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Unique rectification in d-complete posets

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Abstract. The jeu-de-taquin-based Littlewood-Richardson rule of Thomas–Yong (2009) for minuscule varieties has been extended in two orthogonal directions, either enriching the cohomology theory or else expanding the family of varieties considered. In one direction, Buch–Samuel (2016) developed a combinatorial theory of 'unique rectification targets' in minuscule posets to extend the Thomas-Yong rule from ordinary cohomology to K-theory. Separately, Chaput–Perrin (2012) used the combinatorics of Proctor's 'd-complete posets' to extend the Thomas-Yong rule from minuscule varieties to a broader class of Kac–Moody structure constants. We begin to address the unification of these theories. Our main result is the existence of unique rectification targets in a large class of d-complete posets. From this result, we obtain conjectural positive combinatorial formulas for certain K-theoretic Schubert structure constants in the Kac–Moody setting.

1 Introduction

The 1970s saw a major advance in enumerative geometry when Schützenberger proved the Littlewood-Richardson rule for describing the cohomology rings of Grassmannians. Since then, the modern Schubert calculus has turned to extending this understanding in two different directions: on the one hand to replace the Grassmannian with a more complicated homogeneous space, and on the other hand to replace ordinary cohomology with a richer generalized cohomology theory. Along these lines, this abstract summarizes our results from [7], beginning to unravel the K-theoretic Schubert calculus of Kac–Moody homogeneous spaces. Our results are purely combinatorial, but allow us to conjecture explicit Littlewood-Richardson-style rules in this geometric context.

Let G be a complex Kac–Moody group with Borel and opposite Borel subgroups B_+, B_-. Let $B_+ \subseteq P \subset G$ be a parabolic subgroup. The homogeneous space $X = G/P$ is a **Kac–Moody flag variety**. The Zariski closures of the B_--orbits are the **Schubert varieties** $\{X_w\}_{w \in W^P}$ and give a cell decomposition of X; here, W^P denotes the set of minimal-length representatives of the quotient W/W_P, where W is the Weyl group of

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G and W_P is the parabolic Weyl group for P. The cohomology ring $H^\star(G/P)$ thereby has a distinguished Schubert basis $\{\sigma_w\}_{w \in W^P}$ with σ_w Poincaré dual to X_w. Thus, to multiply in $H^\star(X)$, it suffices to determine the **Schubert structure constants** $c_{u,v}^w$ defined by $\sigma_u \cdot \sigma_v = \sum_{w \in W^P} c_{u,v}^w \sigma_w$. If $X = \mathrm{Gr}_k(\mathbb{C}^n)$ is a Grassmannian, this problem is solved in a positive combinatorial manner by any of the various Littlewood-Richardson rules. For a general Kac–Moody flag variety X, these $c_{u,v}^w$ are also non-negative integers, but it is generally open to give an analogous Littlewood-Richardson-style rule for them.

For $X = \mathrm{Gr}_k(\mathbb{C}^n)$, Schützenberger's Littlewood-Richardson rule is stated in terms of *jeu de taquin* for standard Young tableaux [16] fitting inside a $k \times (n-k)$ rectangle. One may realize this rectangle as a subposet of positive roots for $\mathrm{GL}_n(\mathbb{C})$ so that the inversion set of $w \in W^P$ is an order ideal and may then realize standard Young tableaux as linear extensions of intervals in this poset. From this perspective, Thomas-Yong [19] gave a uniform extension of Schützenberger's rule to compute all cohomological Schubert structure constants for the larger family of *minuscule varieties*. This was further extended by Chaput-Perrin [4] to a positive combinatorial formula for certain Λ-*minuscule* Schubert structure constants for general Kac–Moody X. In the Chaput-Perrin rule, the role of the $k \times (n-k)$ rectangle is played by Proctor's *d-complete posets* [11, 12]; these are exactly those posets encoding the containment relations among Λ-minuscule Schubert varieties.

Much work in modern Schubert calculus studies homogeneous spaces through richer cohomology theories. In these theories, there are Schubert bases analogous to the σ_w and the corresponding Schubert structure constants enjoy various positivity properties. Hence, it makes sense to attempt to develop positive combinatorial formulas for these structure constants in the style of classical Littlewood-Richardson rules. For Grassmannians, one has, for example, the K-theory rule of Buch [1]. Our interest is in the K-theory ring $K(X)$ of the Kac–Moody flag variety X, where the K-theoretic Schubert classes $\{[\mathcal{O}_{X_w}]\}_{w \in W^P}$ are represented by the structure sheaves of the Schubert varieties. Specifically, we are interested in the structure constants $K_{u,v}^w$ of $K(X)$ defined by

$$[\mathcal{O}_{X_u}] \cdot [\mathcal{O}_{X_v}] = \sum_{w \in W^P} K_{u,v}^w [\mathcal{O}_{X_w}]. \tag{1.1}$$

For Grassmannians, various alternatives to Buch's original rule [1] for $K_{u,v}^w$ are now known. However, only the rule of Thomas–Yong [20] is currently known to extend to all of the minuscule varieties [2, 3, 5]. This rule is based on a jeu de taquin theory for *increasing tableaux*. This combinatorial theory displays several extra subtleties when compared to Schützenberger's jeu de taquin for standard tableaux. In particular, a key ingredient is the need to identify increasing tableaux with the *unique rectification target* property. (These combinatorial notions are reviewed in Section 2.)

