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Signed Mahonian Identities on Permutations with Subsequence Restrictions

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Abstract. In this paper, we present a number of results surrounding Caselli's conjecture on the equidistribution of the major index with sign over the two subsets of permutations of $\{1, 2, \ldots, n\}$ containing respectively the word $1\,2 \cdots k$ and the word $(n-k+1) \cdots n$ as a subsequence, under a parity condition of n and k. We derive broader bijective results on permutations containing varied subsequences. As a consequence, we obtain the signed mahonian identities on families of restricted permutations, in the spirit of a well-known formula of Gessel and Simion, covering a combinatorial proof of Caselli's conjecture. We also derive an extension of the insertion lemma of Haglund, Loehr, and Remmel which allows us to obtain a signed enumerator of the major-index increments resulting from the insertion of a pair of consecutive numbers in any place of a given permutation.

Keywords: Signed mahonian statistics, major index with sign, subsequence restrictions

1 Introduction

1.1 Signed mahonians

Let $\mathfrak{S}_n$ be the set of permutations of $\{1, 2 \ldots, n\}$. The inversion number and the major index are two well-known mahonian statistics of permutations. Let $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$ be a permutation in $\mathfrak{S}_n$, where $\sigma_i = \sigma(i)$ for $1 \le i \le n$. An *inversion* of σ is a pair (σ_i, σ_j), $1 \le i < j \le n$ such that $\sigma_i > \sigma_j$. The *inversion number* $\mathsf{inv}(\sigma)$ of σ is defined to be the number of inversions of σ. A *descent* of σ is an integer i, $1 \le i \le n-1$ such that $\sigma_i > \sigma_{i+1}$. Let $\mathsf{Des}(\sigma)$ denote the set of descents of σ. The *descent number* (des) and *major index* (maj) of σ are defined by $\mathsf{des}(\sigma) = |\mathsf{Des}(\sigma)|$ and $\mathsf{maj}(\sigma) = \sum_{i \in \mathsf{Des}(\sigma)} i$.

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Percy MacMahon [6] proved that the major index statistic is equidistributed with the inversion number statistic over $\mathfrak{S}_n$, i.e.,

$$\sum_{\sigma\in\mathfrak{S}_n} q^{\mathsf{maj}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathsf{inv}(\sigma)} = [2]_q[3]_q\cdots[n]_q, \tag{1.1}$$

where $[j]_q = 1+q+\cdots+q^{j-1}$ for any positive integer j. This result was extended to the group B_n of signed permutations with respect to the *flag major index* statistic by Adin–Roichman [2].

Gessel and Simion obtained the following formula of the distribution of the major index with sign over $\mathfrak{S}_n$ (see [8, Corollary 2] for an interesting bijective proof)

$$\sum_{\sigma\in\mathfrak{S}_n} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)} = [2]_{-q}[3]_q\cdots[n]_{(-1)^{n-1}q}. \tag{1.2}$$

A type-B analogue of (1.2) was obtained by Adin–Gessel–Roichman [1, Theorem 1.5].

A *word* W on a set X is a finite sequence of elements in X. Unless specified otherwise, we consider only the words without repeated elements. The word W is a permutation of X if W consists of all elements of X. Given a word $W = w_1w_2\cdots w_k$ on the set $\{1,2,\ldots,n\}$, we say that a permutation $\sigma\in\mathfrak{S}_n$ contains the word W as a *subsequence* if there exists a sequence of indices $1\le i_1<i_2<\cdots<i_k\le n$ such that $\sigma_{i_j}=w_j$ for all j, $1\le j\le k$. Let $\mathfrak{S}_n(W)$ denote the subset of $\mathfrak{S}_n$ consisting of the permutations containing the word W as a subsequence, i.e.,

$$\mathfrak{S}_n(W) := \{\sigma\in\mathfrak{S}_n : \sigma^{-1}(w_1)<\sigma^{-1}(w_2)<\cdots\sigma^{-1}(w_k)\}.$$

In particular, for two integers $a,b\in\{1,2,\ldots,n\}$, $a<b$, let $\mathfrak{S}_n(a:b)$ denote the subset of permutations containing the word $a(a+1)\cdots b$ as a subsequence. For example, $\mathfrak{S}_4(2:4) = \{1234, 2134, 2314, 2341\}$.

By a classical result of Stanley [7] and Foata–Schützenberger [4], the statistics maj and inv remain equidistributed on all permutations in $\mathfrak{S}_n$ containing the word $(n-k+1)\cdots n$ as a subsequence, for $1\le k\le n-1$, i.e.,

$$\sum_{\sigma\in\mathfrak{S}_n(n-k+1:n)} q^{\mathsf{maj}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n(n-k+1:n)} q^{\mathsf{inv}(\sigma)} = [k+1]_q[k+2]_q\cdots[n]_q. \tag{1.3}$$

Arising from the study of signed mahonians in parabolic quotients of Coxeter groups, Caselli [3, Corollary 3.4] obtained the following product formula for the distribution of the major index with sign over $\mathfrak{S}_n(n-k+1:n)$, which includes the formula in (1.2) as a special case.

$$\sum_{\sigma\in\mathfrak{S}_n(n-k+1:n)} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)} = [k+1]_{(-1)^{nk+n+k}q}[k+2]_{(-1)^{k+1}q}\cdots[n]_{(-1)^{n-1}q}. \tag{1.4}$$

Caselli remarked that the proof of (1.4) is quite involved, without algebraic or combinatorial insight. He also raised a question [3, Problem 5.8] about giving a bijective proof of the following observation.

