

Triangulations of the product of spheres with few vertices

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Abstract. A small triangulation of sphere products can be found in lower dimensional cases by computer search and is known in few other cases: Klee and Novik constructed a balanced triangulation of $\mathbb{S}^1 \times \mathbb{S}^{d-2}$ with $3d$ vertices and a centrally symmetric triangulation of $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$ with $2d + 2$ vertices for all $d \geq 3$ and $1 \leq i \leq d - 2$. In this paper, we provide an alternative centrally symmetric $(2d + 2)$ -vertex triangulation of $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$. We also construct the first balanced triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ with $4d$ vertices, using a sphere decomposition inspired by handle theory.

Keywords: sphere products, minimal triangulations of manifolds, balanced complexes, centrally symmetric complexes

1 Introduction

Minimal triangulations of manifolds are an important research object in combinatorial and computational topology. What is the minimal number of vertices required to triangulate a given manifold? How do we construct a vertex-minimal triangulation and is this triangulation unique?

In this paper, we focus on the triangulation of sphere products. From a result of [2], it is known that a combinatorial triangulation of $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$ has at least $2d - i + 2$ vertices. In 1986, [7] constructed a triangulation of $\mathbb{S}^1 \times \mathbb{S}^{d-2}$ with $2d + 1$ vertices for odd d . Later, two groups of researchers, [1] as well as [3], found in 2008 that Kühnel's construction is indeed the unique minimal triangulation for odd d . For even d , they showed that the minimal triangulation requires $2d + 2$ vertices and is not unique.

The minimal triangulations of other sphere products are less well-understood. The best general result is from [4], where a centrally symmetric triangulation of $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$ with $2d + 2$ vertices is constructed as a subcomplex of the boundary of the $(d + 1)$ -cross-polytope. In general, a result of [2] states that a triangulation of $\mathbb{S}^i \times \mathbb{S}^j$ requires at least $i + 2j + 4$ vertices for $i \geq j$. In addition, the minimal triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ for $d \leq 6$ as well as the minimal triangulation of $\mathbb{S}^3 \times \mathbb{S}^3$ are found by the computer

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program BISTELLAR [8], which has shown that this lower bound is not always tight, as a triangulation of $S^2 \times S^2$ requires at least 11 vertices. In this paper, we give an alternative centrally symmetric $(2d + 2)$ -vertex triangulation of $S^2 \times S^{d-3}$ for all $d \geq 5$. The construction is based on finding two shellable balls in the d -sphere whose intersection triangulates $S^1 \times \mathbb{D}^{d-2}$, where \mathbb{D}^{d-2} is the $(d - 2)$ -dimensional disk. By an inductive argument, we also obtain the triangulation of other sphere products in higher dimensions, see [Section 3.2](#).

In recent years, balanced triangulated manifolds have caught much attention. A $(d - 1)$ -dimensional simplicial complex is *balanced* provided that its graph is d -colorable. Many important classes of complexes arise as balanced complexes, such as barycentric subdivisions of regular CW complexes and Coxeter complexes. As taking barycentric subdivisions of a complex would generate a lot of new vertices, one would ask if there is a more efficient way to construct the balanced triangulated manifold from a non-balanced one.

In much of the same spirit as Kühnel's construction, [5] provided a balanced triangulation of $S^1 \times S^{d-2}$ with $3d$ vertices for odd d and with $3d + 2$ vertices otherwise. Furthermore, [10] showed that the number of vertices for the minimal triangulation is indeed $3d$ for odd d and $3d + 2$ otherwise. However, as of yet, no small balanced triangulations of $S^i \times S^{d-i-1}$ for $2 \leq i \leq d - 3$ exist in literature. In this paper, we construct the first balanced triangulation of $S^2 \times S^{d-3}$ with $4d$ vertices. The construction uses a sphere decomposition inspired by handle theory.

The extended abstract is structured as follows. In [Section 2](#), we review the basics of simplicial complexes, balanced triangulations, and other relevant definitions. In [Section 3](#), we present our centrally symmetric $(2d + 2)$ -vertex triangulation of $S^2 \times S^{d-3}$ and construct other sphere products inductively. In [Section 4](#), the balanced triangulation of $S^2 \times S^{d-3}$ with $4d$ vertices is constructed, followed by a discussion of its properties.

