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On the Asymptotic Enumeration of Restricted Strip Arrangements of a Chessboard

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Abstract. We are concerned with the number $T(m,n)$ of ways to arrange $1 \times k$ non-overlapping strips on an $m \times n$ chessboard (not necessarily fully covering the chessboard) such that there is at most one strip with its longest side horizontal in each row and at most one vertical strip in each row. While generating functions for $T(m,n)$ have been computed in some cases using the transfer matrix method, a general formula has proved elusive; in lieu of this, we provide a way to estimate $T(m,n)$ asymptotically. In the case where one dimension m of the chessboard is fixed and the other dimension n tends towards infinity, we are able to determine an asymptotically equivalent product formula for $T(m,n)$ as $n \to \infty$.

Keywords: strip arrangements, asymptotic enumeration, convex analysis

1 Introduction

The results discussed here concern a problem related to the general problem of tiling enumeration, or counting the number of ways to divide a region on some lattice into subsets of connected fundamental units. While problems of this type are of intrinsic interest to combinatorial mathematicians, they have also received practical interest, for example, by statistical physicists as models of nearest-neighbor particle bonding [1]. Most often, study has focused on grouping the fundamental regions into edge-adjacent pairs called *dimers* or, colloquially, *dominoes*. Enumerations of domino tilings have been studied on a variety of regions and lattices, including on the square lattice rectangles [5, 9] and Aztec diamonds [3], on the triangular lattice hexagons [6] and Aztec dungeons [2], and a host of other regions, a brief survey of which is given by Propp [7].

We wish to extend these results on *chessboards*, or rectangles in the square lattice, by expanding the pool of tiles from strict 1×2 dominoes to $1 \times k$ strips of any positive integral length k. This expansion allows another generalization by introducing the notion of a strip's natural horizontal or vertical alignment in the ambient chessboard according to its longer side of length k[1], and considering what happens when we impose restrictions on the number of horizontal strips in each row and vertical strips in each column. Our

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[1]For $k = 1$, the alignment of a 1×1 monomer, which we still allow, must be explicitly specified.

specific problem in the sequel, then, is counting the number $T(m, n)$ of ways to arrange non-overlapping strips on an $m \times n$ chessboard such that there is at most one horizontal strip in each row and at most one vertical strip in each column, where we use the word "arrange" to denote the fact that not each fundamental unit square of the chessboard need be covered by a strip, in order to better facilitate the row and column restrictions. This is in contrast to the usual notion of "tiling," which usually calls for the covering of each fundamental region. Examples of such a tiling are given in Figure 1.

There is a certain practical consideration for the given row and column restrictions. Note that each fundamental unit square in any given row must, when "reading" from left to right, occur before, within, or after a horizontal strip (neglecting the vertical strips, as they are unrestricted in each row). The same can be said by "reading" vertical strips within each column, thus giving a pair of these three states for each unit square; however, as strips cannot overlap, a square cannot be both in a horizontal strip and a vertical strip, so we exclude this one pair. This gives a total of eight states for each unit square (see Figure 2), which is termed in statistical physics as an *eight-vertex model*. The true physical significance of our particular model has not been well studied, but there is one example of an eight-vertex model that is well-studied as an extension of a six-vertex model illustrating water molecule bonds in ice [1].

Previous work on the problem of enumerating these restricted strip tilings has applied the transfer matrix method, a technique borrowed from statistical physics per the preceding discussion, to obtain generating functions $\sum_{n\geq 0} T(m, n)x^n$. These generating

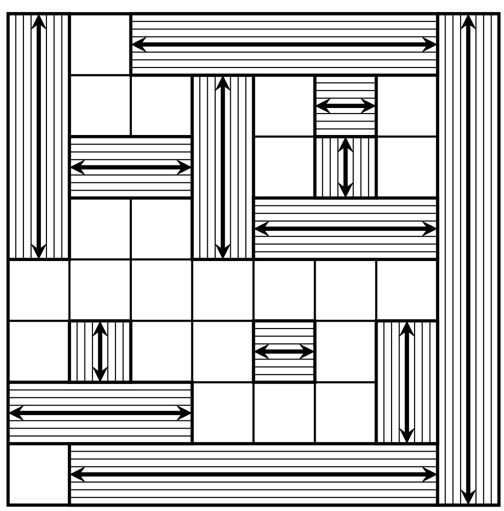

Figure 1: *A strip arrangement on an* 8×8 *chessboard with at most one horizontal strip in each row and at most one vertical strip in each column. We include arrows to easily indicate the direction of each strip; we note that such distinctions are* required *for strips of unit length.*