Both [3] and [20] ask to what extent their combinatorial theory extends to d-complete posets. The missing ingredient is that it is unknown if general d-complete posets have "enough" unique rectification targets. We conjecture that they do:

Conjecture 1. *Let $\mathcal{P}$ be a d-complete poset and let $\lambda \subseteq \mathcal{P}$ be an order ideal. Then there is an (explicitly-defined) unique rectification target supported on λ.*

We study the existence and structure of unique rectification targets in d-complete posets. Every d-complete poset can be constructed by gluing together (in prescribed ways) certain irreducible d-complete posets [11]. These irreducible pieces are classified [11] and include all of the minuscule posets (the posets describing the Schubert stratification of minuscule varieties). Informally, the following is our main result.

Theorem 2. *Conjecture 1 holds in the case that $\mathcal{P}$ is built from minuscule posets.*

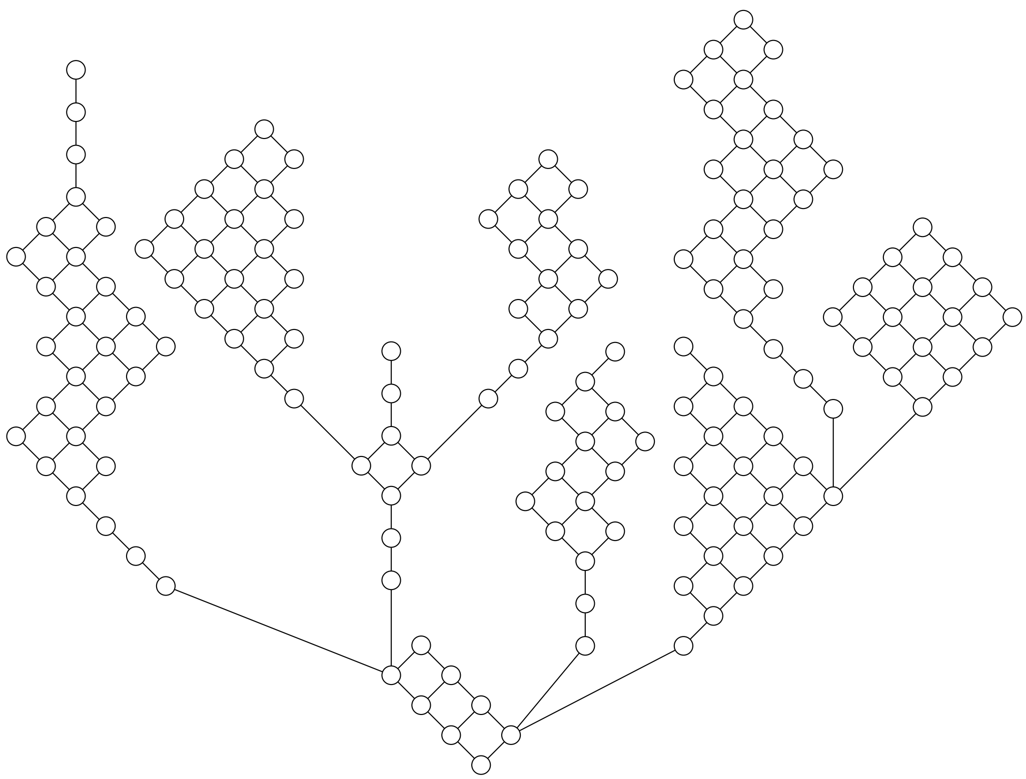

Figure 1: The Hasse diagram of a representative d-complete poset $\mathcal{P}$ that is "built from minuscule posets" in the sense of Theorem 2. In $\mathcal{P}$, every order ideal $\lambda \subseteq \mathcal{P}$ has a unique rectification target, provided by Theorem 2.

For an example of such a poset covered by Theorem 2, see Figure 1. The precise formulation of Theorem 2 appears as Corollary 22. For any $\mathcal{P}$ satisfying Conjecture 1, one obtains (as in [3, §3.5]) a corresponding combinatorially-defined associative commutative unital algebra $K(\mathcal{P})$ with a basis $\{\lambda\}$ indexed by order ideals of $\mathcal{P}$. The structure constants $t^{\nu}_{\lambda,\mu}$ of $K(\mathcal{P})$ are defined in such a way as to transparently alternate in degree.