2 Preliminaries

A *simplicial complex* Δ with vertex set V is a collection of subsets $\sigma \subseteq V$, called *faces*, that is closed under inclusion, such that for every $v \in V$, $\{v\} \in \Delta$. For $\sigma \in \Delta$, let $\dim \sigma := |\sigma| - 1$ and define the *dimension* of Δ , $\dim \Delta$, as the maximum dimension of the faces of Δ . A face $\sigma \in \Delta$ is said to be a *facet* provided that it is a face which is maximal with respect to inclusion. We say that a simplicial complex Δ is *pure* if all of its facets have the same dimension. If Δ is $(d - 1)$ -dimensional and $-1 \leq i \leq d - 1$, then the *f-number* $f_i = f_i(\Delta)$ denotes the number of i -dimensional faces of Δ . The *star* and *link* of a face σ in Δ is defined as follows:

$$\text{st}(\sigma, \Delta) := \{\tau \in \Delta : \sigma \cup \tau \in \Delta\}, \quad \text{lk}(\sigma, \Delta) := \{\tau \in \text{st}_\Delta \sigma : \tau \cap \sigma = \emptyset\}.$$

When the context is clear, we may simply denote the star and link of σ as $\text{st}(\sigma)$ and $\text{lk}(\sigma)$ respectively. We also define the *restriction* of Δ to a vertex set W as $\Delta[W] := \{\sigma \in \Delta : \sigma \subseteq W\}$. A subcomplex $\Omega \subset \Delta$ is said to be *induced* provided that for all faces $F \in \Delta$, if every vertex $v \in F$ is a vertex of Ω , then F is a face in Ω . The *i-skeleton* of a simplicial complex Δ is the subcomplex containing all faces of Δ which have dimension at most i . In particular, the 1-skeleton of Δ is the graph of Δ .

Denote by σ^d the d -simplex. A *combinatorial $(d-1)$ -sphere* (respectively, a *combinatorial $(d-1)$ -ball*) is a simplicial complex PL homeomorphic to $\partial\sigma^d$ (respectively, σ^{d-1}). A closed *combinatorial $(d-1)$ -manifold* is a connected simplicial complex with the property that the link of each vertex is a combinatorial $(d-2)$ -sphere. A simplicial complex Δ is a *simplicial manifold*, if the geometric realization of Δ is homeomorphic to a manifold. The boundary complex of a simplicial d -ball is a simplicial $(d-1)$ -sphere. In general, a simplicial manifold need not be combinatorial.

A $(d-1)$ -dimensional simplicial complex Δ is called *balanced* if the graph of Δ is d -colorable; that is, there exists a coloring map $\kappa : V \rightarrow \{1, 2, \dots, d\}$ such that $\kappa(x) \neq \kappa(y)$ for all edges $\{x, y\} \in \Delta$. A simplicial complex is *centrally symmetric* or *cs* if it is endowed with a free involution $\alpha : V(\Delta) \rightarrow V(\Delta)$ that induces a free involution on the set of all non-empty faces.

Let ∂C_d^* be the boundary complex of the d -cross-polytope. It is cs, and furthermore, it is a balanced vertex-minimal triangulation of the $(d-1)$ -sphere. Label the vertex set of ∂C_d^* as $\{x_1, \dots, x_d, y_1, \dots, y_d\}$ such that x_i, y_i form a pair of antipodal vertices for all i . Every facet of ∂C_d^* can be written in the form $u_1 u_2 \dots u_d$, where each $u_i \in \{x_i, y_i\}$. We say a facet has a switch at position i if u_i and u_{i+1} have different labels. Let $B(i, d)$ be the pure subcomplex of ∂C_d^* that contains all facets with at most i switches. For example, $B(0, d)$ consists of the two disjoint facets $\{x_1, \dots, x_d\}$ and $\{y_1, \dots, y_d\}$. If Γ is a subcomplex of ∂C_d^* , we let the *complement* of Γ in ∂C_d^* be the complex generated by those facets that are not in Γ . Denote by \mathcal{D}_d the dihedral group of order $2d$.

The following lemma is essentially Theorem 1.2 in [4].

Lemma 2.1. *For $0 \leq i < d-1$, the complex $B(i, d)$ satisfies the following properties:*

1. $B(i, d)$ contains the entire i -skeleton of ∂C_d^* as a subcomplex.
2. The boundary of $B(i, d)$ is homeomorphic to $\mathbb{S}^i \times \mathbb{S}^{d-i-2}$.
3. $B(i, d)$ is a balanced centrally symmetric combinatorial manifold whose integral (co)homology groups coincide with those of \mathbb{S}^i . Also, $B(0, d) \cong \mathbb{D}^{d-1} \times \mathbb{S}^0$ and $B(1, d) \cong \mathbb{D}^{d-2} \times \mathbb{S}^1$.
4. The complement of $B(i, d)$ in ∂C_d^* is simplicially isomorphic to $B(d-i-2, d)$.
5. $B(i, d)$ admits a vertex-transitive action of $\mathbb{Z}_2 \times \mathcal{D}_d$ if i is even and of \mathcal{D}_{2d} if i is odd.

Finally, we define shellability.

Definition 2.2. Let Δ be a pure d -dimensional simplicial complex. A *shelling* of Δ is a linear ordering of the facets F_1, F_2, \dots, F_s such that $F_k \cap (\cup_{i=1}^{k-1} F_i)$ is a pure $(d-1)$ -dimensional complex for all $2 \leq k \leq s$, and Δ is called *shellable* if it has a shelling.