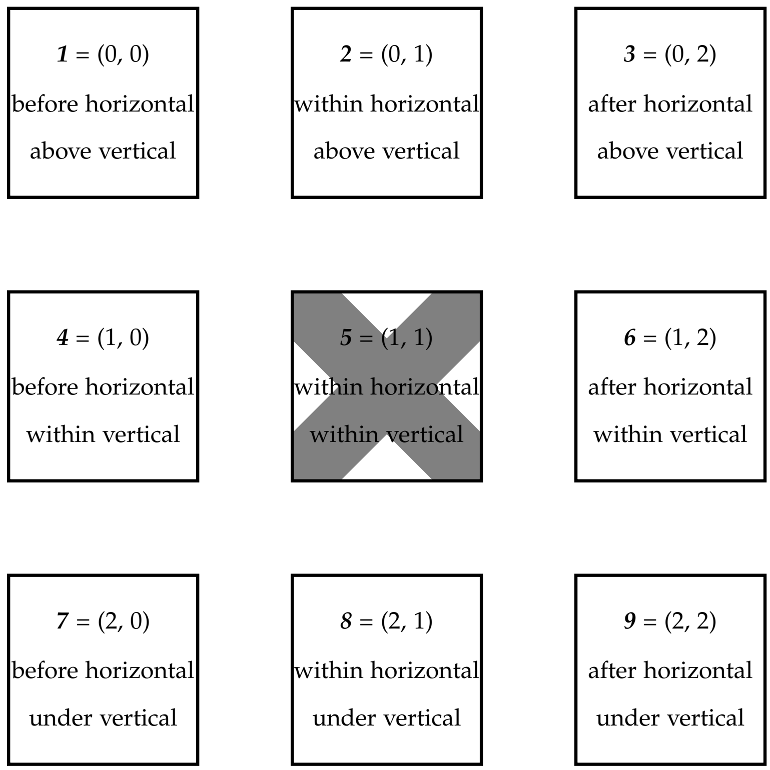

Figure 2: *Translation of fundamental unit squares in a restricted strip arrangement into pairs of 3-colorings of unit squares of a chessboard, giving 8 allowed colors and 1 disallowed color.*

functions have led to exact formulas for $T(m,n)$ as a function of n for certain fixed values of m [4]; however, a general formula for $T(m,n)$ has proven elusive. In lieu of this, we discuss an asymptotic formula for $T(m,n)$ in the above case where one dimension m of the chessboard is fixed and the other length n goes to infinity. As the corresponding generating functions are necessarily rational by the transfer matrix method [8], we see that a formula for $T(m,n)$ as a function of n takes the form $\sum \lambda_k^n P_k(n)$ where $\lambda_k \in \mathbb{C}$ and P_k is a polynomial, so we attempt to identify the largest value of λ and the leading term of the corresponding polynomial. In sum, we produce the following asymptotic equivalence:

Theorem 1. *For a fixed m, as $n \to \infty$,*

$$T(m,n) \sim \frac{1}{(m!)^2} \prod_{j=1}^{m} \left[1 + \binom{j}{2} + \binom{m+1-j}{2} \right] n^m \left(1 + \binom{m+1}{2} \right)^n .$$

2 A Sketch of the Proof

Let one of the dimensions m of the chessboard be fixed. The corresponding generating function $\mathcal{T}_m(x) = \sum_{n \geq 0} T(m,n)x^n$ can be computed using the transfer matrix method [4] that was outlined by Stanley [8], who showed that this forces $\mathcal{T}_m(x)$ to be rational. $T(m,n)$, therefore, is a (finite) sum of terms of the form $C(m)n^{P(m)}(\Lambda(m))^n$, where $C(m)$, $P(m)$, and $\Lambda(m)$ are expressions independent of n, but possibly dependent on m. The dominant term among all of these is the one first with the largest possible $\Lambda(m)$, and then with the largest value of $P(m)$ among these, and this dominant term gives us an asymptotic formula for $T(m,n)$. Table 1 illustrates these values for various values of m, and perhaps the careful reader can discern patterns among these, but we will show how to determine the values of $\Lambda(m)$, $P(m)$, and $C(m)$ in turn, and thereby prove Theorem 1.