(This construction is discussed in Section 4.) For $w \in W^P$ Λ-minuscule, the interval $[\mathrm{id}, w]$ in Bruhat order is isomorphic to the poset of order ideals of a certain d-complete poset $\mathcal{P}_w$ constructed from w. Building on Conjecture 1, we propose the following.

Conjecture 3. *Let $X = G/P$ be a Kac–Moody flag variety and let $m \in W^P$ be Λ-minuscule. Then, for $u, v, w \leq m$ in Bruhat order, we have the equality of structure constants $K_{u,v}^w = t_{\lambda,\mu}^\nu$ between the rings $K(X)$ and $K(\mathcal{P}_m)$, where the order ideals $\lambda, \mu, \nu \subseteq \mathcal{P}_m$ correspond respectively to the Weyl group elements $u, v, w \in W^P$.*

In Section 4, we give the precise versions of Conjecture 1 and Theorem 2, as well as the details necessary for a precise understanding of Conjecture 3. In light of Conjecture 3, Theorem 2 should be understood as giving a conjectural positive combinatorial rule for certain K-theoretic Schubert structure constants of Kac–Moody flag varieties. Several cases of Conjecture 3 are known to be true or have been previously conjectured. If the flag variety X is minuscule, then Conjecture 3 reduces to the main theorem of [3]. If, on the other hand, X is general but $|\nu| - |\lambda| - |\mu| = 0$, then Conjecture 3 reduces to [4, Conjecture 1.1], many cases of which are proved in [4, Theorem 1.3]. Assuming one followed the general structure utilized by [3, 4], the main ingredients one would need in a proof of Conjecture 3 are

(1.) a proof of the remaining cases of Conjecture 1 and

(2.) *ad hoc* geometric verifications of Conjecture 3 for special u lying in a generating set of classes.

For a large class of such Schubert problems, Theorem 2 provides the necessary first ingredient, so it only remains to establish the second in those cases.

Another potential application of Theorem 2 (or more generally Conjecture 1) is to establishing plane partition identities. In [6], the authors use the existence of unique rectification targets in minuscule posets to give bijective proofs of the equinumerosity of various classes of plane partitions, in particular resolving a 1983 question of Proctor [10]. The main technology of [6] applies equally to any d-complete poset satisfying Conjecture 1; hence, we expect Theorem 2 to yield analogous identities.

This abstract is organized as follows. In Section 2, we fix notation for posets, describe the Thomas-Yong theory of jeu de taquin for increasing tableaux, and recall the definition of unique rectification targets (URTs). Section 3 studies the behavior of URTs when two posets are combined via Proctor's *slant sum* operation, introduces the notion of a *p-chain URT*, and establishes the necessary technical fact that all increasing tableaux of straight shape in a double-tailed diamond poset are p-chain URTs. In Section 4, we first recall background on d-complete posets and recall needed notions to make Conjectures 1 and 3 precise. We then apply the results from Section 3 to the study of d-complete posets and prove Theorem 2, our main result.

2 Posets, skew shapes, and rectifications

All posets in this paper will be finite, nonempty, and connected. Although there is now a more general notion of infinite d-complete posets (see the "Added Notes" at the very end of [15] for discussion), we will not explicitly consider such objects. In this section, $\mathcal{P}$ will denote an otherwise arbitrary poset.

We often visualize posets using Hasse diagrams, where each element is represented by a circle, and $a \lessdot b$ if there is a line that goes up from a to b. We denote the minimum element (if it exists) of the poset $\mathcal{P}$ by $\hat{0}_{\mathcal{P}}$. We say a poset $\mathcal{P}$ has a $\hat{0}_{\mathcal{P}}$ to mean that it has a minimum, which is $\hat{0}_{\mathcal{P}}$. We say that a poset $\mathcal{P}$ is a **chain** if all pairs of elements of $\mathcal{P}$ are comparable, that is, if $\mathcal{P}$ is a total order. The size of a chain $\mathcal{P}$ is the number of its elements. A **shape** ν of $\mathcal{P}$ is any subset of $\mathcal{P}$. The shape ν has a natural poset structure given by restricting that of $\mathcal{P}$. A shape ν of $\mathcal{P}$ is called an **order ideal** of $\mathcal{P}$ if it is closed downwards, i.e. if $y \in \nu$ and $x < y$ together imply $x \in \nu$. Similarly, an **order filter** of $\mathcal{P}$ is a subset that is closed upwards. For historical reasons, we will also refer to the order ideals of $\mathcal{P}$ as **straight shapes**. If $\lambda \subseteq \nu$ are straight shapes of $\mathcal{P}$, then the shape $\nu \setminus \lambda$ is called a **skew shape** of $\mathcal{P}$ and is denoted ν/λ. Note that every straight shape λ can also be realized as the skew shape $\lambda/\emptyset$. An element $x \in \lambda$ is called an **inner corner** of the skew shape ν/λ if x is maximal in λ. We write $\mathrm{IC}(\nu/\lambda)$ to denote the set of inner corners of ν/λ. For a skew shape ν/λ of $\mathcal{P}$, a function $T : \nu/\lambda \to \mathbb{Z}_{>0}$ is called a **skew increasing $\mathcal{P}$-tableau** of shape ν/λ if T is a strictly order preserving map, i.e. if $x < y$ implies $T(x) < T(y)$. If, in addition, T is a bijection onto an initial segment of $\mathbb{Z}_{>0}$, we say T is a **skew standard $\mathcal{P}$-tableau**. In both cases, if ν/λ is a straight shape, we drop the word "skew." We depict a skew increasing $\mathcal{P}$-tableau T using Hasse diagrams with labels. For an element $x \in \mathcal{P}$, we put the value of $T(x)$ in the circle of the Hasse diagram corresponding to x. Also, to make clear what the ambient poset is we represent skewed out elements (the elements in λ) with unlabeled hollow circles. For a skew shape ν/λ of $\mathcal{P}$, we say a function $T : \nu/\lambda \to \mathbb{Z}_{>0} \cup \{\bullet\}$ is a **skew dotted increasing $\mathcal{P}$-tableau** of shape ν/λ if there is a rational number q such that T becomes a strictly order preserving map ($\nu/\lambda \to \mathbb{Q}$) when we replace each $\bullet$ with that fixed q.