3 The cs triangulations of the sphere products

It is known that for $i \leq j$, the minimal triangulation of $\mathbb{S}^i \times \mathbb{S}^j$ requires at least $i + 2j + 4$ vertices, see [2]. Such triangulations are constructed by [8] in lower dimensional cases but not known in general. We aim at finding an alternative triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ with $2d + 2$ vertices for $d \geq 5$. The following theorem is Theorem 7 in [6].

Theorem 3.1. *Let M be a simply connected codimension-1 submanifold of \mathbb{S}^d , where $d \geq 5$. If M has the homology of $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$ and $1 < i \leq \frac{d-1}{2}$, then M is homeomorphic to $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$.*

Proposition 3.2. *Fix d and $i \leq \frac{d-1}{2}$. Let D_1 and D_2 be two combinatorial d -balls such that*

1. $\partial(D_1 \cup D_2)$ is $(d-1)$ -dimensional submanifold of a combinatorial d -sphere.
2. $D_1 \cap D_2 = \partial D_1 \cap \partial D_2$ is a path-connected combinatorial $(d-1)$ -manifold (with boundary) that has the same homology as \mathbb{S}^{i-1} .
3. $\partial(D_1 \cap D_2)$ has the same homology as $\mathbb{S}^{i-1} \times \mathbb{S}^{d-i-1}$.

Then $\partial(D_1 \cup D_2)$ triangulates $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$ for $d \geq 5$.

Proof: First note that $D_1 \cup D_2$ is the union of two combinatorial d -balls that intersect along the combinatorial $(d-1)$ -manifold $D_1 \cap D_2$. Hence $D_1 \cup D_2$ is a combinatorial d -manifold, and $\partial(D_1 \cup D_2)$ is a combinatorial $(d-1)$ -manifold.

Since $D_1 \cap D_2 = \partial D_1 \cap \partial D_2$, we have that the intersection of $\overline{\partial D_1 \setminus \partial D_2}$ and $D_1 \cap D_2$ is exactly $\partial(D_1 \cap D_2)$. We apply the Mayer-Vietoris sequence on $(\overline{\partial D_1 \setminus \partial D_2}, D_1 \cap D_2, \partial D_1)$, and by condition (2) we obtain that $\overline{\partial D_1 \setminus \partial D_2}$ has the same homology as \mathbb{S}^{d-i-1} .

Note that the intersection of

$$(\overline{\partial D_1 \setminus \partial D_2}) \cap (\overline{\partial D_2 \setminus \partial D_1}) = \partial(\partial D_1 \cap \partial D_2) = \partial(D_1 \cap D_2).$$

We then apply the Mayer-Vietoris sequence to $(\overline{\partial D_1 \setminus \partial D_2}, \overline{\partial D_2 \setminus \partial D_1}, \partial(D_1 \cup D_2))$. By condition (3), we find that $\partial(D_1 \cup D_2)$ has the same homology as $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$.

Finally, the complex $D_1 \cup D_2$ is simply connected, since the union of two simply connected open subsets $\text{int} D_1, \text{int} D_2$ with path-connected intersection $D_1 \cap D_2$ is simply connected. We conclude from condition (1) and **Theorem 3.1** that $\partial(D_1 \cup D_2)$ triangulates $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$. \square

The above proposition provides us with a general method of constructing a triangulation of $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$.

3.1 A triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$

Let τ be a face of ∂C_d^* and let $\kappa(\tau)$ count the number of y labels in τ . Define Γ_j to be the union of facets τ in ∂C_d^* that have at most 2 switches and with $\kappa(\tau) = j$. Hence for $1 \leq j \leq d-1$, the complex Γ_j consists of d facets $\tau_j^k = \{x_1, \dots, x_d\} \setminus \{x_k, \dots, x_{k+j-1}\} \cup \{y_k, \dots, y_{k+j-1}\}$ for $1 \leq k \leq d$.

Lemma 3.3. *The complex $\cup_{k=0}^i \Gamma_k$ is a shellable $(d-1)$ -ball for all $0 \leq i \leq \lceil \frac{d+1}{2} \rceil$.*

Proof: The $(d-1)$ -ball $\cup_{k=0}^i \Gamma_k$ has a shelling order $\{x_1, \dots, x_d\}, \tau_1^1, \dots, \tau_1^d, \dots, \tau_i^1, \dots, \tau_i^d$. \square

We propose the candidates $D_1, D_2 \subseteq \partial C_{d+1}^*$ that satisfy the conditions in [Proposition 3.2](#).

