Séminaire Lotharingien de Combinatoire **82B** (2019)
Article #29, 12 pp.

Proceedings of the 31st Conference on Formal Power Series and Algebraic Combinatorics (Ljubljana)

Stanley symmetric functions for signed involutions

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Abstract. Involution words are variations of reduced words associated to twisted involutions in Coxeter groups. These words are saturated chains in a partial order first considered by Richardson and Springer in their study of symmetric varieties. In the symmetric group, involution words can be enumerated in terms of tableaux using appropriate analogues of the symmetric functions introduced by Stanley to count reduced words. We adapt this approach to the group of signed permutations. We show that the involution words for the longest element in the Coxeter group C_n are in bijection with reduced words for the longest element in $A_n = S_{n+1}$, which are known to be in bijection with standard tableaux of shape $(n, n-1, \ldots, 2, 1)$.

Keywords: Signed permutations, Stanley symmetric functions, transition equations

1 Introduction

Let W be a Coxeter group with simple generating set S. A *reduced word* for $w \in W$ is a minimal-length sequence $(r_1, r_2, \ldots, r_\ell)$ of simple generators $r_i \in S$ with $w = r_1 r_2 \cdots r_\ell$. Let $\mathcal{R}(w)$ be the set of reduced words for w.

Of primary interest are the finite Coxeter groups of classical types A and B/C. Fix an integer $n \geq 1$ and let $[n] = \{1, 2, \ldots, n\}$ and $[\pm n] = \{\pm 1, \pm 2, \ldots, \pm n\}$. Let $A_n = S_{n+1}$ be the group of permutations of $[n+1]$. Let C_n be the group of permutations w of $[\pm n]$ with $w(-i) = -w(i)$ for all i. Define $s_1, s_2, \ldots, s_n \in A_n$ and $t_0, t_1, \ldots, t_{n-1} \in C_n$ by

$$s_i = (i, i{+}1), \quad t_0 = (-1, 1), \quad \text{and} \quad t_i = (-i{-}1, -i)(i, i{+}1) \text{ for } i \neq 0. \tag{1.1}$$

Then A_n is a Coxeter group relative to the generating set $S = \{s_1, s_2, \ldots, s_n\}$ while C_n is a Coxeter group relative to the generating set $S = \{t_0, t_1, \ldots, t_{n-1}\}$. We refer to elements of C_n as *signed permutations*.

Each finite Coxeter group contains a unique element of maximal length, where the *length* of an element w refers to the common length of any word in $\mathcal{R}(w)$. Let w_n^A and w_n^C denote the longest elements of A_n and C_n. Then w_n^A is the permutation $i \mapsto n+2-i$ while w_n^C is the negation map $i \mapsto -i$. There are attractive product formulas for the number of reduced words for both of these permutations:

$$\left|\mathcal{R}(w_n^A)\right| = \frac{\binom{n+1}{2}!}{\prod_{i=1}^{n}(2i-1)^i} \qquad \text{and} \qquad \left|\mathcal{R}(w_n^C)\right| = \frac{(n^2)!}{n^n \prod_{i=1}^{n-1}[i(2n-i)]^i}. \tag{1.2}$$

Stanley proved the first of these identities [13, Corollary 4.3] and conjectured the second, which was later shown by Haiman [4, Theorem 5.12].

Let $\mathrm{SYT}(\lambda)$ be the set of *standard Young tableaux* of shape λ. Define $\delta_n = (n, n-1, \dots, 2, 1)$ and write (n^n) for the partition with n parts of size n. The identities (1.2) are equivalent to $\left|\mathcal{R}\left(w_n^A\right)\right| = |\mathrm{SYT}(\delta_n)|$ and $\left|\mathcal{R}\left(w_n^C\right)\right| = |\mathrm{SYT}((n^n))|$ via the well-known hook-length formula [14, Corollary 7.21.6]. As one would expect from this formulation, there are natural bijective proofs of the identities (1.2), due to Edelman and Greene [3] in type A and to Haiman [4] and Kraśkiewicz [8] in type C.

The main result of this paper is a product formula similar to (1.2) for the cardinality of a set of reduced-word-like objects associated to w_n^C. Write $\ell : W \to \mathbb{N}$ for the length function of (W, S) and let $\mathcal{I}(W) = \{y \in W : y = y^{-1}\}$ be the set of involutions in W. There is a unique associative product $\circ : W \times W \to W$ satisfying $s \circ s = s$ for any $s \in S$ and $u \circ v = uv$ for any $u, v \in W$ with $\ell(uv) = \ell(u) + \ell(v)$, and it can be shown that every element $y \in \mathcal{I}(W)$ has the form

$$y = r_\ell \circ (\cdots \circ (r_2 \circ (r_1 \circ 1 \circ r_1) \circ r_2) \circ \cdots) \circ r_\ell \tag{1.3}$$

for some sequence of simple generators $r_i \in S$. A sequence $(r_1, r_2, \dots, r_\ell)$ of shortest possible length satisfying (1.3) is an *involution word* for y. Let $\hat{\mathcal{R}}(y)$ be the set of involution words for $y \in \mathcal{I}(W)$. This set is always nonempty, with $\hat{\mathcal{R}}(1) = \{\varnothing\}$.

Example 1.1. In C_2, we have $t_0 \circ (t_1 \circ (t_0 \circ 1 \circ t_0) \circ t_1) \circ t_0 = t_0 \circ (t_1 \circ t_0 \circ t_1) \circ t_0 = t_0 \circ t_1 t_0 t_1 \circ t_0 = t_0 t_1 t_0 t_1 = t_1 t_0 t_1 t_0 = w_2^C$ and $t_1 \circ (t_0 \circ (t_1 \circ 1 \circ t_1) \circ t_0) \circ t_1 = w_2^C$ and it holds that $\hat{\mathcal{R}}(w_2^C) = \{(t_0, t_1, t_0), (t_1, t_0, t_1)\}$.

