begin{array}{ccccc} \mathbb{Z}^n = \mathrm{Div}(G) & \longrightarrow\!\!\!\!\rightarrow & \mathrm{Pic}(G) & \longrightarrow\!\!\!\!\rightarrow & \mathrm{Jac}(G) \\ D & \longmapsto & [D] & \longmapsto & [D - \deg(D)q]\,. \end{array}$$

Apply the functor $\mathrm{Hom}(\,\cdot\,, \mathbb{C}^\times)$ to get a sequence of dual groups

$$\mathrm{Jac}(G)^* \hookrightarrow \mathrm{Pic}(G)^* \hookrightarrow (\mathbb{C}^\times)^n \subset GL(\mathbb{C}^n),$$

identifying $(\mathbb{C}^\times)^n$ with diagonal matrices having nonzero diagonal entries. Define ρ to be the composition of these mappings:

$$\begin{array}{cccc} \rho\colon & \mathrm{Jac}(G)^* & \longrightarrow & \mathrm{GL}(\mathbb{C}^n) \\ & \chi & \mapsto & \mathrm{diag}(\chi([v_1 - q]), \chi([v_2 - q]), \ldots, \chi([v_{n-1} - q]), 1). \end{array}$$

Theorem 4.1. *Consider* $[D] \in \mathrm{Jac}(G)$ *as a character of* $\Gamma := \mathrm{im}(\rho) \subset \mathrm{GL}(\mathbb{C}^n)$ *via* $[D](\rho(\chi)) := \chi([D])$ *for each* $\chi \in \mathrm{Jac}(G)^*$. *Then*

(1)
$$\left\{ x^E := \textstyle\prod_{i=1}^n x_i^{E(v_i)} : E \in \mathbb{E}_{[D]} \right\}$$

is a $\mathbb{C}$*-basis for the relative invariants* $\mathbb{C}[\mathbf{x}]^\Gamma_{[D]}$,

(2) $\mathbb{C}[\mathbf{x}] = \bigoplus_{[D']\in\mathrm{Jac}(G)} \mathbb{C}[\mathbf{x}]^\Gamma_{[D']}$, *and*

(3) *the correspondence $E \mapsto x^E$ for effective divisors E gives a bijection between systems of primary and $[D]$-secondary divisors and systems of monomial primary and $[D]$-relative invariants.*

Computing $\Lambda_{[D]}$ thus becomes an application of Molien's theorem.

Corollary 4.2. *Let $[D] \in \mathrm{Jac}(G)$. The generating function for the λ-sequence for $\mathbb{E}_{[D]}$ is given by the Molien series*

$$\Lambda_{[D]}(z) = \Phi_{\Gamma,[D]}(z) = \frac{1}{|\,\mathrm{Jac}(G)|} \sum_{\chi \in \mathrm{Jac}(G)^*} \frac{\overline{\chi([D])}}{\det(I_n - z\rho(\chi))}.$$

5 Cycle Graphs and Necklaces

We now turn to the special case of cycle graphs. Let C_n be the cycle graph with vertices $v_1, \ldots, v_n$, in order around the cycle, with $q := v_n$. It is well-known that $\mathrm{Jac}(C_n) \simeq \mathbb{Z}/n\mathbb{Z}$, with generator $[v_1 - q]$ and such that $[D_j] := j[v_1 - q] = [v_j - q]$ for $j = 1, \ldots, n$. It follows that $\mathrm{ord}_q(v_i) = n/\gcd(i,n)$ for all i. To apply Theorem 2.1, for convenience take $\ell_{v_i} = n$ for $i = 1, \ldots, n-1$ and $\ell_n = 1$. The primary divisors are then $\mathcal{P} = \{nv_1, nv_2, \ldots, nv_{n-1}, q\}$ and part (4) of Theorem 2.1 may be applied to the Hermite normal form for $\mathrm{im}_{\mathbb{Z}}(\tilde{L})$ to find the secondary divisors for $[D_j] \in \mathrm{Jac}(C_n)$:

$$\mathcal{S}_{[D_j]} = \left\{ (a_1, a_2, \ldots, a_{n-1}, 0) : 0 \le a_i < n \text{ for all } i \text{ and } a_{n-1} = (-j + \textstyle\sum_{i=1}^{n-2} i a_i) \bmod n \right\}.$$

Consider the case where $n = 3$ and $[D] = [D_1] = [(1,0,-1)]$. As elements of $\mathbb{Z}^3 \simeq \mathrm{Div}(C_3)$, the primary divisors are $\mathcal{P} = \{(3,0,0), (0,3,0), (0,0,1)\}$, and the secondary divisors are $\mathcal{S}_{[D]} = \{(1,0,0), (0,2,0), (2,1,0)\}$. By Corollary 2.2, the λ-sequence generating function is

$$\Lambda_{[D]}(z) = \frac{z + z^2 + z^3}{(1-z^3)^2(1-z)} = z + 2z^2 + 3z^3 + 5z^4 + 7z^5 + 9z^6 + 12z^7 + 15z^8 + \ldots$$

See for example the five elements of $|D + 4k|$ corresponding to the term $5z^4$ in Figure 2.