Construction 3.4. For $d \geq 3$, define two simplicial d -balls D_1, D_2 as a subcomplex of the octahedral d -sphere on vertex set $\{x_1, y_1, \dots, x_{d+1}, y_{d+1}\}$ as follows:

1. For d is odd, let $m = \frac{d-1}{2}$. Define $D_1 = (\cup_{k=0}^{m+1} \Gamma_k) * \{x_{d+1}\}$ and $D_2 = (\cup_{k=m}^d \Gamma_k) * \{y_{d+1}\}$. In particular, $D_1 \cap D_2 = \Gamma_m \cup \Gamma_{m+1}$ is cs.
2. For d is even, let $m = \frac{d}{2}$ and $\gamma := \cup_{i=1}^m \tau_{m-1}^i$ be a subcomplex of Γ_{m-1} . By the definition, τ_j^k and τ_{d-j}^{k+j} are antipodal facets for any k, j . So $-\gamma = \cup_{i=m}^{d-1} \tau_{m+1}^i \subseteq \Gamma_{m+1}$. In this case we let

$$D_1 = ((\cup_{k=0}^m \Gamma_k) \cup (-\gamma)) * \{x_{d+1}\}, \quad D_2 = ((\cup_{k=m}^d \Gamma_k) \cup \gamma) * \{y_{d+1}\}.$$

In particular, $D_1 \cap D_2 = \Gamma_m \cup \gamma \cup (-\gamma)$ is centrally symmetric.

Next we show that $\partial(D_1 \cap D_2) \cong \mathbb{S}^1 \times \mathbb{S}^{d-3}$. Given two facets $F_1, F_2 \in \partial C_d^*$, let $d(F_1, F_2)$ be the distance from F_1 to F_2 in the facet-ridge graph of ∂C_d^* .

Lemma 3.5. *Let Δ be a combinatorial $(d-1)$ -manifold in ∂C_d^* whose facet-ridge graph is a $2d$ -cycle. Enumerate its facets as $\sigma_1, \sigma_2, \dots, \sigma_{2d}$ such that σ_i, σ_{i+1} are adjacent for $1 \leq i \leq 2d$. If $\sigma_i = -\sigma_{d+i}$ for all i , then Δ triangulates $\mathbb{S}^1 \times \mathbb{D}^{d-2}$.*

Proof: Let $\sigma_1 = \{u_1, \dots, u_d\}$. By the assumption, $\sigma_{d+1} = \{-u_1, \dots, -u_d\}$. Since $d(\sigma_1, \sigma_{d+1}) = d$ in ∂C_d^* , the sequence $\sigma_1, \sigma_2, \dots, \sigma_{d+1}$ gives the shortest path from σ_1 to σ_{d+1} . So it follows that there is an ordering of the vertices, say (u_1, \dots, u_d) , such that $\sigma_{i+1} = \sigma_i \setminus \{u_i\} \cup \{-u_i\}$. Together with $\sigma_i = -\sigma_{i+d}$ for all i , we see that $\Delta \cong B(1, d)$ as defined in [4]. Hence as $B(1, d)$, Δ also triangulates $\mathbb{S}^1 \times \mathbb{D}^{d-2}$. \square

Lemma 3.6. *The complex $D_1 \cap D_2$ constructed above triangulates $\mathbb{S}^1 \times \mathbb{D}^{d-2}$.*

Proof: For odd d and $m = \frac{d-1}{2}$, we enumerate the facets of $D_1 \cap D_2 = \Gamma_m \cup \Gamma_{m+1}$ as $(\sigma_1, \dots, \sigma_{2d}) :=$

$$(\tau_m^1, \tau_{m+1}^1, \tau_m^2, \tau_{m+1}^2, \dots, \tau_m^d, \tau_{m+1}^d).$$

Each σ_i has exactly two adjacent facets $\sigma_{i-1}, \sigma_{i+1}$, and so the facet-ridge graph of $D_1 \cup D_2$ is a $2d$ -cycle. Furthermore, $\tau_m^j = -\tau_{m+1}^{j+m}$ by the definition. So the claim follows from [Lemma 3.5](#). The proof is similar for d is even. \square

Theorem 3.7. *The complex $\partial(D_1 \cup D_2)$ triangulates $S^2 \times S^{d-3}$ for $d \geq 5$.*

Proof: This follows from [Lemmas 3.3](#) and [3.6](#) and [Proposition 3.2](#). \square

Property 3.8. *For the complex $\partial(D_1 \cup D_2)$ in [Construction 3.4](#):*

1. *It has $2d + 2$ vertices.*
2. *It contains the 2-skeleton of ∂C_{d+1}^* .*
3. *It admits vertex-transitive actions by the group $\mathbb{Z}_2 \times \mathcal{D}_d$ if d is odd, and by \mathbb{Z}_2 if d is even.*

Remark 3.9. For d is odd, three types of vertex-transitive action on $\partial(D_1 \cup D_2)$ are given by

- D maps x_j to y_j , and y_j to x_j , for $1 \leq j \leq d + 1$.
- R fixes x_{d+1}, y_{d+1} , and maps x_j, y_j to x_{d-j+1}, y_{d-j+1} respectively, for $1 \leq j \leq d$.
- S fixes x_{d+1}, y_{d+1} , and maps x_j, y_j to x_{j+1}, y_{j+1} (modulo d) respectively.