Let $p = \lfloor \frac{n}{2} \rfloor$ and $q = \lceil \frac{n}{2} \rceil$. In [7], the authors and Hamaker showed that

$$|\hat{\mathcal{R}}(w_n^A)| = \binom{\binom{p+1}{2} + \binom{q+1}{2}}{\binom{p+1}{2}} |\mathrm{SYT}(\delta_p)| |\mathrm{SYT}(\delta_q)| \tag{1.4}$$

and conjectured the following theorem, which is our main new result.

Theorem 1.2. It holds that $|\hat{\mathcal{R}}(w_n^C)| = |\mathrm{SYT}(\delta_n)| = |\mathcal{R}(w_n^A)|$.

There is an algebraic approach to enumerating $\mathcal{R}(w_n^A)$, $\mathcal{R}(w_n^C)$, $\hat{\mathcal{R}}(w_n^A)$, and $\hat{\mathcal{R}}(w_n^C)$ by means of certain generating functions called *Stanley symmetric functions*. We write $[x_1 x_2 \cdots] f$ for the coefficient of a square-free monomial in a homogeneous symmetric function f. The Stanley symmetric functions of interest have the following properties:

- Each (type A) Stanley symmetric function F_w has $[x_1 x_2 \cdots] F_w = |\mathcal{R}(w)|$.
- Each (type C) Stanley symmetric function G_w has $[x_1 x_2 \cdots] G_w = 2^{\ell(w)} |\mathcal{R}(w)|$.

- Each (type A) involution Stanley symmetric function $\hat{F}_y$ is a multiplicity-free sum of certain instances of F_w, and it holds that $[x_1 x_2 \cdots]\hat{F}_y = |\hat{\mathcal{R}}(y)|$.
- Each (type C) involution Stanley symmetric function $\hat{G}_y$ is a multiplicity-free sum of certain instances of G_w, and it holds that $[x_1 x_2 \cdots]\hat{G}_y = 2^{\hat{\ell}(y)}|\hat{\mathcal{R}}(y)|$.

There are expressions for $F_{w_n^A}$, $G_{w_n^C}$, and $\hat{F}_{w_n^A}$ as *Schur functions* s_λ, *Schur Q-functions* Q_λ, and *Schur S-functions* S_λ. For the definitions of these symmetric functions, see Section 2. The identities (1.2) and (1.4) are corollaries of the following formulas:

Theorem 1.3 (Stanley [13])**.** $F_{w_n^A} = s_{\delta_n}$.

Theorem 1.4 (Worley [15]; Billey and Haiman [2])**.** $G_{w_n^C} = Q_{(2n-1,2n-3,\ldots,3,1)} = S_{(n^n)}$.

Theorem 1.5 (Hamaker, Marberg, and Pawlowski [6])**.** $\hat{F}_{w_n^A} = 2^{-q}Q_{(n,n-2,n-4,\ldots)} = s_{\delta_p}s_{\delta_q}$.

Theorem 1.2, in turn, is an immediate corollary of the following result, which adds an entry for $\hat{G}_{w_n^C}$ to the preceding sequence of identities.

Theorem 1.6. $\hat{G}_{w_n^C} = G_{w_n^A} = S_{\delta_n}$.

We relegate most technical arguments in this extended abstract to the full length article [12], but sketch the outline of some proofs. In particular, our strategy for proving Theorem 1.6 is as follows.

One can write $\hat{G}_{w_n^C}$ as a sum $\sum_{v\in\mathcal{A}_n} G_v$ indexed by a certain set $\mathcal{A}_n$ of signed permutations $v \in C_n$, the *atoms* of w_n^C. The transition equations of Lascoux-Schützenberger as adapted by Billey [1] generate various identities between sums of type C Stanley symmetric functions. Work of Lam [10] implies that $G_{w_n^A} = S_{\delta_n}$, and we show that one can apply a specific sequence of transition equations to rewrite $G_{w_n^A}$ as exactly the sum $\sum_{v\in\mathcal{A}_n} G_v$. The fact that this is possible is somewhat miraculous. It is an intriguing open problem to find bijective or geometric proofs of our results.

2 Preliminaries

Fix a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$. The *Young diagram* of λ is the set of pairs $D_\lambda = \{(i,j) : i \in [k] \text{ and } j \in [\lambda_i]\}$, which we envision as a collection of left-justified boxes oriented as in a matrix. A *semistandard tableau* of shape λ is a filling of the boxes of the Young diagram D_λ by positive integers, such that each row is weakly increasing from left to right and each column is (strictly) increasing from top to bottom. Such a tableau is *standard* if its boxes contain exactly the numbers $1, 2, \ldots, |\lambda|$.

Similarly, a *marked semistandard tableau* of shape λ is a filling of the Young diagram of λ by numbers from the alphabet of primed and unprimed positive integers $\{1, 2, 3, \ldots\} \sqcup$

$\{1', 2', 3', \dots\}$ such that (i) the rows and columns are weakly increasing under the order $1' < 1 < 2' < 2 < \cdots$, (ii) no unprimed letter i appears twice in the same column, and (iii) no primed letter i' appears twice in the same row.

Assume λ is a strict partition, i.e., has all distinct parts. A *marked semistandard shifted tableau* of shape λ is a filling of the *shifted Young diagram* $\{(i, i+j-1) : (i,j) \in D_\lambda\}$ with primed and unprimed positive integers satisfying properties (i)-(iii) from the previous paragraph. A semistandard marked (shifted) tableau T of shape λ is *standard* if exactly one of i or i' appears in T for each $i = 1, 2, \dots, |\lambda|$.

Given a tableau T, write x^T for the monomial formed by replacing the boxes in T containing i or i' by x_i and then multiplying the resulting variables.

Example 2.1. If T, U, and V are the tableaux of shape $\lambda = (4,3,1)$ given by

$$T = \begin{array}{|c|c|c|c|}\hline 2 & 2 & 2 & 3 \\ \hline 3 & 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-3} 5 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array} \qquad \text{and} \qquad U = \begin{array}{|c|c|c|c|}\hline 1' & 1 & 1 & 3 \\ \hline 1' & 3 & 4' & \multicolumn{1}{c}{} \\ \cline{1-3} 5 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array} \qquad \text{and} \qquad V = \begin{array}{|c|c|c|c|}\hline 1 & 2' & 3 & 3 \\ \hline \multicolumn{1}{c|}{} & 2' & 4 & 6 \\ \cline{2-4} \multicolumn{2}{c|}{} & 5 & \multicolumn{1}{c}{} \\ \cline{3-3}\end{array}$$

then T is semistandard, U is marked and semistandard, and V is marked, semistandard, and shifted. We have $x^T = x_2^3 x_3^3 x_4 x_5$ and $x^U = x_1^4 x_3^2 x_4 x_5$ and $x^V = x_1 x_2^2 x_3^2 x_4 x_5 x_6$.