Definition 4. Let T be a skew increasing $\mathcal{P}$-tableau of shape ν/λ. If γ is a nonempty set of inner (or outer) corners of ν/λ, then $\mathsf{AddDots}_{\gamma}(T)$ is the skew dotted increasing $\mathcal{P}$-tableau S of shape $\nu/\lambda \cup \gamma$ defined by

$$S(x) = \begin{cases} T(x), & \text{if } x \in \nu/\lambda; \\ \bullet, & \text{if } x \in \gamma. \end{cases}$$

Definition 5. Let T be a skew dotted increasing $\mathcal{P}$-tableau. For $n \in \mathbb{Z}_{>0}$, $\mathsf{Swap}_{\bullet,n}(T)$ is the skew dotted increasing $\mathcal{P}$-tableau S defined by

$$S(x) = \begin{cases} n, & \text{if } T(x) = \bullet \text{ and } T(y) = n \text{ for some } y \lessdot x; \\ \bullet, & \text{if } T(x) = n \text{ and } T(y) = \bullet \text{ for some } y \gtrdot x; \\ T(x), & \text{otherwise.} \end{cases}$$

Definition 6. Let T be a skew dotted increasing $\mathcal{P}$-tableau. Let $\mathcal{Q}$ be the subset of $\mathcal{P}$ which T maps to an integer: $\mathcal{Q} = \{x : T(x) \in \mathbb{Z}_{>0}\}$. We define $\mathsf{RemoveDots}(T) = T|_{\mathcal{Q}}$.

Definition 7. Let T be a skew increasing $\mathcal{P}$-tableau of shape ν/λ and let $\gamma \subseteq \mathrm{IC}(\nu/\lambda)$. Let $n = \max(\mathrm{Range}(T))$. The **slide** of γ in T is the skew increasing $\mathcal{P}$-tableau $\mathsf{Slide}_\gamma(T) = \mathsf{RemoveDots} \circ \mathsf{Swap}_{\bullet,n} \circ \cdots \circ \mathsf{Swap}_{\bullet,1} \circ \mathsf{AddDots}_\gamma(T)$. We also use $\mathsf{Slide}_{\gamma_1,\ldots,\gamma_n}$ to denote iterated slides: $\mathsf{Slide}_{\gamma_1,\ldots,\gamma_n}(T) = \mathsf{Slide}_{\gamma_n} \circ \cdots \circ \mathsf{Slide}_{\gamma_1}(T)$.

For a tableau T of shape ν/λ, we use the notation $\mathrm{IC}(T)$ to mean $\mathrm{IC}(\nu/\lambda)$.

Definition 8. Let T be a skew increasing $\mathcal{P}$-tableau. We define its rectification step sets, S_i, recursively. First, $S_0 = \{T\}$. Next, $S_{n+1} = \{\mathsf{Slide}_\gamma(S) : S \in S_n \text{ and } \varnothing \neq \gamma \subseteq \mathrm{IC}(S)\}$. The **rectifications** of T are the elements of the **rectification set** $\mathsf{rects}(T) = \{U : U \in S_n$ for some $n \in \mathbb{Z}_{\geq 0}$ and U is of straight shape$\}$. To denote that U is a rectification of T given by sliding the sequence of sets of inner corners $(\gamma_1, \ldots, \gamma_n)$, we write $T \xrightarrow{\gamma_1,\ldots,\gamma_n} U$.

We say a skew increasing $\mathcal{P}$-tableau **rectifies uniquely** if it has exactly one rectification. We say an increasing $\mathcal{P}$-tableau T of straight shape is a **unique rectification target (URT)** if every skew increasing $\mathcal{P}$-tableau which rectifies to T rectifies uniquely.