Comparing with the group actions on $B(2, d + 1)$ in [\[4\]](#), we see that $\partial(D_1 \cup D_2)$ and $\partial B(2, d + 1)$ are combinatorially distinct.

Remark 3.10. There are many other ways to construct D'_1, D'_2 as the subcomplex of ∂C_{d+1}^* that satisfies the conditions in [Proposition 3.2](#). For example, when $d = 2m + 1$, let $\tau = \{x_1, \dots, x_d\}$, $-\tau = \{y_1, \dots, y_d\}$. It is possible to construct a simplicial $(d - 1)$ -ball B in ∂C_d^* and simplicial d -balls

$$D'_1 = (B \cup \Gamma_m \cup \Gamma_{m+1}) * \{x_{d+1}\}, \quad D'_2 = ((-B) \cup \Gamma_m \cup \Gamma_{m+1}) * \{y_{d+1}\}$$

such that

$$B \cup (-B) = \partial C_d^*, \quad B \supseteq \{\sigma : |\sigma \cap \tau| \geq m\}, \quad -B = \{\sigma : |\sigma \cap (-\tau)| \geq m\}.$$

Furthermore, $D'_1 \cap D'_2 = \Gamma_m \cup \Gamma_{m+1}$. When $d = 2m$, D'_1, D'_2 can also be defined in the same spirit as [Construction 3.4](#).

3.2 The triangulation of other sphere products

The goal of this section is to construct a triangulation of $S^i \times S^{d-i-1}$ as a subcomplex of ∂C_{d+2}^* from a given triangulation of $S^{i-1} \times S^{d-i-1}$ in ∂C_{d+1}^* , for $i \leq \frac{d-1}{2}$.

Proposition 3.11. *Let D_1 and D_2 be cones over two combinatorial $(d-1)$ -balls in ∂C_d^* whose coning points are x_{d+1}, y_{d+1} respectively. Furthermore,*

1. *The union of $D_1 \setminus \{x_{d+1}\}$ and $D_2 \setminus \{y_{d+1}\}$ covers ∂C_d^* .*
2. *$D_1 \cap D_2$ is a path-connected combinatorial $(d-1)$ -manifold that has the same homology as S^{i-1} for some $2 \leq i \leq d-2$.*
3. *$\partial(D_1 \cap D_2)$ has the same homology as $S^{i-1} \times S^{d-i-1}$.*

Let

$$E_1 = (\text{st}(y_{d+1}, \partial C_{d+1}^*) \cup D_1) * \{x_{d+2}\}, \quad E_2 = (\text{st}(x_{d+1}, \partial C_{d+1}^*) \cup D_2) * \{y_{d+2}\}.$$

Then the union of $E_1 \setminus \{x_{d+2}\}$ and $E_2 \setminus \{y_{d+2}\}$ covers ∂C_{d+1}^* , $E_1 \cap E_2$ is a combinatorial d -manifold that has the same homology as S^i and $\partial(E_1 \cup E_2)$ triangulates $S^{i+1} \times S^{d-i-1}$.

Proof: By condition (1), $E_1 \cap E_2 = D_1 \cup D_2$. Then we use the Mayer-Vietoris sequence on the triple $(D_1, D_2, D_1 \cup D_2)$ and conclude from [Theorem 3.1](#). The proof is similar to that of [Proposition 3.2](#). \square

Construction 3.12. We take our base construction $\partial(D'_1 \cup D'_2)$ as given in [Remark 3.10](#) (Note that the complexes D_1, D_2 in [Construction 3.4](#) does not satisfy the condition that the union of $D_1 \setminus \{x_{d+1}\}$ and $D_2 \setminus \{y_{d+1}\}$ covers ∂C_d^* .) and apply the above proposition inductively. This gives us a family of cs triangulations of sphere products $S^2 \times S^{d-3}$, $S^3 \times S^{d-3}, \dots, S^{d-3} \times S^{d-3}$. Each triangulation of $S^i \times S^{d-3}$ has $2d + 2i - 2$ vertices.

4 A balanced triangulation of $S^2 \times S^{d-3}$

In this section, we present our main construction for a balanced triangulation of $S^2 \times S^{d-3}$. The geometric intuition of our construction comes from handle theory. The sphere S^d admits the following decomposition, see [\[9\]](#):

$$S^{d-1} = (S^1 \times \mathbb{D}^{d-2}) \cup (\mathbb{D}^2 \times S^{d-3}).$$

Let S be a triangulated $(d-1)$ -sphere that has the decomposition $S = B_1 \cup_{\partial B_1 = \partial B_2} B_2$, where $B_1 \cong S^1 \times \mathbb{D}^{d-2}$, $B_2 \cong \mathbb{D}^2 \times S^{d-3}$, and $\partial B_1 \cong \partial B_2 \cong S^1 \times S^{d-3}$. Note that $S^2 \times S^{d-3}$ admits the decomposition into $(\mathbb{D}^2 \times S^{d-3}) \cup (\mathbb{D}^2 \times S^{d-3}) \cong B_2 \cup B_2$. Then, from S we