Definition 2.2. Let λ be a partition and let μ be a strict partition. The *Schur function* of λ, the *Schur S-function* of λ, and the *Schur Q-function* of μ are then the respective sums

$$s_\lambda = \sum_T x^T, \qquad S_\lambda = \sum_U x^U, \qquad \text{and} \qquad Q_\mu = \sum_V x^V$$

where T runs over semistandard tableaux of shape λ, U runs over semistandard marked tableaux of shape λ, and V runs over marked semistandard shifted tableaux of shape μ.

It is well-known that the Schur functions s_λ, with λ ranging over all partitions, form a basis for the algebra Λ of symmetric functions. Similarly, the Schur Q-functions Q_μ, with μ ranging over all strict partitions, form a basis for the subalgebra $\Gamma \subset \Lambda$ generated by the odd-indexed power sum symmetric functions. Each Schur Q-function is itself Schur-positive, i.e., an $\mathbb{N}$-linear combination of Schur functions.

The set of Schur S-functions, with λ ranging over all partitions, is not linearly independent, but also spans the subalgebra Γ. The set $\{S_\lambda : \lambda \text{ is a strict partition}\}$ is a basis for Γ. For more properties of these functions, see [11, Chaper I, §3] (for s_λ), [11, Chapter III, §8] (for Q_λ), and [11, Chapter III, §8, Ex. 7] (for S_λ).

Next, we review the definitions of F_w, G_w, and $\hat{F}_y$ from [1, 2, 7, 13].

Definition 2.3. The *type A Stanley symmetric function* associated to $w \in A_n = S_{n+1}$ is

$$F_w = \sum_{\mathbf{a} \in \mathcal{R}(w)} \sum_{(i_1 \le i_2 \le \cdots \le i_l) \in \mathcal{C}(\mathbf{a})} x_{i_1} x_{i_2} \cdots x_{i_l}$$

where if $\mathbf{a} = (s_{a_1}, s_{a_2}, \cdots, s_{a_l})$, then $\mathcal{C}(\mathbf{a})$ is the set of all weakly increasing sequences of positive integers $i_1 \le i_2 \le \cdots \le i_l$ such that if $a_j > a_{j+1}$ for $1 \le j < l$ then $i_j < i_{j+1}$.

Definition 2.4. The *type C Stanley symmetric function* associated to $w \in C_n$ is

$$G_w = \sum_{\mathbf{a}\in\mathcal{R}(w)} \sum_{(i_1 \le i_2 \le \cdots \le i_l)\in\mathcal{D}(\mathbf{a})} 2^{|\{i_1,i_2,\ldots,i_l\}|} x_{i_1} x_{i_2} \cdots x_{i_l}$$

where if $\mathbf{a} = (t_{a_1}, t_{a_2}, \cdots, t_{a_l})$, then $\mathcal{D}(\mathbf{a})$ is the set of all weakly increasing sequences of positive integers $i_1 \le i_2 \le \cdots \le i_l$ such that if $a_{j-1} < a_j > a_{j+1}$ for $1 < j < l$ then either $i_{j-1} < i_j \le i_{j+1}$ or $i_{j-1} \le i_j < i_{j+1}$.

Let (W, S) be any Coxeter system. Recall the definition of $\circ$ from the introduction.

Definition 2.5. For each $y \in \mathcal{I}(W) = \{z \in W : z = z^{-1}\}$ let $\mathcal{A}(y)$ be the set of elements $w \in W$ of minimal length with $w^{-1} \circ w = y$. We refer to elements of this set as *atoms*.

Definition 2.6. The *type A* and *type C involution Stanley symmetric functions* associated to $y \in \mathcal{I}(A_n)$ and $z \in \mathcal{I}(C_n)$ are $\hat{F}_y = \sum_{w\in\mathcal{A}(y)} F_w$ and $\hat{G}_z = \sum_{w\in\mathcal{A}(z)} G_w$, respectively.

Each F_w is an $\mathbb{N}$-linear combination of Schur functions [3]. Each G_w is an $\mathbb{N}$-linear combination of Schur Q-functions [10, Theorem 3.12]. It follows that $\hat{F}_y$ and $\hat{G}_z$ are likewise Schur-positive and Schur-Q-positive. For $\hat{F}_y$, a stronger statement holds: if $\kappa(y)$ is the number of 2-cycles in $y \in \mathcal{I}(A_n)$, then $2^{\kappa(y)}\hat{F}_y$ is also Schur-Q-positive [6, Corollary 4.62]. We do not know if $\hat{G}_z$ has any stronger positivity property along these lines.

Let $\iota : A_{n-1} \hookrightarrow C_n$ be the unique group homomorphism with $\iota(s_i) = t_i$ for $i \in [n-1]$, and define $G_w = G_{\iota(w)}$ for $w \in A_{n-1}$. To relate F_w and G_w for $w \in A_{n-1}$, recall that $\Lambda = \mathbb{Q}\text{-span}\{s_\lambda\}$ and $\Gamma = \mathbb{Q}\text{-span}\{S_\lambda\}$ where both spans are over all partitions λ. The second space is also given by $\Gamma = \{Q_\mu : \mu \text{ is a strict partition}\}$. The *superfication map* is the linear map $\phi : \Lambda \to \Gamma$ with $\phi(s_\lambda) = S_\lambda$ for all partitions λ.

Theorem 2.7 (Lam [9])**.** If $w \in A_{n-1}$ then $G_w = \phi(F_w)$.