3 Unique rectification in slant sums and double-tailed diamonds

We first explore the structure of URTs in posets that are built out of smaller posets.

Definition 9 ([11]). Let $\mathcal{P}, \mathcal{Q}$ be disjoint posets. Assume $\mathcal{Q}$ has a minimum element $\hat{0}_\mathcal{Q}$. Let $p \in \mathcal{P}$. The **slant sum** of $\mathcal{Q}$ to $\mathcal{P}$ at p, denoted $\mathcal{P}\ {}_p/^{\hat{0}_\mathcal{Q}}\ \mathcal{Q}$ is the poset on $\mathcal{P} \sqcup \mathcal{Q}$ induced by imposing $\hat{0}_\mathcal{Q} \gtrdot p$ together with the orders on $\mathcal{P}$ and $\mathcal{Q}$. Because a poset's minimum is unique, we will usually drop the $\hat{0}_\mathcal{Q}$ and denote the slant sum as $\mathcal{P}\ {}_p/\ \mathcal{Q}$. If $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ are pairwise disjoint posets which each have minima, then $\mathcal{P}\ {}_p/\ (\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$ denotes the iterated slant sum of posets at p. Finally, given $p_1, \ldots, p_m \in \mathcal{P}$ and pairwise disjoint posets $\mathcal{Q}_i^j$ with minima, we write $\mathcal{P}\ {}_{p_1}/\ (\mathcal{Q}_1^1, \ldots, \mathcal{Q}_{r_1}^1)\ {}_{p_2}/\ \cdots\ {}_{p_m}/\ (\mathcal{Q}_1^m, \ldots, \mathcal{Q}_{r_m}^m)$ to denote the result of slant summing each $\mathcal{Q}_i^j$ onto p_j (in any order).

Definition 10. Let $\mathcal{P}$ be a poset, let $\mathcal{Q}$ be a subset of $\mathcal{P}$, and let T be a skew increasing $\mathcal{P}$-tableau. Then, the **rectifications of T restricted to $\mathcal{Q}$** are the elements of the set $\mathsf{rects}|_{\mathcal{Q}}(T) := \{U|_{\mathcal{Q}} : U \in \mathsf{rects}(T)\}$.

Definition 11. Let $\mathcal{P}$ be a poset and fix $p \in \mathcal{P}$. Let U be a URT in $\mathcal{P}$. Then U is a **p-chain unique rectification target** in $\mathcal{P}$ if U is a URT in $\mathcal{P}\ {}_{p}/\ \mathcal{C}$ for every chain poset $\mathcal{C}$. More generally, U is a **$\{p_1, \ldots, p_n\}$-chain URT** in $\mathcal{P}$ if U is a URT in $\mathcal{P}\ {}_{p_1}/\ \mathcal{C}_1\ {}_{p_2}/\ \cdots\ {}_{p_n}/\ \mathcal{C}_n$ for all pairwise disjoint chains $\mathcal{C}_1, \ldots, \mathcal{C}_n$.

We note that being a p-chain URT is a strictly stronger notion than being a URT.

Proposition 12. *Let $\mathcal{R}$ be the slant sum $\mathcal{R} := \mathcal{P}\ {}_{p_1}/\ (\mathcal{Q}_1^1, \ldots, \mathcal{Q}_{m_1}^1)\ {}_{p_2}/\ \cdots\ {}_{p_n}/\ (\mathcal{Q}_1^n, \ldots, \mathcal{Q}_{m_n}^n)$, for p_i distinct and Q_i^j all pairwise disjoint with minimum elements. Let T be a skew increasing $\mathcal{R}$-tableau with rectifications U and V. If $U|_{\mathcal{P}}$ is a $\{p_1, .., p_n\}$-chain URT in $\mathcal{P}$, then $U|_{\mathcal{P}} = V|_{\mathcal{P}}$.*

Proposition 13. *Let $\mathcal{P}$ be a poset. Let $\mathcal{R} := \mathcal{P}\ {}_{p_1}/\ (\mathcal{Q}_1^1, \ldots, \mathcal{Q}_{m_1}^1)\ {}_{p_2}/\ \cdots\ {}_{p_n}/\ (\mathcal{Q}_1^n, \ldots, \mathcal{Q}_{m_n}^n)$ for $p_i \in \mathcal{P}$ distinct and Q_i^j all pairwise disjoint with minimum elements.*

Let U be an increasing $\mathcal{R}$-tableau of straight shape. Suppose $\{p_1, \ldots, p_n\} \subseteq A \subseteq \mathcal{P}$ and $B_i^j \subseteq \mathcal{Q}_i^j$. Set $D := A \cup \left(\bigcup_{i,j} B_i^j\right)$. If $U|_{\mathcal{P}}$ is an A-chain URT in $\mathcal{P}$ and $U|_{\mathcal{Q}_i^j}$ is a B_i^j-chain URT in $\mathcal{Q}_i^j$ for each i, j, then U is a D-chain URT in $\mathcal{R}$.