Conjecture 1.1. *If n is even or k is odd then*

$$\sum_{\sigma\in\mathfrak{S}_n(1:k)} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n(n-k+1:n)} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)}.$$

It is curious that the above equidistribution of signed major index depends on the parities of n and k. The motivation of this paper is to solve Caselli's problem. We prove much broader results on permutations with varied subsequence restrictions.

1.2 Main results

Given a word $W = w_1 w_2 \cdots w_k$ on the set $\{1, 2, \ldots, n\}$ and an integer t, let $W + t$ denote the word $w'_1 w'_2 \cdots w'_k$ on the set $\{t+1, t+2, \ldots, t+n\}$ obtained from W by incrementing each element by t, i.e., $w'_j = w_j + t$. Our first main result gives a sign-preserving and descent set-preserving bijection between the two subsets of permutations containing respectively the word W and the word $W + 2$ as a subsequence.

Theorem 1.2. *For any word W on the set $\{1, 2, \ldots, n-2\}$, there is a bijection $\phi : \sigma \to \sigma'$ of $\mathfrak{S}_n(W)$ onto $\mathfrak{S}_n(W+2)$ such that*

$$\mathsf{Des}(\sigma') = \mathsf{Des}(\sigma) \quad \text{and} \quad \mathsf{inv}(\sigma') \equiv \mathsf{inv}(\sigma) \pmod 2.$$

Hence we have the following identity

$$\sum_{\sigma\in\mathfrak{S}_n(W)} (-1)^{\mathsf{inv}(\sigma)} t^{\mathsf{des}(\sigma)} q^{\mathsf{maj}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n(W+2)} (-1)^{\mathsf{inv}(\sigma)} t^{\mathsf{des}(\sigma)} q^{\mathsf{maj}(\sigma)}.$$

An immediate consequence of this result is that it proves the case of Conjecture 1.1 when n and k have the same parity.

Our next result establishes a connection between the two parts of the symmetric difference of $\mathfrak{S}_n(W)$ and $\mathfrak{S}_n(W+1)$ when the word W is an increasing sequence of consecutive numbers.

Theorem 1.3. *For $2 \le k \le n-1$ and $1 \le b \le n-k$, let U and V be the words of k consecutive numbers respectively given by*

$$U = b(b+1)\cdots(b+k-1) \qquad \text{and} \qquad V = (b+1)(b+2)\cdots(b+k).$$

Then there is a bijection $\gamma : \sigma \to \sigma'$ of $\mathfrak{S}_n(U) - \mathfrak{S}_n(V)$ onto $\mathfrak{S}_n(V) - \mathfrak{S}_n(U)$ such that

$$\mathsf{Des}(\sigma') = \mathsf{Des}(\sigma) \quad \text{and} \quad \mathsf{inv}(\sigma') - \mathsf{inv}(\sigma) \equiv k-1 \pmod 2.$$

This result explains the case of Conjecture 1.1 when n and k have the opposite parities. Notice that Theorem 1.2 and Theorem 1.3 lead to the following analogous results of (1.4) for families of the permutations. This gives a complete picture of the Conjecture 1.1 for all parity cases of n and k.

Corollary 1.4. *For $2 \le k \le n-1$ and $1 \le b \le n-k+1$, the following results hold.*

1. *If k is odd then we have*

$$\sum_{\sigma \in \mathfrak{S}_n(b:b+k-1)} (-1)^{inv(\sigma)} q^{maj(\sigma)} = [k+1]_{-q}[k+2]_q \cdots [n]_{(-1)^{n-1}q}.$$

2. *If k is even and n is even then we have*

$$\sum_{\sigma \in \mathfrak{S}_n(b:b+k-1)} (-1)^{inv(\sigma)} q^{maj(\sigma)} = \begin{cases} [k+1]_q[k+2]_{-q}[k+3]_q \cdots [n]_{-q} & \text{for } b \text{ odd} \\ \left(2-[k+1]_q\right)[k+2]_{-q}[k+3]_q \cdots [n]_{-q} & \text{for } b \text{ even.} \end{cases}$$