A *reflection* in a Coxeter group is an element conjugate to a simple generator. With our notation as in [1, §3], the reflections in C_n are the elements $s_{ii} = (i, \bar{i})$ for $i \in [n]$ along with $s_{ij} = s_{ji} = (i, \bar{j})(\bar{i}, j)$ and $t_{ij} = t_{ji} = (i, j)(\bar{i}, \bar{j})$ for $i, j \in [n]$ with $i < j$. If $t \in C_n$ is a reflection and $u, v \in C_n$ are such that $v = ut$ and $\ell(v) = \ell(u) + 1$, then we write $u \lessdot v$; then $\lessdot$ is the covering relation of the *Bruhat order* of C_n. For $w \in C_n$ and $j \in [n]$, define

$$\mathcal{S}_j(w) = \{ws_{ij} : i \in [n],\ w \lessdot ws_{ij}\} \subseteq C_n,$$

$$\mathcal{T}_j^-(w) = \{wt_{ij} : 1 \le i < j,\ w \lessdot wt_{ij}\} \subseteq C_n,$$

$$\mathcal{T}_j^+(w) = \{wt_{jk} : j < k \le n+1,\ w \lessdot wt_{jk}\} \subseteq C_{n+1}.$$

We refer to the following technical result as the *transition formula* for G_w.

Theorem 2.8 (Billey [1])**.** If $w \in C_n$ and $j \in [n]$ then $\displaystyle\sum_{v\in\mathcal{T}_j^+(w)} G_v = \sum_{v\in\mathcal{S}_j(w)} G_v + \sum_{v\in\mathcal{T}_j^-(w)} G_v.$

3 Atoms and quasi-atoms

Let $S \subset \mathbb{Z}$ be a set of integers. A *perfect matching* on a set S is a set of pairwise disjoint 2-element subsets $\{i, j\}$, referred to as *blocks*, whose union is S. A perfect matching M is *symmetric* if $\{i, j\} \in M$ implies $-\{i, j\} \overset{\text{def}}{=} \{-i, -j\} \in M$, and *noncrossing* if it does not occur that $i < a < j < b$ for any blocks $\{i, j\}, \{a, b\} \in M$. Let $\mathsf{NCSP}(n)$ denote the set of noncrossing, symmetric, perfect matchings on the set $[\pm n]$. The three elements of $\mathsf{NCSP}(3)$ are $\{\{\pm 1\}, \{\pm 2\}, \{\pm 3\}\}$, $\{\pm\{1, 2\}, \{\pm 3\}\}$, and $\{\{\pm 1\}, \pm\{2, 3\}\}$.

For us, a *word* is a finite sequence of nonzero integers. The *one-line representation* of $w \in C_n$ is the word $w_1 w_2 \cdots w_n$ where $w_i = w(i)$. We usually write $\overline{m}$ in place of $-m$ so that, for example, the elements of C_2 are 12, $\overline{1}2$, $1\overline{2}$, $\overline{1}\,\overline{2}$, 21, $\overline{2}1$, $2\overline{1}$, and $\overline{2}\,\overline{1}$. If $w = w_1 w_2 \cdots w_n$ is a word then we write $[[w]]$ for the subword formed by omitting each repeated letter after its first appearance. For example, $[[312311243]] = 3124$. Suppose M is a symmetric, noncrossing, perfect matching on a subset of $[\pm n]$. Define

$$\mathrm{Pair}(M) = \{(a, -b) : \{a, b\} \in M, 0 < a < b\} \sqcup \{(-a, -a) : \{-a, a\} \in M, 0 < a\}.$$

Let $(a_1, b_1), (a_2, b_2), \ldots, (a_l, b_l)$ and $(c_1, d_1), (c_2, d_2), \ldots, (c_l, d_l)$ be the elements of $\mathrm{Pair}(M)$ listed in order such that $b_1 < b_2 < \cdots < b_l$ and $c_1 < c_2 < \cdots < c_l$, and define

$$\alpha_{\min}(M) = [[a_1 b_1 a_2 b_2 \cdots a_l b_l]] \quad \text{and} \quad \alpha_{\max}(M) = [[c_1 d_1 c_2 d_2 \cdots c_l d_l]].$$

If $M \in \mathsf{NCSP}(n)$, then $\alpha_{\min}(M)$ and $\alpha_{\max}(M)$ contain exactly one letter from $\{\pm i\}$ for each $i \in [n]$, so are the one-line representations of elements of C_n. If u and v are words, both with $n \geq i + 2$ letters, then we write $u \lessdot_i v$ to mean that

$$u_i u_{i+1} u_{i+2} = cab, \quad v_i v_{i+1} v_{i+2} = bca, \quad \text{and} \quad u_j = v_j \text{ for } j \notin \{i, i+1, i+2\} \tag{3.1}$$

for some $a < b < c$. Let $<_{\mathcal{A}}$ be the transitive closure of the relations $\lessdot_i$ for all $i \geq 1$. We apply $<_{\mathcal{A}}$ to signed permutations in one-line notation. Finally, define $\mathcal{A}_n = \mathcal{A}(w_n^{\mathsf{C}}) \subset C_n$ and $\mathcal{A}_M = \{w \in C_n : \alpha_{\min}(M) \leq_{\mathcal{A}} w \leq_{\mathcal{A}} \alpha_{\max}(M)\}$ for $M \in \mathsf{NCSP}(n)$.

Example 3.1. If $M = \{\{\pm 1\}, \pm\{2, 3\}, \pm\{4, 5\}\} \in \mathsf{NCSP}(5)$ then the interval $\mathcal{A}_M$ is

$$\begin{tikzcd}[column sep=small]
& \overline{1}2\overline{3}4\overline{5} \arrow[dl, dash] \arrow[dr, dash] & \\
\overline{1}245\overline{5}\overline{3} \arrow[d, dash] & & 2\overline{3}\,\overline{1}4\overline{5} \arrow[d, dash] \\
\overline{1}4\overline{5}2\overline{3} \arrow[d, dash] & & 2\overline{3}4\overline{5}\,\overline{1} \arrow[d, dash] \\
4\overline{5}\,\overline{1}2\overline{3} \arrow[dr, dash] & & 24\overline{5}\,\overline{3}\,\overline{1} \arrow[dl, dash] \\
& 4\overline{5}2\overline{3}\,\overline{1} &
\end{tikzcd}$$

Theorem 3.2 (Hamaker and Marberg [5]). It holds that $\mathcal{A}_n = \bigsqcup_{M \in \mathsf{NCSP}(n)} \mathcal{A}_M$.