We now investigate the p-chain unique rectification targets of double-tailed diamonds. This special family of d-complete posets plays a central role in the study of general d-complete posets. We will apply Proposition 14 to the general case in Section 4.

For $k \geq 3$, a **double-tailed diamond** $\mathcal{D}(k)$ has $2k - 2$ elements, two of which are incomparable elements in the middle with chains of size $k - 2$ above and below them. It is easy to work out that any increasing tableau on any order ideal of a double-tailed diamond is a URT. (This is explicitly observed in [3, Proof of Theorem 3.12].) For application in Section 4, we strengthen this observation to the setting of p-chain URTs.

Proposition 14. *Every increasing $\mathcal{D}(n)$-tableau of straight shape is an $\{\ell_1, r_1\}$-chain URT.*

4 d-complete posets and minuscule posets

In this section, we recall the context of the d-complete posets of [12], and prove our main result Theorem 2 regarding slant sum trees of minuscule posets. (Note that the paper [11] uses a dual convention, so the posets in [11] are the duals of ours.) We also develop appropriate terminology here to give precise interpretations of Conjectures 1 and 3.

Briefly, the algebraic context of d-complete posets is as follows. (For further details, see [4, 12, 17].) Let Λ be a dominant integral weight of a Kac–Moody Lie algebra $\mathfrak{g}$ with (generally infinite) Weyl group W. We call $w \in W$ **Λ-minuscule** if it can be written as a reduced word in the simple reflections $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$, so that for all j, we have $(s_{i_{j+1}} \cdots s_{i_\ell} - s_{i_j} \cdots s_{i_\ell})\Lambda = \alpha_{i_j}$, where α_{i_j} is the simple root for s_{i_j}. (In fact, this property is independent of the choice of reduced word [18, Proposition 2.1].) Now, if w is Λ-minuscule, then the interval $[\mathrm{id}, w]$ in Bruhat order is a distributive lattice. A poset $\mathcal{P}$ is d-complete if and only if it is isomorphic to the poset of join irreducibles of such a 'Λ-minuscule distributive lattice'; equivalently, a poset $\mathcal{Q}$ is isomorphic to a Bruhat interval $[\mathrm{id}, w]$ for some Λ-minuscule w if and only if $\mathcal{Q}$ is isomorphic to the poset of order ideals of a d-complete poset. Since Bruhat order on W also describes containment of Schubert varieties in the Kac–Moody homogeneous space $X = G/B$, we have for $u, v \leq w$ all Λ-minuscule that the inclusion of Schubert varieties $X_u \subseteq X_v$ is equivalent to the reverse inclusion $\lambda_v \subseteq \lambda_u$ of the corresponding order ideals in the d-complete poset for w. In addition to their algebraic relations, d-complete posets enjoy a number of beautiful combinatorial properties, including an analogue of the classical hook-length formula (see, e.g., [8, 9, 13, 14, 15]); for a bibliography of work on d-complete posets, see [15, §12]. Figure 1 shows an example of a d-complete poset.

Say a d-complete poset is **irreducible** if it is not the slant sum of two d-complete posets. Proctor [11] showed that all d-complete posets can be uniquely decomposed as a slant sum of irreducible d-complete components. In this decomposition, irreducible components are only slant summed onto special nodes of other irreducible components, called *acyclic nodes* [11]; that is, if $\mathcal{P} = \mathcal{Q}\,{}_q/\,\mathcal{R}$ is d-complete and $\mathcal{R}$ is irreducible, then q is an acyclic node of its irreducible component. (It is sufficient for our purposes to use Proctor's explicit identification [11] of all acyclic nodes of all irreducible d-complete posets.) The irreducible d-complete posets are classified into 15 (mostly infinite) families by Proctor and Stembridge [11, 18]; we follow Proctor's numbering and naming conventions for these families from [11]. Of these 15 families, only the components from families 1–9 and 11 have acyclic nodes.

For a poset $\mathcal{P}$, we say an increasing $\mathcal{P}$-tableau T of straight shape $\lambda \subseteq \mathcal{P}$ is **minimally-labeled** if it is minimal among all increasing $\mathcal{P}$-tableaux of shape λ under nodewise comparison of labels; that is, if U is another increasing tableau of shape λ, then $U(x) \geq T(x)$ for all $x \in \lambda$. It is easy to see that there exists a unique minimally-labeled $\mathcal{P}$-tableau of each straight shape λ. We write M_λ for this unique tableau. The precise version of Conjecture 1 is the following.