2.1 Computing the Exponential Base $\Lambda(m)$

The largest value for the exponential base follows directly from the transfer matrix method. We give a very brief overview here; for full details, see [4].

The transfer matrix used to compute $T(m,n)$ has as its basis m-tuples of "before," "during," or "after" states for horizontal strips mentioned previously; these represent each possible state in which a column of the chessboard can be. An entry of the matrix, then, between two columns is zero if it breaks the before-during-after paradigm (i.e., it tries to skip from "before" to "after" immediately or it tries to go backwards), and otherwise is weighted by the number of ways to place a vertical strip in the target column.

Typically, the exponential bases of the transfer matrix are the eigenvalues of the trans-

m	1	2	3	4	5	6
Dominant term of $T(m,n)$	$n2^n$	$n^2 4^n$	$\frac{4}{3}n^3 7^n$	$\frac{1225}{576}n^4 11^n$	$\frac{847}{225}n^5 16^n$	$\frac{64}{9}n^6 22^n$
$\Lambda(m)$	2	4	7	11	16	22
$P(m)$	1	2	3	4	5	6
$C(m)$	1	1	4/3	1225/576	847/225	64/9

Table 1: *For $m = 1, \ldots, 6$, the dominant terms in the sum expression for $T(m,n)$ as a function of n – that is, the summand that is a product of the exponential $\Lambda(m)^n$ with the largest base and the monomial $C(m)n^{P(m)}$ with greatest degree. The values of $\Lambda(m)$, $P(m)$, and $C(m)$ are expressly listed for each m for clarity.*

fer matrix [8]. In our case, the forbiddance of going backwards in the before-during-after paradigm implies that the matrix is upper triangular, so we need only compute the largest possible weight, which corresponds to those columns that are not crossed by a horizontal strip. In such cases, we can refuse to place a vertical strip in exactly one way, or we can place a vertical strip on the m squares of the column by picking any two of the corresponding $m+1$ gridlines.

Therefore, $\Lambda(m) = 1 + \binom{m+1}{2}$.

2.2 Computing the Polynomial Degree $P(m)$

Next, we determine the degree of the polynomial associated with $\Lambda(m) = 1 + \binom{m+1}{2}$. Perhaps this is the most apparent pattern in Table 1, but we prove this rigorously via a bounding argument. Indeed, we have the lower bound

$$T(m,n) \geq \binom{n}{m} m! \left(1 + \binom{m+1}{2}\right)^{n-m} \left\{\prod_{j=1}^{m} \left[1 + \binom{j}{2} + \binom{m+1-j}{2}\right]\right\} \tag{2.1}$$

by first choosing m of the n columns to contain a horizontal strip, then choosing distinct rows for the horizontal strips in each of the chosen columns (this is akin to selecting a placement of m non-attacking rooks on a $m \times m$ chessboard), and finally determining the number of ways to place a vertical strip in each column – for the $n - m$ columns without any horizontal strip, there are $1 + \binom{m+1}{2}$ possible placements of a vertical strip, whereas, for each $j = 1, \ldots, m$, the column that contains a horizontal strip in row j allows $1 + \binom{j}{2} + \binom{m+1-j}{2}$ possible placements of a vertical strip. From inequality (2.1), we deduce that $P(m) \geq \deg \binom{n}{m} = m$.

Conversely, instead of spreading around the horizontal strips, we can find an upper bound for $T(m,n)$ by consolidating them to overlap the same set of columns. A summation argument [4] then gives the upper bound

$$T(m,n) \leq n^m \left(1 + \binom{m+1}{2}\right)^n \left\{1 - \left[\frac{1 + \binom{m}{2}}{1 + \binom{m+1}{2}}\right]^{\frac{1}{m}}\right\}^{-m}. \tag{2.2}$$

We are interested solely in the degree of the polynomial; therefore, combining the results of inequalities (2.1) and (2.2) gives $P(m) = m$.