3. *If k is even and n is odd then we have*

$$\sum_{\sigma \in \mathfrak{S}_n(b:b+k-1)} (-1)^{inv(\sigma)} q^{maj(\sigma)} = \begin{cases} \left(2-[k+1]_{-q}\right)[k+2]_{-q}[k+3]_q \cdots [n]_q & \text{for } b \text{ odd} \\ [k+1]_{-q}[k+2]_{-q}[k+3]_q \cdots [n]_q & \text{for } b \text{ even.} \end{cases}$$

For any element $r \in \{1,2,\ldots,n\}$ and any permutation W of the set $\{1,2,\ldots,n\}-\{r\}$, Haglund–Loehr–Remmel [5] derived an insertion lemma which describes the increment of major index resulting from the insertion of the element r in W, and proved that no matter what the element r is with respect to other elements

$$\sum_{\sigma \in \mathfrak{S}_n(W)} q^{\mathsf{maj}(\sigma)} = q^{\mathsf{maj}(W)}[n]_q. \tag{1.5}$$

We derive an extension of the insertion lemma which allows us to obtain the following signed analogue.

Theorem 1.5. *For $1 \le r \le n-1$ and any permutation W of the set $\{1,2,\ldots,n\}-\{r,r+1\}$, we have*

$$\sum_{\sigma \in \mathfrak{S}_n(W)} (-1)^{inv(\sigma)} q^{maj(\sigma)} = (-1)^{inv(W)} q^{maj(W)} [n-1]_{(-1)^n q}[n]_{(-1)^{n-1}q}.$$

We derive some extended results from our main results, over the permutations with subsequence restrictions defined by an injective labeling of a poset, and by an pattern-avoiding condition within a given underlying set.

2 A proof of Theorem 1.2

In this section, we shall establish a sign-preserving and descent set-preserving map $\phi : \mathfrak{S}_n(W) \to \mathfrak{S}_n(W+2)$ for any word W on the set $\{1,2,\ldots,n-2\}$.

2.1 The construction of the map $\phi : \mathfrak{S}_n(W) \to \mathfrak{S}_n(W+2)$.

Given a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n(W)$, we shall construct the corresponding permutation $\phi(\sigma)$ by removing the elements $n-1, n$ from σ, increment each of the remaining elements by 2, and then insert the elements $1, 2$ at appropriate positions so that $\phi(\sigma)$ satisfies the requested conditions. Let y_1, y_2 denote the entries for the elements $1, 2$ in $\phi(\sigma)$, i.e., $\{y_1, y_2\} = \{1, 2\}$, where y_2 appears to the right of y_1. Described in algorithm A, the construction of $\phi(\sigma)$ is given by case analysis, where the cases I, II, III, and IV describe the construction when the elements $n-1, n$ of σ are not adjacent, and the cases V and VI describe the construction when $n-1, n$ of σ are adjacent.

In the following algorithm, we assume $\sigma_0 = \sigma_{n+1} = 0$, and let σ_j^+ denote the entry σ_j incremented by 2.

Algorithm A.

Find the elements $n-1, n$ of σ. Let $\{\sigma_a, \sigma_b\} = \{n-1, n\}$ for some integers a, b with $1 \le a < b \le n$. We construct the permutation $\phi(\sigma)$ according to the following cases.

I. $\sigma_{a-1} > \sigma_{a+1}$ and $\sigma_{b-1} > \sigma_{b+1}$ for $a > 1$ and $a+1 < b \le n$.

Starting with σ_a (σ_b, respectively), search to the left and find the maximal increasing sequence of consecutive entries $\sigma_t < \sigma_{t+1} < \cdots < \sigma_a$ ($\sigma_s < \sigma_{s+1} < \cdots < \sigma_b$, respectively). Then remove the elements σ_a, σ_b from σ, increment each of the remaining elements by 2, and insert y_1 (y_2, respectively) on the immediate left of σ_t^+ (σ_s^+, respectively). Note that if $s = a+1$ then y_2 is between σ_{a-1}^+ and σ_{a+1}^+ is on the right of y_2.

II. $\sigma_{a-1} < \sigma_{a+1}$ and $\sigma_{b-1} < \sigma_{b+1}$ for $a \ge 1$ and $a+1 < b < n$.

Starting with σ_a (σ_b, respectively), search to the right and find the maximal decreasing sequence of consecutive entries $\sigma_a > \sigma_{a+1} > \cdots > \sigma_t$ ($\sigma_b > \sigma_{b+1} > \cdots > \sigma_s$, respectively). Then remove the elements σ_a, σ_b from σ, increment each of the remaining elements by 2, and insert y_1 (y_2, respectively) on the immediate right of σ_t^+ (after σ_s^+, respectively). Note that if $t = b-1$ then y_1 is between σ_{b-1}^+ and σ_{b+1}^+.