Using Theorem 3.2, we may then compute the cone for D with respect to q:

$$\mathcal{K}_D = \{(-2/3, -1/3, 0) + \lambda_1(2,1,3) + \lambda_2(1,2,3) + \lambda_3(0,0,1) : \lambda_1, \lambda_2, \lambda_3 \ge 0\}.$$

There are three integer points in the fundamental parallelepiped:

$$\Pi \cap \mathbb{Z}^3 = \{(0,0,1), (0,1,2), (1,1,3)\}.$$

Therefore, the integer-point transform of $\mathcal{K}_D$ is

$$\sigma_{\mathcal{K}}(z_1, z_2, z_3) = \frac{z_3 + z_2 z_3^2 + z_1 z_2 z_3^3}{(1 - z_1^2 z_2 z_3^3)(1 - z_1 z_2^2 z_3^3)(1 - z_3)},$$

and we find $\sigma_{\mathcal{K}}(1,1,z) = \Lambda_{[D]}(z)$ in accordance with Theorem 3.2 (3).

Considering that $(0,0,1)$ is an extreme ray, $\mathcal{K}_D$ is determined by its lower face, which we project onto its first two coordinates to define the cone $\widetilde{\mathcal{K}}_D \subset \mathbb{R}^2$. Recalling that the fundamental parallelepiped Π of $\mathcal{K}_D$ is half-open, we see that $\Pi \cap \mathbb{Z}^3$ is in bijection with the integer points of the corresponding fundamental parallelogram $\widetilde{\Pi}$ for $\widetilde{\mathcal{K}}_D$. Figure 1 depicts $\widetilde{\mathcal{K}}_D$ and $\widetilde{\Pi}$. Note the three integer points in $\widetilde{\Pi} \cap \mathbb{Z}^2$ corresponding to the three secondary divisors for $[D]$. The intersection of $\mathcal{K}_D$ with the plane at height k has integer points in bijection with the elements of the complete linear system $|D + kq|$, and its projection into $\mathbb{R}^2$ is the polytope P_{D+kq} defined in Section 3. The case $k = 7$ is illustrated in Figure 1.

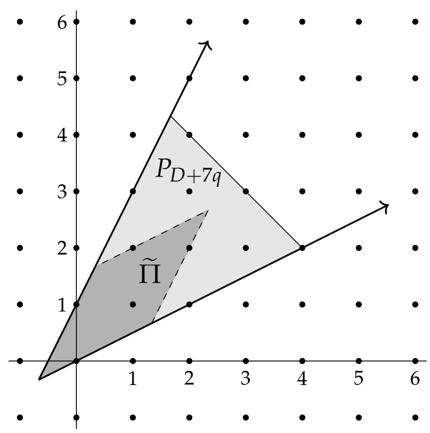

Figure 1: The cone $\widetilde{\mathcal{K}}_D$, its fundamental parallelogram, and the polytope P_{D+7q} for the divisor $D = (1,0,-1)$ on C_3.

Invariant theory. Let ω be a primitive n-th root of unity. Since $\mathrm{Jac}(C_n)$ is the cyclic group of order n generated by $[D_1] = [v_1 - q]$, the dual group $\mathrm{Jac}^*(C_n)$ is also cyclic of order n, generated by the character ψ determined by $\psi([v_1 - q]) = \omega$. As explained in Section 4, we identify ψ with the diagonal matrix $\mathrm{diag}(\omega, \omega^2, \ldots, \omega^{n-1}, 1)$, and by Corollary 4.2,

$$\Lambda_{[D_j]}(z) = \frac{1}{n} \sum_{k=0}^{n-1} \frac{\omega^{-jk}}{\prod_{i=0}^{n-1}(1 - \omega^{ik} z)}$$

for $j = 1, \ldots, n$. In particular,

$$\Lambda_{[0]}(z) = \Lambda_{[D_n]}(z) = \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{(1 - z^{n/\gcd(n,k)})^{\gcd(n,k)}} = \frac{1}{n} \sum_{d|n} \frac{\phi(d)}{(1 - z^d)^{n/d}}$$

where ϕ is the Euler totient function. The formula on the right is well-known from Polya counting theory and yields the first part of the following surprising result.

Theorem 5.1. *On a cycle graph with n vertices, $\lambda_{[0]}(k)$ counts the number of binary necklaces with n black beads and k white beads. More generally, $\lambda_{[D_j]}(k)$ is the number of binary necklaces with n black beads and k white beads and with period divisible by $(n+k)/\gcd(n,k,j)$.*

In the case where n and k are coprime, we can say more. Let $N_{n,k}$ denote the set of necklaces with n black beads and k white beads. Represent a necklace in $N_{n,k}$ by a rotational equivalence class $[(w_1,\ldots,w_k)]$ where each nonnegative integer w_i represents w_i white beads followed by a single black bead running counterclockwise in a cycle. For each j and k, define

$$\begin{aligned}\varphi_{j,k} : |D_j + kq| \to \quad & N_{n,k}\\ E \quad \mapsto \; & [(E(v_1),\ldots,E(v_n))].\end{aligned}$$

Theorem 5.2. *If $\gcd(k,n)=1$, then $\varphi_{j,k}$ is a bijection.*

Figure 2 illustrates all three bijections $\varphi_{1,4}$, $\varphi_{2,4}$, and $\varphi_{3,4}$ for the case $n=3$ and $k=4$ depending upon which vertex is designated as q.

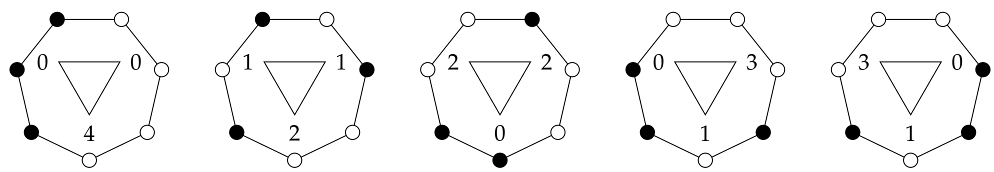

Figure 2: A complete linear system on C_3 and the corresponding necklaces.

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