For example, we have $\mathcal{A}_L = \{\overline{1}2\overline{3}, 2\overline{3}\overline{1}\}$, $\mathcal{A}_M = \{\overline{3}1\overline{2}\}$, and $\mathcal{A}_N = \{\overline{3}\overline{2}\overline{1}\}$ for the matchings $L = \{\{\pm 1\}, \pm\{2,3\}\}$, $M = \{\pm\{1,2\}, \{\pm 3\}\}$, and $N = \{\{\pm 1\}, \{\pm 2\}, \{\pm 3\}\}$, and $\mathcal{A}_3 = \mathcal{A}_L \sqcup \mathcal{A}_M \sqcup \mathcal{A}_N$. For an atom $w \in \mathcal{A}_n$, define $M(w)$ to be the unique noncrossing, symmetric, perfect matching in $\mathsf{NCSP}(n)$ with $w \in \mathcal{A}_{M(w)}$.

Given a word $w = w_1 w_2 \cdots w_n$ such that $|w_1|$, $|w_2|$, ..., $|w_n|$ are distinct and nonzero, define $\mathrm{fl}_\pm(w) \in C_n$ to be the signed permutation whose one-line representation is formed by replacing each letter of w by its image under the order-preserving bijection $\{\pm w_1, \pm w_2, \ldots, \pm w_n\} \to [\pm n]$. For example, $\mathrm{fl}_\pm(3\overline{2}5\overline{7}) = 2\overline{1}3\overline{4} \in C_4$. If M is a partition of a symmetric $2n$-element subset $X = -X \subset [\pm m]$, then define $\mathrm{fl}_\pm(M)$ to be the partition of $[\pm n]$ formed by replacing each element of each block of M by its image under the order-preserving bijection $X \to [\pm n]$.

Suppose $w \in C_n$ and $v = \mathrm{fl}_\pm(w_2 w_3 \cdots w_n) \in \mathcal{A}_{n-1}$. Define $M'(w)$ to be the unique perfect matching on $[n] \setminus \{\pm w_1\}$ with $\mathrm{fl}_\pm(M'(w)) = M(v)$. Since $M(v)$ is symmetric and noncrossing, $M'(w)$ is symmetric and noncrossing.

The matching $M'(w)$ may be read off directly from the one-line representation of w by the following procedure. Let $w^0, w^1, w^2, \ldots, w^l$ be any sequence of words whose first term is $w^0 = w_2 w_3 \cdots w_n$ (note the deliberate omission of w_1) and whose final term is strictly increasing, in which w^i for $i > 0$ is formed from w^{i-1} by removing a consecutive subword $q_i p_i$ with $p_i < q_i$. Let $\{c_1, c_2, \ldots, c_k\}$ be the set of letters in w^l. Then $M'(w)$ is the matching whose blocks consist of $\{p, -q\}$, $\{-p, q\}$, and $\{\pm c\}$ for each $(q,p) \in \{(q_1,p_1), (q_2,p_2), \ldots, (q_l,p_l)\}$ and each $c \in \{c_1, c_2, \ldots, c_k\}$. It is a nontrivial fact that this construction is independent of the choices of descents [5]. One can read off $M(v)$ from the one-line representation of $v \in \mathcal{A}_{n-1}$ by a similar procedure.

Example 3.3. Let $w = 316\overline{7}4\overline{5}\overline{2}$. One sequence of words $w^0, w^1, \ldots, w^l$ as above is

$$w^0 = 16\overline{7}4\overline{5}\overline{2}, \quad w^1 = 14\overline{5}\overline{2}, \quad w^2 = 1\overline{2}, \quad w^3 = \varnothing,$$

so $M'(w) = \{\pm\{1,2\}, \pm\{4,5\}, \pm\{6,7\}\}$. Setting $v = \mathrm{fl}_\pm(w_2 w_3 \cdots w_n)$, we have

$$\alpha_{\min}(M) = 5\overline{6}3\overline{4}1\overline{2} \lhd_3 5\overline{6}13\overline{4}\overline{2} \lhd_1 15\overline{6}3\overline{4}\overline{2} = v$$

for $M = \{\pm\{1,2\}, \pm\{3,4\}, \pm\{5,6\}\} = \mathrm{fl}_\pm(M'(w))$, so $v \in \mathcal{A}_M$.

Definition 3.4. An element $w \in C_n$ is a *quasi-atom* if the following conditions hold:

(a) One has $w_1 > 0$ and $\mathrm{fl}_\pm(w_2 w_3 \cdots w_n) \in \mathcal{A}_{n-1}$, where $\mathcal{A}_0 = \{\varnothing\}$.

(b) At most one block $\{a, b\} \in M'(w)$ has $0 < a < w_1 < b$.

(c) No symmetric block $\{\pm c\} \in M'(w)$ has $0 < w_1 < c$.

A quasi-atom w is *odd* if no block $\{a, b\} \in M'(w)$ exists with $0 < a < w_1 < b$; otherwise, w is *even*. We write $\mathcal{Q}^+_n$ and $\mathcal{Q}^-_n$ for the sets of even and odd quasi-atoms in C_n, and define $\mathcal{Q}_n = \mathcal{Q}^+_n \sqcup \mathcal{Q}^-_n$. By convention $\mathcal{A}_0 = \{\varnothing\}$, $\mathcal{Q}^+_1 = \varnothing$, and $\mathcal{Q}^-_1 = \{1\}$. In rank two we have $\mathcal{A}_1 = \{\bar{1}\}$, $\mathcal{Q}^+_2 = \varnothing$, and $\mathcal{Q}^-_2 = \{2\bar{1}\}$. In rank three we have $\mathcal{A}_2 = \{\bar{2}\bar{1}, 1\bar{2}\}$, $\mathcal{Q}^+_3 = \{21\bar{3}\}$, and $\mathcal{Q}^-_3 = \{3\bar{2}\bar{1}, 31\bar{2}, 12\bar{3}\}$. The following is not obvious:

Proposition 3.5. The sets $\mathcal{A}_n$ and $\mathcal{Q}_n$ are disjoint.