can form a triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-2}$ in the following way: take two copies of B_2 and denote them as B_2 and B'_2 . If ∂B_2 is an induced subcomplex in B_2 , then we glue B_2 and B'_2 along their boundaries. The resulting complex is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^{d-3}$. However, if ∂B_2 is not an induced subcomplex of B_2 , then usually we cannot glue B_2 and B'_2 by identifying their boundaries directly and still obtain a triangulated manifold. An alternative method is to find a complex $N \cong \partial B_2 \times \mathbb{D}^1$ with $\partial N = \partial B_2 \cup \partial B'_2$ so that N serves as a tubular neighborhood of both ∂B_2 and $\partial B'_2$. Finally the complex $B_2 \cup N \cup B'_2$ is a triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$.

Our approach of constructing a balanced triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ is by finding suitable balanced candidates of B_2 and N as described above.

Definition 4.1. Consider (Γ_1, σ_1) and (Γ_2, σ_2) , where Γ_i is the boundary complex of the d -cross-polytope, and σ_i is a fixed facet of Γ_i . Let κ be the coloring map on $\Gamma_1 \cup \Gamma_2$. If e_i is an edge in Γ_i but not in $\pm\sigma_i$ and $\kappa(e_1) = \kappa(e_2)$, then the \diamond -connected sum $(\Gamma_1 \# \Gamma_2, \sigma_1 \# \sigma_2)$ is obtained by deleting e_i from Γ_i , and gluing $\Gamma_1 - e_1$ with $\Gamma_2 - e_2$ by identifying $\text{st}(e_1)[V(\sigma_1)]$ with $\text{st}(e_2)[V(\sigma_2)]$, and $\text{st}(e_1)[V(-\sigma_1)]$ with $\text{st}(e_2)[V(-\sigma_2)]$.

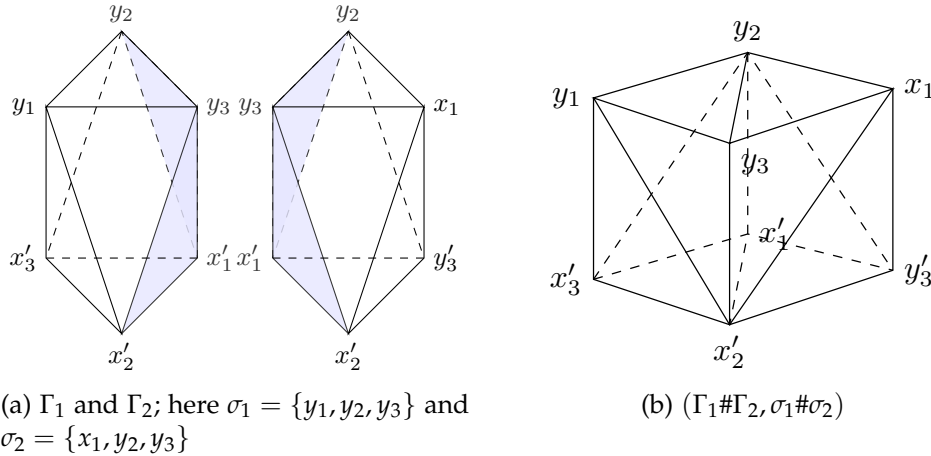


Figure 1: The \diamond -connected sum $(\Gamma_1 \# \Gamma_2, \sigma_1 \# \sigma_2)$: delete the edge $\{y_3, x'_1\}$ in both Γ_1 and Γ_2 , then glue Γ_1 and Γ_2 along the 4-cycle (y_3, x'_2, x'_1, y_2) .

The following properties of the \diamond -connected sum justify the notation $(\Gamma_1 \# \Gamma_2, \sigma_1 \# \sigma_2)$ in the definition.

Property 4.2. Let Γ_1 and Γ_2 be two d -crosspolytopes. Furthermore, Γ_1 has antipodal facets $\sigma_1 = \{x_1, \dots, x_d\}$, $-\sigma_1 = \{y_1, \dots, y_d\}$, and Γ_2 has antipodal facets $\sigma_2 = \{x_{d+1}, \dots, x_{2d}\}$, $-\sigma_2 = \{y_{d+1}, \dots, y_{2d}\}$. Then $(\Gamma_1 \# \Gamma_2, \sigma_1 \# \sigma_2)$ satisfies the following properties:

1. The complex is a balanced triangulation of \mathbb{S}^{d-1} .

2. The restriction of $(\Gamma_1 \# \Gamma_2, \sigma_1 \# \sigma_2)$ to $V(\sigma_1) \cup V(\sigma_2)$ is the usual connected sum of simplices $\sigma_1 \# \sigma_2$.
3. The link of every edge $e = \{x_i, y_j\}$ in $(\Gamma_1 \# \Gamma_2, \sigma_1 \# \sigma_2)$ is the boundary complex of a $(d-2)$ -crosspolytope.