III. $\sigma_{a-1} > \sigma_{a+1}$ and $\sigma_{b-1} < \sigma_{b+1}$ for $a > 1$ and $a+1 < b < n$.

Starting with σ_a (σ_b, respectively), search to the left (right, respectively) and find the maximal increasing (decreasing, respectively) sequence of consecutive entries $\sigma_t < \sigma_{t+1} < \cdots < \sigma_a$ ($\sigma_b > \sigma_{b+1} > \cdots > \sigma_s$, respectively). Then remove the elements σ_a, σ_b from σ, increment each of the remaining elements by 2, and insert y_1 (y_2, respectively) immediately before σ_t^+ (after σ_s^+, respectively).

Notice that in the above three cases the elements y_1, y_2 are not adjacent. Choose either $(y_1, y_2) = (1, 2)$ or $(y_1, y_2) = (2, 1)$ such that $\mathsf{inv}(\phi(\sigma)) \equiv \mathsf{inv}(\sigma) \pmod 2$.

IV. $\sigma_{a-1} < \sigma_{a+1}$ and $\sigma_{b-1} > \sigma_{b+1}$ for $1 \le a < b \le n$.

Starting with σ_a (σ_b, respectively), search to the right (left, respectively) and find the maximal decreasing (increasing, respectively) sequence of consecutive entries $\sigma_a > \sigma_{a+1} > \cdots > \sigma_t$ ($\sigma_s < \sigma_{s+1} < \cdots < \sigma_b$, respectively). Then remove the elements σ_a, σ_b from σ, increment each of the remaining elements by 2. To preserve the descent set and the parity of the inversion number, the insertion of y_1, y_2 is determined as follows.

If $t \neq s$ then there exists at least one element between σ_t and σ_s. We insert y_1 (y_2, respectively) immediately after σ_t^+ (before σ_s^+, respectively). Since y_1, y_2 are not adjacent, choose either $(y_1, y_2) = (1,2)$ or $(y_1, y_2) = (2,1)$ such that $\mathsf{inv}(\phi(\sigma)) \equiv \mathsf{inv}(\sigma) \pmod 2$.

Otherwise, $t = s$. The insertion and assignment of y_1, y_2 are determined according the following possibilities.

(i) $a+1 < t < b-1$. If $(\sigma_a, \sigma_b) = (n-1, n)$ and $a+b$ is odd, or $(\sigma_a, \sigma_b) = (n, n-1)$ and $a+b$ is even then insert $(y_1, y_2) = (1,2)$ adjacently on the immediate right of σ_t^+; otherwise, insert $(y_1, y_2) = (2,1)$ adjacently on the immediate left of σ_t^+.

(ii) $a+1 = t < b-1$. If $(\sigma_a, \sigma_b) = (n-1, n)$ and $a+b$ is odd, or $(\sigma_a, \sigma_b) = (n, n-1)$ and $a+b$ is even then insert $(y_1, y_2) = (1,2)$ adjacently on the immediate right of σ_t^+. Otherwise, find the maximal increasing sequence of consecutive entries $\sigma_r < \sigma_{r+1} < \cdots < \sigma_a$ (set $r = 1$ if $a = 1$). We insert y_1 (y_2, respectively) immediately before σ_r^+ (σ_t^+, respectively), where $(y_1, y_2) = (2,1)$ if $a+r$ is even, and $(y_1, y_2) = (1,2)$ if $a+r$ is odd.

(iii) $a+1 < t = b-1$. If $(\sigma_a, \sigma_b) = (n-1, n)$ and $a+b$ is even, or $(\sigma_a, \sigma_b) = (n, n-1)$ and $a+b$ is odd then insert $(y_1, y_2) = (2,1)$ adjacently on the immediate left of σ_t^+. Otherwise, find the maximal decreasing sequence of consecutive entries $\sigma_b > \sigma_{b+1} > \cdots > \sigma_r$ (set $r = n$ if $b = n$). We insert y_1 (y_2, respectively) immediately after σ_t^+ (σ_r^+, respectively), where $(y_1, y_2) = (1,2)$ if $b+r$ is even, and $(y_1, y_2) = (2,1)$ if $b+r$ is odd.

(iv) $a+1 = t = b-1$. If $(\sigma_a, \sigma_b) = (n, n-1)$ then to the right of σ_b find the maximal decreasing sequence of consecutive entries $\sigma_b > \sigma_{b+1} > \cdots > \sigma_r$ (set $r = n$ if $b = n$). We insert y_1 (y_2, respectively) immediately after σ_t^+ (σ_r^+, respectively), where $(y_1, y_2) = (1,2)$ if $b+r$ is even, and $(y_1, y_2) = (2,1)$ if $b+r$ is odd.

Otherwise, $(\sigma_a, \sigma_b) = (n-1, n)$. Then to the left of σ_a find the maximal increasing sequence of consecutive entries $\sigma_r < \sigma_{r+1} < \cdots < \sigma_a$ (set $t = 1$ if $a = 1$). We insert y_1 (y_2, respectively) immediately before σ_r^+ (σ_t^+, respectively), where $(y_1, y_2) = (2,1)$ if $a+r$ is even, and $(y_1, y_2) = (1,2)$ if $a+r$ is odd.