4 Transition graphs

In this section, we construct a directed bipartite graph $\overrightarrow{\mathcal{L}_n}$ with vertex set $\mathcal{A}_n \sqcup \mathcal{Q}_n$. We use the letter $\mathcal{L}$ to denote this graph since it will later serve as one "layer" in a larger graph of interest.

Explicitly, define $\overrightarrow{\mathcal{L}_n}$ to be the smallest directed graph containing each element of $\mathcal{A}_n \sqcup \mathcal{Q}_n$ as a vertex and containing each of the following directed edges:

- If $v \in \mathcal{Q}^+_n$ and $b = v_1 > 0$, and $\{a, c\} \in M'(v)$ is the unique block with $0 < a < b < c$, then $\overrightarrow{\mathcal{L}_n}$ has an incoming edge $t_{bc}v \to v$ and an outgoing edge $v \to t_{ab}v$.

- If $v \in \mathcal{A}_n$ then v has a single incoming edge $u \to v$ in $\overrightarrow{\mathcal{L}_n}$, where if $v_1 < 0$ then $u = vt_0$, and if $v_1 > 0$ then $u = t_{bc}v$ where $b = v_1$ and c is such that $\{b, -c\} \in M(v)$.

See Figure 1 for an example. It is not obvious from this description of $\overrightarrow{\mathcal{L}_n}$ that each vertex incident to $v \in \mathcal{A}_n \sqcup \mathcal{Q}^+_n$ is contained in $\mathcal{A}_n \sqcup \mathcal{Q}_n$, but this turns out to be the case.

$$
\begin{tikzcd}
& 4\bar{3}1\bar{2} \arrow[dl] \arrow[dr] & \\
\bar{4}\bar{3}1\bar{2} & & 3\bar{4}1\bar{2}
\end{tikzcd}
\quad
\begin{tikzcd}
& & 4\bar{1}2\bar{3} \arrow[dll] \arrow[d] \arrow[drr] & & \\
\bar{4}\bar{1}2\bar{3} & & 3\bar{1}2\bar{4} \arrow[d] & & 1\bar{4}2\bar{3} \\
& & 2\bar{1}3\bar{4} \arrow[dl] \arrow[dr] & & \\
& \bar{2}\bar{1}3\bar{4} & & 1\bar{2}3\bar{4} &
\end{tikzcd}
\quad
\begin{tikzcd}
& 4\bar{3}\bar{2}\bar{1} \arrow[dl] \arrow[dr] & \\
\bar{4}\bar{3}\bar{2}\bar{1} & & 3\bar{4}\bar{2}\bar{1}
\end{tikzcd}
\quad
\begin{tikzcd}
& 42\bar{3}\bar{1} \arrow[dl] \arrow[dr] & \\
32\bar{4}\bar{1} \arrow[d] & & \bar{4}2\bar{3}\bar{1} \\
23\bar{4}\bar{1} \arrow[d] \arrow[dr] & & \\
\bar{2}3\bar{4}\bar{1} & 13\bar{4}\bar{2} &
\end{tikzcd}
$$

Figure 1: The directed bipartite graph $\overrightarrow{\mathcal{L}_4}$. The vertices in the top row are all odd.

Theorem 4.1. The vertices in $\overrightarrow{\mathcal{L}_n}$ are precisely the signed permutations $\mathcal{A}_n \sqcup \mathcal{Q}_n$. Each edge in $\overrightarrow{\mathcal{L}_n}$ goes from an even quasi-atom to an odd quasi-atom, from an odd quasi-atom to an even quasi-atom, or from an odd quasi-atom to an atom.

Corollary 4.2. The directed graph $\overrightarrow{\mathcal{L}_n}$ is bipartite.

Recall the definitions of $\mathcal{S}_j(w) \subset C_n$ and $\mathcal{T}_j^+(w) \subset C_{n+1}$ for $w \in C_n$ from Theorem 2.8. Our next theorem relates the edges of $\overrightarrow{\mathcal{L}_n}$ to Billey's transition formula for G_w.

Theorem 4.3. Suppose $u, w \in \mathcal{Q}_n^-$ and $w_1 < n$. Then $v \in \mathcal{S}_1(u)$ if and only if $u \to v$ is an edge in $\overrightarrow{\mathcal{L}_n}$, and $v \in \mathcal{T}_1^+(w)$ if and only if $v \to w$ is an edge in $\overrightarrow{\mathcal{L}_n}$. Hence $\mathcal{T}_1^+(w) \subset C_n$.

A *sink*/*source* in a directed graph is a vertex with no outgoing/incoming edges.

Theorem 4.4. The set of sinks in $\overrightarrow{\mathcal{L}_n}$ is precisely $\mathcal{A}_n$. The set of sources in $\overrightarrow{\mathcal{L}_n}$ is precisely the set of odd quasi-atoms of the form $nv_1v_2\cdots v_{n-1} \in \mathcal{Q}_n^-$ where $v_1v_2\cdots v_{n-1} \in \mathcal{A}_{n-1}$.

The preceding result indicates a natural way of packaging the graphs $\overrightarrow{\mathcal{L}_n}$ together for successive values of n. For $0 < m < n$, write $\uparrow_m^n : C_m \to C_n$ for the transformation

$$\uparrow_m^n(v_1v_2\cdots v_m) = n(n-1)\cdots(m+3)(m+2)v_1v_2\cdots v_m(m+1) \in C_n.$$