2.3 Computing the Leading Coefficient $C(m)$

It remains only to determine the leading coefficient $C(m)$ exactly. We have been concentrating in the preceding arguments on the placements of the horizontal strips. The

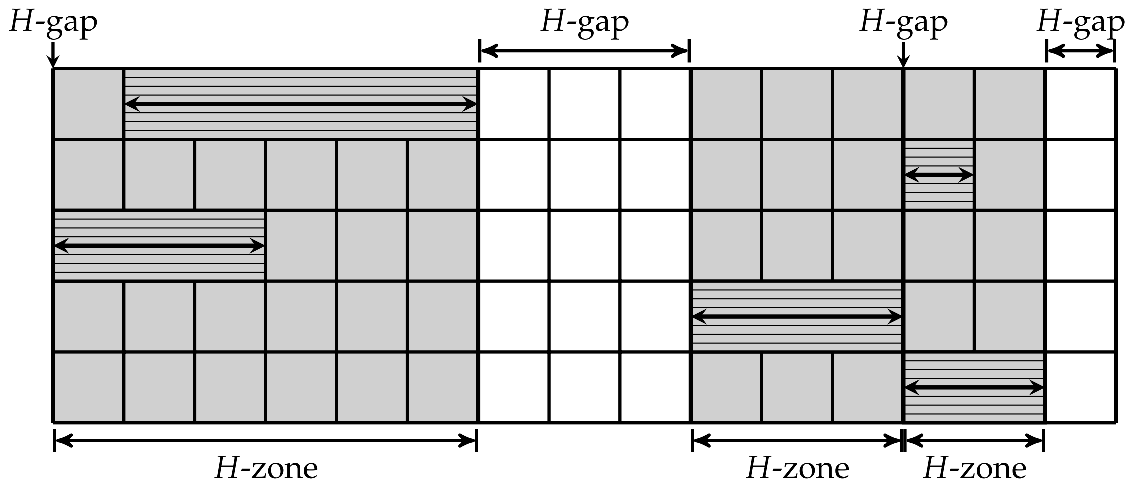

Figure 3: *An arrangement of horizontal strips with H-gaps and H-zones labeled. Intuitively, H-gaps appear after one horizontal strip ends and another horizontal strip begins; any horizontal strips that intersect the same columns belong to the same H-zone. Note that there are always H-gaps at either end of the chessboard, and that it is possible for an H-gap to be of length 0.*

biggest insight here was to instead consider the placement of *gaps* between these horizontal strips. We formally define an *H-gap of size s* to be a contiguous set of s columns on the chessboard that are not crossed by any horizontal strip, and are such that the leftmost gridline of the gap is either the leftmost gridline of the chessboard or coincides with the rightmost boundary of some horizontal strip, and similarly the rightmost gridline of the gap is either the rightmost gridline of the chessboard or coincides with the leftmost boundary of some horizontal strip. Dually, we also define an *H-zone of size z* to be a set of contiguous columns of the chessboard that are covered by at least one horizontal strip, and are such that the leftmost gridline of the zone is the rightmost gridline of some H-gap, and the rightmost gridline of the zone coincides with the leftmost gridline of some other H-gap. We illustrate a configuration of H-gaps and H-zones in Figure 3.

The great advantage of partitioning strip arrangements into H-zones and H-gaps is that the placement of strips (both horizontal and vertical) in the H-gaps and the placement of strips in the H-zones are independent of one another. By definition, no horizontal strip can cross between H-zones and H-gaps (indeed, horizontal strips are forbidden in H-gaps); and, as both zones and gaps are collections of columns, no vertical strip can cross between them, since in fact no vertical strip can cross between any two columns. It therefore follows that

$$T(m,n) = Z(0,0)G(1,m) + \sum_{g=2}^{m+1} \sum_{\kappa=g-1}^{n} Z(g-1,\kappa)G(g,n-\kappa), \tag{2.3}$$

where Z and G are defined as follows:

$Z(z,\kappa)$ = the number of ways to place strips in a number z of H-zones of total size κ
$G(g,\lambda)$ = the number of ways to place strips in a number g of H-gaps of total size λ

The sum of (2.3) is indexed by the possible values κ of the number of columns in the strip arrangement that coincide with at least one horizontal strip and the possible values g of the number of H-gaps in the strip arrangement.