V. $b = a+1$ and $(\sigma_a, \sigma_b) = (n-1, n)$ for $1 \le a < n$.

Starting with σ_b, find the maximal increasing sequence of consecutive entries $\sigma_t < \sigma_{t+1} < \cdots < \sigma_a < \sigma_b$ to the left (set $t = 1$ if $a = 1$), and find the maximal decreasing sequence of consecutive entries $\sigma_b > \sigma_{b+1} > \cdots > \sigma_s$ to the right (set $s = n$ if $b = n$). Then remove the elements σ_a, σ_b from σ, increment each of the remaining elements by 2.

(i) If $\sigma_{a-1} > \sigma_{b+1}$ then $a \neq 1$ and we insert $(y_1, y_2) = (1,2)$ adjacently on the immediate left of σ_t^+.

(ii) Otherwise, $\sigma_{a-1} < \sigma_{b+1}$. We insert y_1 (y_2, respectively) immediately before σ_t^+ (after σ_s^+, respectively), where $(y_1, y_2) = (1,2)$ if $t+s$ is odd, and $(y_1, y_2) = (2,1)$ otherwise.

VI. $b = a+1$ and $(\sigma_a, \sigma_b) = (n, n-1)$ for $1 \le a < n$.

Starting with σ_a, find the maximal increasing sequence of consecutive entries $\sigma_t < \sigma_{t+1} < \cdots < \sigma_a$ to the left (set $t = 1$ if $a = 1$), and find the maximal decreasing sequence of consecutive entries $\sigma_a > \sigma_b > \sigma_{b+1} > \cdots > \sigma_s$ to the right (set $s = n$ if $b = n$). Then remove the elements σ_a, σ_b from σ, increment each of the remaining elements by 2.

(i) If $\sigma_{a-1} < \sigma_{b+1}$ then $b \neq n$ and we insert $(y_1, y_2) = (2,1)$ adjacently on the immediate right of σ_s^+.

(ii) Otherwise, $\sigma_{a-1} > \sigma_{b+1}$. We insert y_1 (y_2, respectively) immediately before σ_t^+ (after σ_s^+, respectively), where $(y_1, y_2) = (1,2)$ if $t+s$ is even, and $(y_1, y_2) = (2,1)$ otherwise.

Proposition 2.1. *The map $\phi : \mathfrak{S}_n(W) \to \mathfrak{S}_n(W+2)$ constructed by algorithm A preserves the descent set and the parity of the inversion number of a permutation.*

Example 2.2. In the following, we demonstrate the construction of the map ϕ in case VI, using some permutations in $\mathfrak{S}_9$ containing the word $W = 3\,1\,7\,5$.

Let $\sigma = 3\,1\,9\,8\,6\,4\,2\,7\,5 \in \mathfrak{S}_9(W)$. We have $\mathsf{inv}(\sigma) = 18$ and $(\sigma_3, \sigma_4) = (9,8)$. By case VI, to the left of σ_3 find the maximal increasing sequence of consecutive entries $(1,9)$, and to the right of σ_3 find the maximal decreasing sequence of consecutive entries $(9,8,6,4,2)$. Remove the elements $8,9$ from σ and increment the other elements by 2. Since $\sigma_2 < \sigma_5$, by V(i) with $(y_1, y_2) = (2,1)$ inserted, we obtain $\phi(\sigma) = 5\,3\,8\,6\,4\,2\,1\,9\,7 \in \mathfrak{S}_9(W+2)$ with $\mathsf{inv}(\phi(\sigma)) = 18$.

Moreover, if $\sigma' = 3\,6\,9\,8\,4\,2\,1\,7\,5 \in \mathfrak{S}_9(W)$, we have $\mathsf{inv}(\sigma') = 21$ and $(\sigma'_3, \sigma'_4) = (9,8)$. Since $\sigma_2 > \sigma_5$, by VI(ii) with $(y_1, y_2) = (1,2)$ inserted, we obtain $\phi(\sigma') = 1\,5\,8\,6\,4\,3\,2\,9\,7 \in \mathfrak{S}_9(W+2)$ with $\mathsf{inv}(\phi(\sigma')) = 15$.

2.2 The construction of the map $\phi^{-1} : \mathfrak{S}_n(W+2) \to \mathfrak{S}_n(W)$.

For a word $V = v_1 v_2 \cdots v_d$ on the set $\{1, 2, \ldots, n\}$, let $\tau_n(V)$ denote the *n-complement* of V defined by $\tau_n(V) = (n+1-v_1)(n+1-v_2)\cdots(n+1-v_d)$. For any word W on the set $\{1, 2, \ldots, n-2\}$, we observe that the n-complement of the word $W+2$, say $W' = \tau_n(W+2)$, is also a word on the set $\{1, 2, \ldots, n-2\}$ and, moreover, $W = \tau_n(W'+2)$.