The above properties ensure that it is possible to take the \diamond -connected sum inductively. Also recall that if Γ is a pure simplicial complex, then as long as there exist two facets F and F' on Γ and a map $\phi : F \rightarrow F'$ so that v and $\phi(v)$ do not have a common neighbor for every $v \in F$, then we can remove F, F' and identify ∂F with $\partial F'$ to get Γ^ϕ . This is called a *handle addition*. Similarly, assume that there are two edges e_1 and e_2 of the same color in $(\Gamma_1 \# \dots \# \Gamma_k, \sigma_1 \# \dots \# \sigma_k)$ but not in $A := \sigma_1 \# \sigma_2 \dots \# \sigma_k$ or $-A$. Note that $\text{st}(e_i)$ is a cross-polytope with antipodal facets $\text{st}(e_i)[V(A)]$ and $\text{st}(e_i)[V(-A)]$ for $i = 1, 2$. If the identification maps

$$\phi : \text{st}(e_1)[V(A)] \rightarrow \text{st}(e_2)[V(A)] \quad \text{and} \quad \phi' : \text{st}(e_1)[V(-A)] \rightarrow \text{st}(e_2)[V(-A)]$$

are well-defined, then the maps ϕ and ϕ' naturally extend to a map

$$\bar{\phi} : \text{st}(e_1) \rightarrow \text{st}(e_2)$$

if for every $v \in \text{st}(e_1)$, v and $\phi(v)$ (or $\phi'(v)$) do not have a common neighbor. In this way we obtain a balanced simplicial complex $((\Gamma_1 \# \Gamma_2 \dots \# \Gamma_k)^\phi, (\sigma_1 \# \sigma_2 \dots \# \sigma_k)^\phi)$ by removing e_1, e_2 and identifying $\text{lk}(e_1)$ with $\bar{\phi}(\text{lk}(e_1)) = \text{lk}(e_2)$. We call this the \diamond -handle addition.

We are now ready to construct a balanced triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ with $4d$ vertices. We will write $\Gamma_1 \# \Gamma_2$ to denote the \diamond -connected sum if σ_1 and σ_2 are clear from the context. Also, to simplify notation, we will sometimes write $x_1 \dots x_m$ to denote the face $\{x_1, \dots, x_m\}$.

Construction 4.3. Let $d \geq 3$. Take two d -crosspolytopes P and P' . The vertex sets of P and P' are $\{x_1, \dots, x_d, y_1, \dots, y_d\}$ and $\{x'_1, \dots, x'_d, y'_1, \dots, y'_d\}$ respectively. We let $\sigma_i = x_1 \dots x_i y_{i+1} \dots y_d$ for $1 \leq i \leq d$ and let $\sigma_i = y_1 \dots y_i x_{i+1} \dots x_d$ for $d+1 \leq i \leq 2d$. Then the complex $\Delta_1 := \cup_{i=1}^{2d} \sigma_i$ is exactly $B(1, d)$. We further partition the boundary of P as $\partial P = \Delta_1 \cup_{\partial \Delta_1} \Delta_2$. By Lemma 1.1, $\Delta_2 \cong B(d-3, d)$ and $\Delta_1 \cap \Delta_2$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^{d-3}$.

Next, define a simplicial map $f : \partial P \rightarrow \partial P'$ induced by the following bijection on the vertex sets:

$$x_i \mapsto x'_{i+1}, \quad y_i \mapsto y'_{i+1} \quad \text{for } 1 \leq i \leq d-1; \quad x_d \mapsto y'_1, \quad y_d \mapsto x'_1.$$

By Lemma 2.1, the complex Δ_1 admits a vertex-transitive action by the dihedral group \mathcal{D}_{2d} of order $4d$, where a generator is given by the map we have chosen (Theorem 1.2 of [4]). Hence f is a simplicial isomorphism and $f(\Delta_1) \cong B(1, d)$. For each i , there

is a unique d -cross-polytope Γ_i containing σ_i and $f(\sigma_i)$ as antipodal facets. Next, we check that we can take the \diamond -connected sum of Γ_i and Γ_{i+1} inductively. Without loss of generality, assume that $1 \leq i \leq d$; otherwise, we can relabel by switching x and y . Note that for $i \leq d - 2$,

$$\sigma_i \cap \sigma_{i+1} = x_1 x_2 \dots x_i y_{i+2} \dots y_d, \text{ and } f(\sigma_i) \cap f(\sigma_{i+1}) = x'_2 x'_3 \dots x'_{i+1} y'_{i+3} \dots y'_d y'_1.$$

The missing indices are $i + 1$ and $i + 2$ respectively, so we let $e_i = x'_{i+1} y_{i+2}$. It follows that $\Gamma_i \cap \Gamma_{i+1} = \text{st}(e_i, \Gamma_i) = \text{st}(e_i, \Gamma_{i+1})$ and hence the \diamond -connected sum is well defined. Similarly, $\Gamma_{d-1} \cap \Gamma_d = \text{st}(\{x'_d, x_1\}, \Gamma_d)$ and $\Gamma_d \cap \Gamma_{d+1} = \text{st}(\{y'_1, x_2\}, \Gamma_d)$. Inductively, we form a complex $\Gamma = ((\Gamma_1 \# \Gamma_2 \dots \# \Gamma_{2d})^\phi, \Delta_1)$ which contains Δ_1 and $f(\Delta_1)$ as subcomplexes.