Conjecture 15. *Let $\mathcal{P}$ be d-complete and let $\lambda \subseteq \mathcal{P}$ be an order ideal. Then, the minimally-labeled increasing $\mathcal{P}$-tableau M_λ of shape λ is a unique rectification target.*

In light of the slant sum structure of d-complete posets, Conjecture 15 would follow from Proposition 13 together with information about (p-chain) URTs in the 15 families of irreducible d-complete posets. Specifically, it remains to show that

- for each irreducible d-complete poset $\mathcal{Q}$ with acyclic nodes, that M_λ is a p-chain URT for each order ideal $\lambda \subseteq \mathcal{Q}$ and each acyclic node $p \in \mathcal{Q}$, and that

- for each irreducible d-complete poset $\mathcal{Q}$ without acyclic nodes, that M_λ is a URT for each order ideal $\lambda \subseteq \mathcal{Q}$.

Unfortunately, we are unable to establish the necessary results for some of these families; hence, we can only leverage Proposition 13 to prove a weaker version of Conjecture 15, namely Theorem 2. First, we recall the *minuscule posets*, a special subset of d-complete posets. Except for some trivial instances, all minuscule posets are irreducible.

Algebraically, one obtains the minuscule posets as follows. Suppose the Kac–Moody group G is in fact complex reductive. Put a partial order on the positive roots Φ^+ of G by taking the transitive closure of the covering relation $\alpha \lessdot \beta$ if and only if $\beta - \alpha$ is a simple root. The simple root δ is a **minuscule root** if for every positive root $\alpha \in \Phi^+$, the multiplicity of $\delta^\vee$ in the simple coroot expansion of $\alpha^\vee$ is at most 1. For each minuscule root, one obtains a corresponding **minuscule poset** $\mathcal{P}_\delta$ by restricting the partial order on $(\Phi^+)^\vee$ to those positive coroots that use $\delta^\vee$ in their simple coroot expansion. There is also a corresponding **minuscule variety** obtained as the quotient G/P_δ, where P_δ is the maximal parabolic subgroup associated to the minuscule root δ. The minuscule poset $\mathcal{P}_\delta$ encodes the Schubert stratification of G/P_δ; specifically, the Schubert varieties are naturally indexed by the order ideals of $\mathcal{P}_\delta$, and inclusions of order ideals correspond to reverse inclusions of Schubert varieties.

Combinatorially, the minuscule posets are completely classified. Minuscule posets consist of three infinite families together with a pair of exceptional examples. This classification is given in Table 1, with examples shown in Figure 2. One infinite family of minuscule posets is the **rectangles**; combinatorially, these are the products $\mathcal{C}_i \times \mathcal{C}_j$ of two chain posets. Another infinite family is the double-tailed diamonds studied earlier. The final infinite family is the **shifted staircases**; identifying the chain $\mathcal{C}_i$ with the natural order on $\{1, \ldots, i\}$, shifted staircases are of the form $\{(x_1, x_2) \in \mathcal{C}_i \times \mathcal{C}_i : x_1 \geq x_2\}$, with the order structure restricted from $\mathcal{C}_i \times \mathcal{C}_i$. For convenience, we will assume that shifted staircases have at least 10 nodes, as the smaller shifted staircases coincide with small rectangles/double-tailed diamonds. Lastly, for the definitions of the exceptional **Cayley-Moufang swivel** and **bat**, see their Hasse diagrams depicted in the Figure 2. The acyclic nodes of the minuscule posets are also shown in Figure 2; we will use the indexing of these nodes as L and R, as in that figure.

Here, we will only use the following lemma in the case $k = 1$ of rectangles; however, we note that it is equally true for four of Proctor's other families, as listed.

Lemma 16. *Let $k \in \{1, 3, 5, 6, 7\}$. Let $\mathcal{P}$ be an irreducible d-complete poset from family k and let $A \subseteq \mathcal{P}$ be the set of all acyclic nodes in $\mathcal{P}$. If a straight-shaped increasing $\mathcal{P}$-tableau U is a URT for all posets in family k, then U is a A-chain URT in all such posets.*

Theorem 17 ([3])**.** *Let $\mathcal{P}$ be a minuscule poset. Then, for every order ideal $\lambda \subseteq \mathcal{P}$, the minimally-labeled increasing $\mathcal{P}$-tableau M_λ of shape λ is a URT in $\mathcal{P}$.* □