To find ϕ^{-1}, given a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n(W+2)$, we shall construct the corresponding permutation $\phi^{-1}(\sigma)$ by removing the elements $1,2$ from σ, decrement each of the remaining elements by 2, and then insert the elements $n-1, n$ at appropriate positions so that $\phi^{-1}(\sigma)$ satisfies the requested conditions. The construction of the map

ϕ^{-1} is exactly the reverse operation of ϕ, which is essentially established from ϕ by $\phi^{-1} = \tau_n \circ \phi \circ \tau_n$. We omit a detailed construction.

$$\begin{array}{ccc} \mathfrak{S}_n(W) & \xrightarrow{\phi} & \mathfrak{S}_n(W+2) \\ \tau_n \uparrow & & \downarrow \tau_n \\ \mathfrak{S}_n(W'+2) & \xleftarrow{\phi} & \mathfrak{S}_n(W') \end{array}$$

3 A proof of Theorem 1.3

In the following, we shall establish a bijection between the two parts of the symmetric difference of $\mathfrak{S}_n(U)$ and $\mathfrak{S}_n(V)$, where $U = b(b+1)\cdots(b+k-1)$ and $V = (b+1)(b+2)\cdots(b+k)$ for $2 \le k \le n-1$ and $1 \le b \le n-k$.

Notice that the set $\mathfrak{S}_n(U) \cap \mathfrak{S}_n(V)$ consists of all permutations containing the word $b(b+1)\cdots(b+k)$ as a subsequence.

Given a $\sigma \in \mathfrak{S}_n(U) - \mathfrak{S}_n(V)$, notice that the element $b+k$ appears to the left of the element $b+k-1$ in σ, i.e., $\sigma^{-1}(b+k) < \sigma^{-1}(b+k-1)$. We shall rearrange the elements $b, b+1, \ldots, b+k$ of σ to construct a permutation $\sigma' \in \mathfrak{S}_n(V) - \mathfrak{S}_n(U)$ satisfying the requested conditions. The construction is given in algorithm C below. In the following algorithm, we compose permutations right to left.

Algorithm C.

(C1) If $b+k$ appears to the left of b in σ, then set

$$\sigma' = \begin{pmatrix} b+k & b & b+1 & \cdots & b+k-1 \\ b+1 & b & b+2 & \cdots & b+k \end{pmatrix} \sigma.$$

Notice that $\mathsf{inv}(\sigma') - \mathsf{inv}(\sigma) = 1-k$.

(C2) Otherwise, $b+k$ appears between $b+j-1$ and $b+j$ for some j $(1 \le j \le k-1)$. Then if in particular $j = k-1$ then set

$$\sigma' = \begin{pmatrix} b & \cdots & b+k-2 & b+k & b+k-1 \\ b+1 & \cdots & b+k-1 & b+k & b \end{pmatrix} \sigma,$$

otherwise, $1 \le j \le k-2$ and set

$$\sigma' = \begin{pmatrix} b & \cdots & b+j-1 & b+k & b+j & b+j+1 & \cdots & b+k-1 \\ b+1 & \cdots & b+j & b+j+1 & b & b+j+2 & \cdots & b+k \end{pmatrix} \sigma.$$

Notice that in the former case $\mathsf{inv}(\sigma') - \mathsf{inv}(\sigma) = k-1$, while in the latter case $\mathsf{inv}(\sigma') - \mathsf{inv}(\sigma) = 2j-k+1$.

We observe that $\mathsf{Des}(\sigma') = \mathsf{Des}(\sigma)$ and $\mathsf{inv}(\sigma') - \mathsf{inv}(\sigma) \equiv k-1 \pmod 2$. On the other hand, it is straightforward to construct the inverse map by the reverse operation.

4 A proof of Theorem 1.5

For a fixed $r \in \{1,2,\ldots,n-1\}$, let $W = w_1w_2\cdots w_{n-2}$ be a permutation of the set $\{1,2,\ldots,r-1,r+2,\ldots,n\}$. Let $\mathfrak{S}^*_n(W) \subset \mathfrak{S}_n(W)$ denote the subset consisting of the permutations in which the elements $r, r+1$ are adjacent. Note that interchanging the elements $r, r+1$ is a sign-reversing involution on the difference set $\mathfrak{S}_n(W) - \mathfrak{S}^*_n(W)$ which preserves descent sets. Hence

$$\sum_{\sigma\in\mathfrak{S}_n(W)} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)} = \sum_{\sigma\in\mathfrak{S}^*_n(W)} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)}. \tag{4.1}$$

Any permutation $\sigma \in \mathfrak{S}^*_n(W)$ can be obtained from W by inserting the elements $r, r+1$ adjacently to the left of W, between two elements of W, or to the right of W, i.e., one of the $n-1$ *spaces* of W. These spaces are indexed by $0,1,\ldots,n-2$ from left to right. We shall study the major-index increment of such an insertion by extending the insertion lemma of Haglund–Loehr–Remmel [5, Lemma 4.1].