We can easily deduce

$$\begin{aligned} G(g,n-\kappa) &= \binom{n-\kappa+g-1}{g-1}\left(1+\binom{m+1}{2}\right)^{n-\kappa} \\ &= \frac{1}{(g-1)!}\left(1+\binom{m+1}{2}\right)^{n-\kappa} O(n^{g-1}), \end{aligned} \tag{2.4}$$

and it turns out that, when n is large, this is the only contributor to the polynomial factor. Since we are concerned with the degree m term, we look only at the situation when $m = g-1$, which corresponds to the situation where each of the m rows has a horizontal strip, and each such horizontal strip defines its own H-zone (i.e., no two of the horizontal strips intersect the same columns). In this case, we can compute

$$Z(m,\kappa) = m! \sum_{\substack{x_1+\cdots+x_m=\kappa \\ x_i \ge 1}} \left\{ \prod_{j=1}^{m} \left(1+\binom{j}{2}+\binom{m+1-j}{2}\right)^{x_j} \right\}. \tag{2.5}$$

Combining the results of equations (2.3), (2.4), and (2.5), we conclude that

$$\begin{aligned} C(m) &= \lim_{n\to\infty} \frac{T(m,n)}{n^m(1+\binom{m+1}{2})^n} \\ &= \sum_{\kappa=m}^{\infty} \sum_{\substack{x_1+\cdots+x_m=\kappa \\ x_i \ge 1}} \left\{ \prod_{j=1}^{m} \left(\frac{1+\binom{j}{2}+\binom{m+1-j}{2}}{1+\binom{m+1}{2}} \right)^{x_j} \right\} \\ &= \prod_{j=1}^{m} \left(\frac{1+\binom{j}{2}+\binom{m+1-j}{2}}{1+\binom{m+1}{2}} \right) \left(1 - \frac{1+\binom{j}{2}+\binom{m+1-j}{2}}{1+\binom{m+1}{2}} \right)^{-1} \\ &= \frac{1}{(m!)^2} \prod_{j=1}^{m} \left[1+\binom{j}{2}+\binom{m+1-j}{2} \right], \end{aligned}$$

and therefore deduce the asymptotic formula of Theorem 1.

3 Conclusion

We remark in closing that there are a variety of other methods to determine the asymptotics of $T(m, n)$. One of particular interest is to see what happens when the ambient chessboard is square – i.e., when $m = n \to \infty$. In this regard, it has been shown that, as $n \to \infty$, $\log T(n, n) = 4n \log n - 2n \log 8 + O(n^{2/3})$ [4]. The method of proof involves many repeated and lengthy analytic computations, and is therefore outside the scope of this paper, but full details are provided in the reference.

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References

[1] R. J. Baxter. *Exactly Solved Models in Statistical Mechanics*. Reprint of the 1982 original. Dover Publications, Mineola, NY, 2007.

[2] M. Ciucu and T. Lai. "Proof of Blum's conjecture on hexagonal dungeons". *J. Combin. Theory Ser. A* **125** (2014), pp. 273–305. Link.

[3] N. Elkies, G. Kuperberg, M. Larsen, and J. Propp. "Alternating-sign matrices and domino tilings. I". *J. Algebraic Combin.* **1**.2 (1992), pp. 111–132. Link.

[4] A. Gershon. "New Directions in the Enumeration of Tilings on a Chessboard". PhD thesis. Indiana University, 2015.

[5] P. W. Kasteleyn. "The statistics of dimers on a lattice: I. The number of dimer arrangements on a quadratic lattice". *Physica* **27**.12 (1961), pp. 1209–1225. Link.

[6] P. A. MacMahon. *Combinatory Analysis*. Vol. I & II. Reprint of the 1915 & 1916 originals. AMS Chelsea Publishing, Providence, RI, 1984.

[7] J. Propp. "Enumeration of Matchings: Problems and Progress". *New Perspectives in Algebraic Combinatorics*. MSRI Publications 38. Cambridge University Press, New York, 1999.

[8] R. P. Stanley. *Enumerative Combinatorics*. Vol. 1. Cambridge Studies in Advanced Mathematics 49. Cambridge University Press, New York, 2012.

[9] H. N. V. Temperley and M. E. Fisher. "Dimer problem in statistical mechanics—an exact result". *Philos. Mag. (8)* **6** (1961), pp. 1061–1063.