We partition Γ as $\Gamma = \Delta_1 \cup f(\Delta_1) \cup N$, so that $N \cap \Delta_1 = \partial \Delta_1$ and $N \cap f(\Delta_1) = \partial f(\Delta_1)$. N is then the tubular neighborhood that we would like to construct. Finally, let $\Sigma = \Delta_2 \cup_{\partial \Delta_1} N \cup_{\partial f(\Delta_1)} f(\Delta_2)$. (This is well defined as by [Lemma 2.1](#), $\partial \Delta_1 \cong \partial \Delta_2$.) As shown in [Figure 2](#), when $d = 3$, σ gives the minimal balanced triangulation of $\mathbb{S}^0 \times \mathbb{S}^2$.

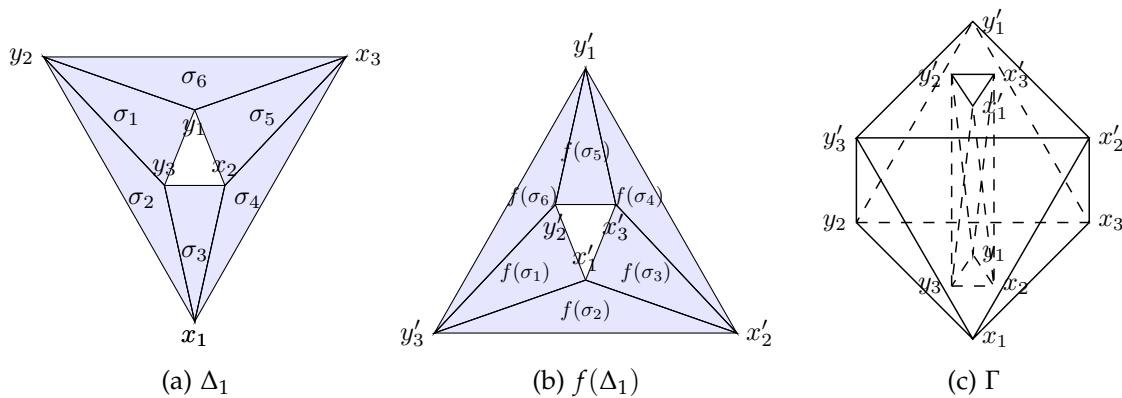


Figure 2: The complexes Δ_1 and $f(\Delta_1)$ when $d = 3$, and the resulting Γ constructed using the previously described sequence of connected sums.

Next to prove Σ indeed triangulates $\mathbb{S}^2 \times \mathbb{S}^{d-3}$, we check that Σ satisfies all the conditions as described in [Theorem 3.1](#).

Lemma 4.4. *The complex Σ in [Construction 4.3](#) is simply connected codimension-1 submanifold of \mathbb{S}^d for $d \geq 5$.*

Proposition 4.5. *The complex Σ in [Construction 4.3](#) is a balanced triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ for $d \geq 3$.*

Proof: Applying the Mayer-Vietoris sequence to the triple $(\Delta_2 \cup f(\Delta_2), N, \Sigma)$, we find that Σ has the same homology as $\mathbb{S}^2 \times \mathbb{S}^{d-3}$, and so the result follows by [Lemma 4.4](#) and [Theorem 3.1](#). \square

Property 4.6. For $d \geq 5$, the complex Σ in [Construction 4.3](#) satisfies:

1. $f_0(\Sigma) = 4d$;
2. $f_1(\Sigma) = 4d(2d - 3)$;
3. $f_{d-1}(\Sigma) = (d + 2)2^d - 8d$;
4. $\text{Aut}(\Sigma)$ admits a vertex-transitive action of $\mathbb{Z}_2 \times \mathcal{D}_{2d}$.

Remark 4.7. Working with Lorenzo Venturello, we created a CROSSFLIP program for balanced complexes to attempt to reduce the number of vertices of a given triangulation. However, the complexity of finding shellable subcomplexes in the d -cross-polytope grows exponentially with d , and so the program is highly inefficient for $d > 4$. [\[4\]](#) showed that a balanced triangulation of a non-sphere $(d - 1)$ -manifold requires at least $3d$ vertices. It is not known that apart from the sphere bundle over the circle, if there are other manifolds that admit balanced triangulations with $3d$ vertices.

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