Assume $w_0 = 0$ and $w_{n-1} = n+1$. For $0 \le j \le n-2$, the jth space, which is between w_j and w_{j+1}, is called an *RL-space of W relative to r* if it satisfies one of the following conditions:

- $w_j > w_{j+1} > r$,
- $r > w_j > w_{j+1}$, or
- $w_j < r < w_{j+1}$.

Notice that the space to the left (right, respectively) of W is an RL-space if $r < w_1$ ($w_{n-2} < r$, respectively). Any space which is not an RL-space is called an *LR-space* (relative to r). In fact, an RL-space is a space where the insertion of r in W creates no 'new descent', while an LR-space is one where a new descent is created. Suppose there are d RL-spaces of W relative to r, we label the RL-spaces from right to left with $0,1,\ldots,d-1$ and label the LR-spaces from left to right with $d, d+1,\ldots,n-2$, called the *canonical labeling* of W. Let $\alpha(W) = (a_0, a_1, \ldots, a_{n-2})$ denote the vector of the labeling, where a_j is the label the jth space receives.

Example 4.1. Suppose $r = 4$ and W is the permutation $W = 8361297$ of the set $\{1,2,\ldots,9\} - \{4,5\}$. As shown below, the RL-spaces of W relative to 4 are the spaces with labels a_0, a_2, a_5, a_6 and the LR-spaces are the ones with labels a_1, a_3, a_4, a_7. The vector of the canonical labeling of W is $\alpha(W) = (3,4,2,5,6,1,0,7)$.

$${}_{a_0}\,8\,{}_{a_1}\,3\,{}_{a_2}\,6\,{}_{a_3}\,1\,{}_{a_4}\,2\,{}_{a_5}\,9\,{}_{a_6}\,7\,{}_{a_7} \quad\longrightarrow\quad \alpha(W) = (3,4,2,5,6,1,0,7)$$

The result of Haglund–Loehr–Remmel [5, Lemma 4.1] can be expressed as follows.

Lemma 4.2. (Insertion Lemma [5]) *If π is the word obtained from W by inserting the element r at the jth space of W then we have*

$$maj(\pi) = maj(W) + a_j.$$

We associate W with another vector $\beta(W) = (b_0, b_1, \ldots, b_{n-2})$, where b_j is the number of RL-spaces of W relative to r appearing to the right of the jth space for $0 \le j \le n-2$. For example, given $r = 4$ and the word $W = 8\,3\,6\,1\,2\,9\,7$ in Example 4.1, the associated vector is $\beta(W) = (3,3,2,2,2,1,0,0)$.

$$_{b_0}\,8\,_{b_1}\,3\,_{b_2}\,6\,_{b_3}\,1\,_{b_4}\,2\,_{b_5}\,9\,_{b_6}\,7\,_{b_7} \quad \longrightarrow \quad \beta(W) = (3,3,2,2,2,1,0,0)$$

We derive the following extension of the Lemma 4.2.

Lemma 4.3. *If σ is the word obtained from W by inserting the word z_1z_2 at the jth space of W, where z_1z_2 is either $r(r+1)$ or $(r+1)r$, then we have*

$$maj(\sigma) = \begin{cases} maj(W) + a_j + b_j & \text{if } z_1z_2 = r\,(r+1) \\ maj(W) + a_j + b_j + j + 1 & \text{if } z_1z_2 = (r+1)\,r. \end{cases}$$

Notice that σ has the same (opposite, respectively) sign of W if $z_1z_2 = r(r+1)$ ($z_1z_2 = (r+1)r$, respectively). By Lemma 4.3 and (4.1), we have the following result.

Corollary 4.4. *For any element $r \in \{1,2,\ldots,n-1\}$ and any permutation W of the set $\{1,2,\ldots,n\} - \{r, r+1\}$ with the associated vectors $\alpha(W) = (a_0,\ldots,a_{n-2})$ and $\beta(W) = (b_0,\ldots,b_{n-2})$, we have*

$$\sum_{\sigma \in \mathfrak{S}_n(W)} (-1)^{inv(\sigma)} q^{maj(\sigma)} = (-1)^{inv(W)} q^{maj(W)} \left(\sum_{j=0}^{n-2} q^{a_j+b_j} (1 - q^{j+1}) \right).$$

To prove Theorem 1.5, we prove the following result:

$$f(W;q) := \sum_{j=0}^{n-2} q^{a_j+b_j}(1-q^{j+1}) = [n-1]_{(-1)^n q}[n]_{(-1)^{n-1} q}. \tag{4.2}$$

5 Applications

5.1 An extended result of Theorem 1.5

We derive a product formula of the signed enumerator of the major index over the permutations in $\mathfrak{S}_n$ containing a permutation W of the set $\{2k+1, 2k+2, \ldots, n\}$ as a subsequence, in the spirit of Theorem 1.5.