Let C_0 be the set consisting of just the empty word $\emptyset$ and define $\uparrow_0^n\colon C_0 \to C_n$ to be the map $\emptyset \mapsto n\cdots 321$. View $\overrightarrow{\mathcal{L}_0}$ as the edgeless graph with vertex set C_0. Define $\overrightarrow{\mathcal{L}}_{m,n}$ for $0 \le m < n$ to be the directed graph given by replacing each vertex in $\overrightarrow{\mathcal{L}_m}$ by its image under $\uparrow_m^n$. Interpret $\uparrow_n^{n+1}$ as the identity map $C_n \to C_n \hookrightarrow C_{n+1}$ and identify $\overrightarrow{\mathcal{L}}_{n,n+1}$ with $\overrightarrow{\mathcal{L}_n}$. Finally, define $\overrightarrow{\mathcal{G}_n}$ to be the graph given by the disjoint union

$$\overrightarrow{\mathcal{L}}_{0,n+1} \sqcup \overrightarrow{\mathcal{L}}_{1,n+1} \sqcup \overrightarrow{\mathcal{L}}_{2,n+1} \cdots \sqcup \overrightarrow{\mathcal{L}}_{n,n+1}$$

with these additional edges: for $m \in [n]$ and $w \in \mathcal{A}_{m-1}$, include an edge from the sink

$$\uparrow_{m-1}^{n+1}(w_1w_2\cdots w_{m-1}) = (n+1)n\cdots(m+2)(m+1)w_1w_2\cdots w_{m-1}m \tag{4.1}$$

in $\overrightarrow{\mathcal{L}}_{m-1,n+1}$ to the source

$$\uparrow_m^{n+1}(mw_1w_2\cdots w_{m-1}) = (n+1)n\cdots(m+3)(m+2)mw_1w_2\cdots w_m(m+1) \tag{4.2}$$

in $\overrightarrow{\mathcal{L}}_{m,n+1}$. A vertex in $\overrightarrow{\mathcal{L}}_{m,n+1}$ is *odd* it is the image under $\uparrow_m^{n+1}$ of an odd quasi-atom in $\overrightarrow{\mathcal{L}_m}$. All other vertices in $\overrightarrow{\mathcal{L}}_{m,n+1}$ or $\overrightarrow{\mathcal{G}_n}$ are *even*; in particular, the unique vertex $\uparrow_0^{n+1}(\emptyset)$ in $\overrightarrow{\mathcal{L}}_{0,n+1}$ is even. Figure 2 shows the graph $\overrightarrow{\mathcal{G}_4}$. The resulting division into even and odd vertices affords a bipartition of $\overrightarrow{\mathcal{G}_n}$. The following key properties of $\overrightarrow{\mathcal{G}_n}$ are straightforward corollaries of Theorems 2.8, 4.3 and 4.4.

Corollary 4.5. The unique source in $\overrightarrow{\mathcal{G}_n}$ is $\uparrow_0^{n+1}(\emptyset) = w_n^A$. The set of sinks is $\overrightarrow{\mathcal{G}_n}$ is $\uparrow_n^{n+1}(\mathcal{A}_n) = \mathcal{A}_n$. If v is any odd vertex in $\overrightarrow{\mathcal{G}_n}$, then $\sum_{\{u\to v\}\in\overrightarrow{\mathcal{G}_n}} G_u = \sum_{\{v\to w\}\in\overrightarrow{\mathcal{G}_n}} G_w$.

Finally, using these results we can prove Theorem 1.6.

Proof of Theorem 1.6. For each vertex $v \in \overrightarrow{\mathcal{G}_n}$, define $f(v)$ to be 0 if v is odd and G_v if v is even. The last property in Corollary 4.5 implies $\sum_{\{u\to v\}\in\overrightarrow{\mathcal{G}_n}}(f(v)-f(u)) = 0$. On the other hand, $\sum_{\{u\to v\}\in\overrightarrow{\mathcal{G}_n}}(f(v)-f(u)) = \sum_{v\in\overrightarrow{\mathcal{G}_n}} \mathsf{sdeg}(v)f(v)$ where $\mathsf{sdeg}(v)$ is the outdegree of v minus its indegree. The unique source in $\overrightarrow{\mathcal{G}_n}$ has outdegree 1; every sink in $\overrightarrow{\mathcal{G}_n}$ has indegree 1; and every even vertex that is not a source or a sink has indegree 1 and outdegree 1. Hence $0 = \sum_{u\in\mathsf{Source}(\overrightarrow{\mathcal{G}_n})} G_u - \sum_{v\in\mathsf{Sink}(\overrightarrow{\mathcal{G}_n})} G_v = G_{w_n^A} - \sum_{v\in\mathcal{A}_n} G_v$. The last term is $S_{(n,n-1,\dots,3,2,1)} - \hat{G}_{w_n^C}$ by Theorems 1.3 and 2.7 and the definition of $\hat{G}_w$. $\square$

5 Conjectures in type D

Assume $n \geq 3$ and let D_n be the subgroup of signed permutations in C_n whose one-line representations have an even number of negative letters. This subgroup is the finite Coxeter group of classical type D relative to the generating set $S = \{t_1', t_1, t_2, \dots, t_{n-1}\}$ where $t_1' = t_0t_1t_0$. To conclude this article, we describe a conjecture that gives a type D analogue of Theorem 1.2.

For $w \in D_n$ and $a \in \mathcal{R}(w)$, let $\underline{a}$ be the word obtained from a by replacing each t_1' with t_1, and define $\underline{\mathcal{R}}(w) = \{\underline{a} : a \in \mathcal{R}(w)\}$. For instance, $\mathcal{R}(\bar{1}3\bar{2}) = \{(t_1, t_1', t_2), (t_1', t_1, t_2)\}$ while $\underline{\mathcal{R}}(\bar{1}3\bar{2}) = \{(t_1, t_1, t_2)\}$. In type D it is the sets $\underline{\mathcal{R}}(w)$ that have simple tableau enumerations. Let w_n^D be the longest element of D_n.

Theorem 5.1 (Billey and Haiman [2])**.** It holds that $|\underline{\mathcal{R}}(w_n^D)| = |\mathsf{SYT}((n-1)^n)|$, which is also the number of unmarked shifted standard tableaux of shape $(2n-2, 2n-4, \dots, 2)$.

Let (W, S) be a Coxeter system with a group automorphism $\theta : W \to W$ such that $\theta(S) = S$ and $\theta = \theta^{-1}$. Let $\mathcal{I}_\theta(W) = \{w \in W : \theta(w) = w^{-1}\}$. The set of *(twisted) atoms* $\mathcal{A}_\theta(y)$ of $y \in \mathcal{I}_\theta(W)$ consists of the minimal-length elements $w \in W$ with $\theta(w)^{-1} \circ w = y$. The set of *(twisted) involution words* for $y \in \mathcal{I}_\theta(W)$ is $\hat{\mathcal{R}}_\theta(y) = \bigsqcup_{w\in\mathcal{A}_\theta(y)} \mathcal{R}(w)$.