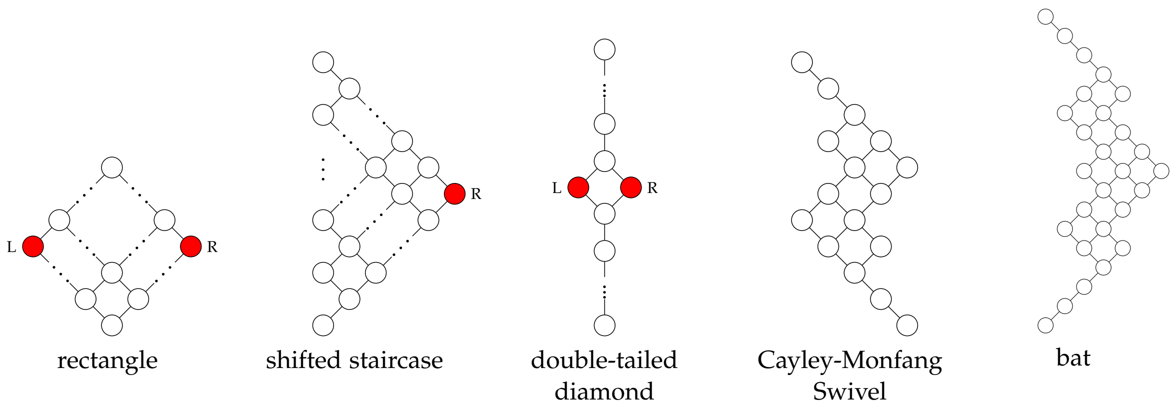

Figure 2: Examples of the 5 families of minuscule posets. The labeled red nodes mark the acyclic nodes of these posets. The last two posets have no acyclic nodes.

Minuscule poset	Minuscule variety	Irreducible d-complete classification
rectangle	Grassmannian	shapes (family 1)
shifted staircase	orthogonal Grassmannian	shifted shapes (family 2)
double-tailed diamond	quadric hypersurface	insets (family 4–special case)
Cayley-Moufang swivel	octonion projective plane	swivels (family 8–special case)
bat	Freudenthal variety	bat (family 15)

Table 1: The 5 families of minuscule posets are named in the first column. The second column identifies the corresponding minuscule homogeneous space. The third column shows how the minuscule posets fall into R. Proctor's classification of irreducible d-complete posets from [11].

Corollary 18. *Let $\mathcal{P}$ be a rectangle. Let M_λ be an minimally-labeled $\mathcal{P}$-tableau of straight shape. Then, M_λ is an $\{L, R\}$-chain URT in $\mathcal{P}$.*

Corollary 19. *Let $\mathcal{P}$ be a shifted staircase with at least 10 nodes. If M_λ is a minimally-labeled $\mathcal{P}$-tableau of straight shape, then M_λ is an $\{R\}$-chain URT in $\mathcal{P}$.*

In order to state the following, we adopt the convention that an *$\emptyset$-chain URT* in $\mathcal{P}$ is just a URT in $\mathcal{P}$.

Proposition 20. *Let $\mathcal{P}$ be a minuscule poset. Let A be the set of acyclic nodes in $\mathcal{P}$. Let M_λ be a minimally-labeled increasing $\mathcal{P}$-tableau of straight shape. Then, M_λ is an A-chain URT in $\mathcal{P}$.*

Proposition 13 allows us to extend Proposition 20 to show that minimally-labeled tableaux are unique rectification targets in iterated slant sums of minuscule posets.

Theorem 21. *Let $\mathcal{P}$ be a d-complete poset. If $\mathcal{P}$ is an iterated slant sum of minuscule posets, then all minimally-labeled increasing $\mathcal{P}$-tableaux of straight shape are unique rectification targets.*

The following is the precise version of Theorem 2.

Corollary 22. *Let $\mathcal{P}$ be a d-complete poset. If $\mathcal{P}$ is an iterated slant sum of minuscule posets and $\mathcal{Q} \subseteq \mathcal{P}$ is an order ideal, then all minimally-labeled increasing $\mathcal{Q}$-tableaux of straight shape are unique rectification targets.*

Finally, we recall the construction necessary to make precise sense of Conjecture 3. Let $\mathcal{P}$ be any poset satisfying the conclusion of Conjecture 15. Then, as in [3, §3.5], we construct a *combinatorial K-theory ring* associated to $\mathcal{P}$. Let $K(\mathcal{P})$ be the free abelian group on the set of order ideals of $\mathcal{P}$. Define a product structure on $K(\mathcal{P})$ by setting $\lambda \cdot \mu := \sum_\nu t^\nu_{\lambda,\mu}\, \nu$, where the Greek letters denote order ideals of $\mathcal{P}$ and $t^\nu_{\lambda,\mu}$ is defined to be $(-1)^{|\nu|-|\lambda|-|\mu|}$ times the number of skew increasing $\mathcal{P}$-tableaux of shape ν/λ that rectify to the minimally-labeled tableau M_μ. (Since M_μ is by hypothesis a URT in $\mathcal{P}$, this number is well-defined.) By [3, Proposition 3.17], this product structure makes $K(\mathcal{P})$ into a commutative associative algebra with the empty order ideal as multiplicative identity. Conjecture 3 claims then that, when $\mathcal{P}$ is d-complete, the structure constants of the algebra $K(\mathcal{P})$ coincide with corresponding Λ-minuscule Schubert structure constants of the K-theory ring $K(X)$, where $X = G/P$ is a Kac–Moody homogeneous space, $w \in W^P$ is a Λ-minuscule Weyl group element for P, and $\mathcal{P}$ is the poset of join irreducibles of the distributive lattice $[\mathrm{id}, w]$.

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