Theorem 5.1. *For $k \geq 1$ and any permutation W of the set $\{2k+1, 2k+2, \dots, n\}$, we have*

$$\begin{aligned}\sum_{\sigma\in\mathfrak{S}_n(W)} (-1)^{inv(\sigma)} q^{maj(\sigma)} &= (-1)^{inv(W)} q^{maj(W)} \sum_{\sigma\in\mathfrak{S}_n(2k+1:n)} (-1)^{inv(\sigma)} q^{maj(\sigma)} \\ &= (-1)^{inv(W)} q^{maj(W)} [n-2k+1]_{(-1)^{n-2k}q} \cdots [n]_{(-1)^{n-1}q}.\end{aligned}$$

5.2 Labelings of a poset

Given a poset $(P, <)$ on a set $P = \{x_1, x_2, \dots, x_k\}$ with $k \leq n-2$, by an *injective labeling* of $(P, <)$ we mean an injection $f : P \to \{1, 2, \dots, n-2\}$. Let $f+2$ be the labeling of $(P, <)$ obtained from f by incrementing the label of each element by 2, which is an injection $P \to \{3, 4, \dots, n\}$. Define

$$\mathfrak{S}_n(f) := \{\sigma \in \mathfrak{S}_n : \sigma^{-1}(f(x_i)) < \sigma^{-1}(f(x_j)), \text{ for } x_i < x_j \text{ in } (P, <)\}.$$

We prove the following result.

Theorem 5.2. *For any poset $(P, <)$ with at most $n-2$ elements and any injective labeling $f : P \to \{1, 2, \dots, n-2\}$, we have*

$$\sum_{\sigma\in\mathfrak{S}_n(f)} (-1)^{inv(\sigma)} q^{maj(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n(f+2)} (-1)^{inv(\sigma)} q^{maj(\sigma)}.$$

5.3 Pattern avoidance

Putting Theorem 1.2 in the realm of pattern-avoidance, one can consider the π-avoiding words of a given underlying set $S \subseteq \{1, 2, \dots, n-2\}$ for a certain pattern π. Let $\mathfrak{S}_n(\pi; S)$ be the set consisting of the permutations $\sigma \in \mathfrak{S}_n$ containing no π-pattern restricted to the elements of S. Let $S+2 := \{z+2 | z \in S\} \subset \{3, 4, \dots, n\}$. For example, for $n = 4$, consider a pattern $\pi = 2\,1$ and a set $S = \{1, 2\}$. Then $S+2 = \{3, 4\}$ and we have

$$\begin{aligned}\mathfrak{S}_4(\pi; S) &= \{1234, 1243, 1324, 1342, 1423, 1432, 3124, 3142, 3412, 4123, 4132, 4312\}, \\ \mathfrak{S}_4(\pi; S+2) &= \{1234, 1324, 1342, 2134, 2314, 2341, 3124, 3142, 3214, 3241, 3412, 3421\}.\end{aligned}$$

$$\sum_{\sigma\in\mathfrak{S}_4(\pi;S)} (-1)^{\mathrm{inv}(\sigma)} q^{\mathrm{maj}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_4(\pi;S+2)} (-1)^{\mathrm{inv}(\sigma)} q^{\mathrm{maj}(\sigma)} = 1 + q^2 - q^3 - q^5.$$

Making use of the map ϕ in Theorem 1.2, we have the following result.

Theorem 5.3. *For a pattern π and an underlying set $S \subseteq \{1, 2, \dots, n-2\}$, we have*

$$\sum_{\sigma\in\mathfrak{S}_n(\pi;S)} (-1)^{inv(\sigma)} q^{maj(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n(\pi;S+2)} (-1)^{inv(\sigma)} q^{maj(\sigma)}.$$

6 Concluding remarks

In this paper, we study the signed distributions of the major index on permutations with subsequence restrictions. Recall that a signed permutation in the group B_n is a bijection σ of the set $\{-n, -n+1, \ldots, -1, 1, 2, \ldots, n\}$ onto itself such that $\sigma(-i) = -\sigma(i)$ for all $1 \leq i \leq n$. The flag major index of σ, denoted by fmaj, is defined as $\mathsf{fmaj}(\sigma) := 2\mathsf{maj}(\sigma) + \mathsf{neg}(\sigma)$, where $\mathsf{maj}(\sigma)$ is the major index of the sequence $(\sigma(1), \ldots, \sigma(n))$ with respect to the order $-1 < \cdots < -n < 1 < \cdots < n$, and $\mathsf{neg}(\sigma)$ is the number negative elements in the sequence. Adin–Gessel–Roichman obtained the following type-B analogue of (1.2)

$$\sum_{\sigma \in B_n} \mathsf{sign}(\sigma) q^{\mathsf{fmaj}(\sigma)} = [2]_{-q}[4]_q \cdots [2n]_{(-1)^n q}. \tag{6.1}$$

In the realm of parabolic quotients of Coxeter groups, we are interested in whether our main results can be extended to the signed permutations in B_n with subsequence restrictions, on the basis of (6.1).

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