Assume W is finite with longest element w_0. If W is A_n, C_n, or D_{2n+1} for $n > 1$, then the only possibilities for θ are the identity map and $w \mapsto w_0ww_0$, and it holds that $|\hat{\mathcal{R}}_\theta(w_0)| = |\hat{\mathcal{R}}(w_0)|$ and (in type D) $|\hat{\underline{\mathcal{R}}}_\theta(w_0)| = |\hat{\underline{\mathcal{R}}}(w_0)|$. Define $*$ to be the automorphism of D_n that interchanges t_1 and t_1' and fixes t_i for $1 < i < n$. When n is odd, $*$ is the inner automorphism $w \mapsto w_0ww_0$. Computations support the following:

Conjecture 5.2. It holds that $\left|\hat{\underline{\mathcal{R}}}(w_n^D)\right| = |\mathsf{SYT}(\lambda)|$ and $\left|\hat{\underline{\mathcal{R}}}_*(w_n^D)\right| = |\mathsf{SYT}(\mu)|$ for $\lambda = (n-1, n-2, \dots, \lfloor\frac{n}{2}\rfloor, \lfloor\frac{n}{2}\rfloor, \dots, 2, 1)$ and $\mu = (n-1, n-2, \dots, \lceil\frac{n}{2}\rceil - 1, \lceil\frac{n}{2}\rceil - 1, \dots, 2, 1)$.

Acknowledgements

This work was partially supported by HKUST grant IGN16SC11. We are grateful to Zach Hamaker for helpful feedback and discussions.

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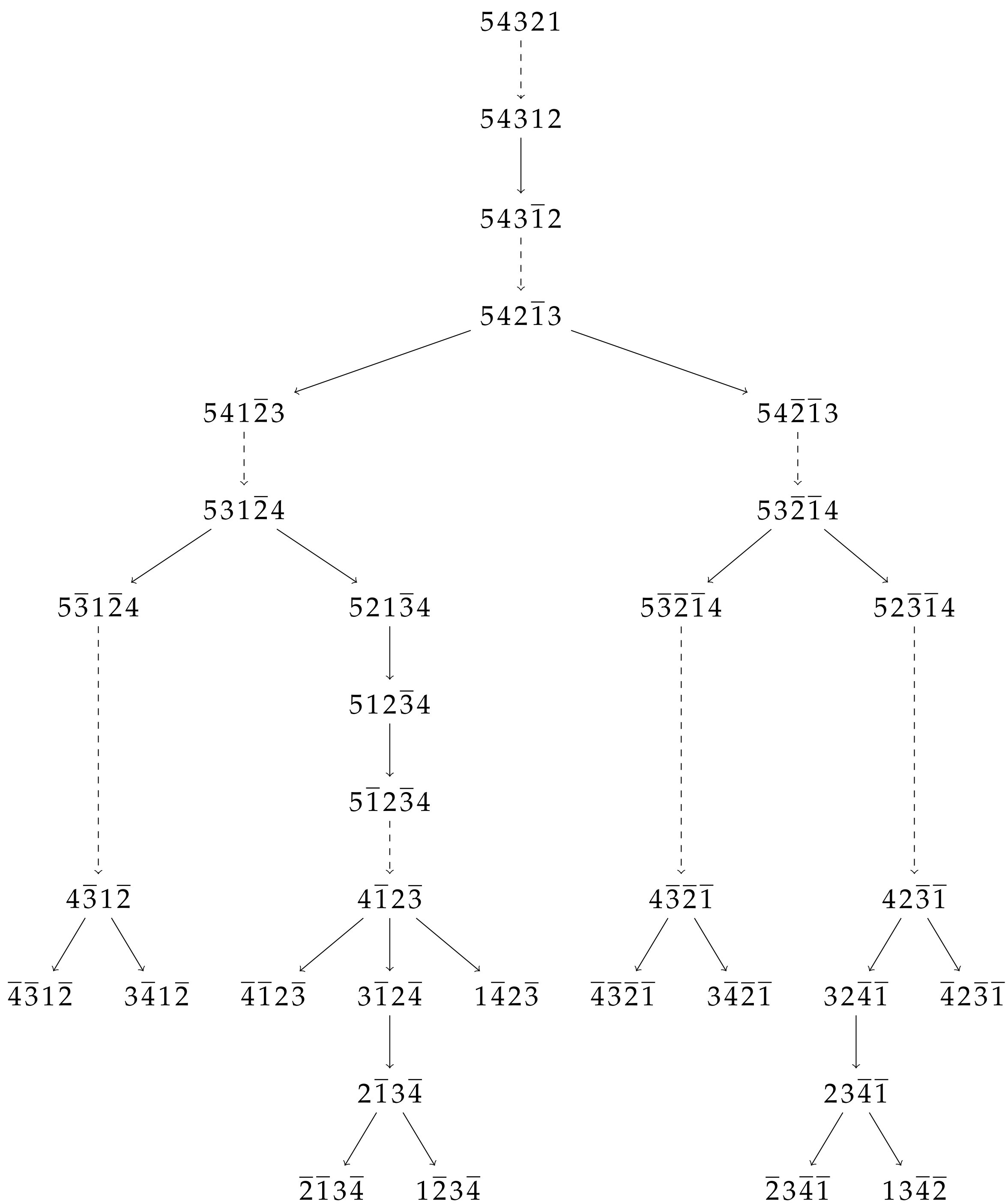

Figure 2: The directed bipartite graph $\overrightarrow{\mathcal{G}_4}$. The dashed arrows correspond to edges between vertices of the form (4.1) and (4.2). We have omitted the terminal 5 from all vertices in the final layer $\overrightarrow{\mathcal{L}}_{4,5} \subset \overrightarrow{\mathcal{G}_4}$. In contrast to what we see in this example, the graph $\overrightarrow{\mathcal{G}_n}$ is not a directed tree for $n \geq 5$.