

# Atomic decomposition of characters and crystals

Cédric Lecouvey<sup>\*1</sup> and Cristian Lenart<sup>†2</sup>

<sup>1</sup>Faculté des Sciences et Techniques, Université de Tours Tours, France

<sup>2</sup>Department of Mathematics and Statistics, State University of New York at Albany, 1400  
Washington Avenue, Albany, NY 12222, U.S.A.

**Abstract.** Lascoux stated that the type  $A$  Kostka–Foulkes polynomials  $K_{\lambda,\mu}(t)$  (Lusztig’s  $t$ -analogue of weight multiplicity) expand positively in terms of so-called atomic polynomials. We define, in arbitrary type, a combinatorial version of the atomic decomposition, based on the connected components of a modified crystal graph. We prove this property in type  $A$ , as well as in types  $B$ ,  $C$ , and  $D$  in a stable range for  $t = 1$ . We also discuss other cases, applications, and a geometric interpretation.

**Résumé.** Lascoux a formulé en type  $A$  la propriété des polynômes de Kostka d’admettre une décomposition positive en polynômes dits atomiques. Nous formulons une version combinatoire de la décomposition atomique pour tous les types, réalisée à l’aide des cristaux modifiés. Nous démontrons cette propriété en type  $A$ , et en type  $B$ ,  $C$  et  $D$  pour  $t = 1$  lorsque le rang est suffisamment grand. Finalement, nous donnons des applications et une interprétation géométrique de nos résultats.

**Keywords:** Kostka–Foulkes polynomial, atomic decomposition, crystal graph, dominance order.

## 1 Introduction

The starting point of this paper is a result of Lascoux on the (type  $A$ ) Kostka–Foulkes polynomials  $K_{\lambda,\mu}(t)$ , which are well-known  $t$ -analogues of the Kostka numbers  $K_{\lambda,\mu}$ , i.e., the number of semistandard Young tableaux of shape  $\lambda$  and content  $\mu$ . Lascoux [5] stated the decomposition of the Kostka–Foulkes polynomials into so-called *atomic polynomials*. Some arguments of the proof in [5] remained elusive, and it was not until the work of Shimozono [12] that the type  $A$  atomic decomposition was completely accepted, this time in larger generality (for the so-called *generalized Kostka–Foulkes polynomials*). However, the latter proof involves several intricate combinatorial arguments and related concepts, such as *plactic monoid*, *cyclage*, and *catabolism*. We provide a simpler, more conceptual approach, which has the additional advantage of extending beyond type  $A$ .

We formulate the  $t$ -atomic decomposition property in arbitrary Lie type as a nonnegative expansion for both a Kostka–Foulkes polynomial  $K_{\lambda,\mu}(t)$  (Lusztig’s  $t$ -analogue

---

\*cedric.lecouvey@lmpt.univ-tours.fr

†clenart@albany.edu

of weight multiplicity [9]), and a  $t$ -analogue  $\chi_\lambda^+(t)$  of the dominant part of an irreducible character (defined in terms of  $K_{\lambda,\mu}(t)$ ). The  $t$ -atomic decomposition property is a strengthening of the monotonicity of  $K_{\lambda,\mu}(t)$  [1].

As opposed to the above algebraic approach, we also define a  $t$ -atomic decomposition property at the combinatorial level of the highest weight crystal  $B(\lambda)$  [3]. This property involves a partition of the dominant part  $B(\lambda)^+$  of  $B(\lambda)$ , and a statistic on  $B(\lambda)^+$ .

We prove the combinatorial  $t$ -atomic decomposition in type  $A$ , thus realizing combinatorially the classical result, while also providing a simple, conceptual proof of it. We also prove this property in types  $B$ ,  $C$ , and  $D$  for  $t = 1$  in a stable range. Our main ingredients are: the *partial order on dominant weights*, and a *modified crystal graph* structure on  $B(\lambda)^+$ , whose connected components define the needed partition. See [7] for the details.

We conjecture that our result in types  $B$ ,  $C$ ,  $D$  holds without specializing  $t$ . Furthermore, in type  $C$ , we conjecture that this result, together with our combinatorial formula for the corresponding  $K_{\lambda,0}(t)$  [8], leads to a statistic which computes any  $K_{\lambda,\mu}(t)$ . Such a statistic (charge) has been long sought. We conclude with a geometric interpretation of the atomic decomposition in terms of the *geometric Satake correspondence* [10].

## 2 The atomic decomposition: definitions and basic facts

### 2.1 Characters and $t$ -deformations

Let  $\mathfrak{g}$  be a simple Lie algebra over  $\mathbb{C}$  of rank  $r$ . Let  $R \supset R^+ \supset S$  be the corresponding sets of roots, positive roots, and simple roots, realized in a real Euclidean space of dimension  $r$  with inner product  $\langle \cdot, \cdot \rangle$ . For any  $\alpha \in R$ , we write  $\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$  for the corresponding coroot. The set  $P$  of integral weights for  $\mathfrak{g}$  consists of vectors  $\beta$  satisfying  $\langle \beta, \alpha^\vee \rangle \in \mathbb{Z}$  for any  $\alpha \in R$ . We write  $P^+ = \{\beta \in P \mid \langle \beta, \alpha^\vee \rangle \geq 0 \text{ for any } \alpha \in S\}$  for the cone of dominant weights of  $\mathfrak{g}$ , and denote by  $\omega_1, \dots, \omega_r$  the fundamental weights. Let  $W$  be the Weyl group of  $\mathfrak{g}$ , and write  $\ell(\cdot)$  for the corresponding length function. The dominance order  $\leq$  on  $P^+$  is defined by  $\alpha < \beta$  if and only if  $\beta - \alpha$  decomposes as a sum of positive roots (or equivalently, simple roots) with nonnegative integer coefficients.

Let  $\chi_\lambda$  be the character of the finite-dimensional irreducible representation  $V(\lambda)$  of  $\mathfrak{g}$  with highest weight  $\lambda \in P^+$ . Let  $K_{\lambda,\gamma}$  denote the multiplicity of the weight  $\gamma$  in  $V(\lambda)$ . Let  $P(\lambda)$  be the set of weights of  $V(\lambda)$ , i.e., the set of  $\gamma$  such that  $K_{\lambda,\gamma} > 0$ . Set  $P^+(\lambda) = P(\lambda) \cap P^+$ , and note that  $P^+(\lambda) = \{\mu \in P^+ \mid \mu \leq \lambda\}$ . Since  $K_{\lambda,\gamma} = K_{\lambda,w(\gamma)}$  for any  $w \in W$ , the character  $\chi_\lambda$  is determined by its dominant part

$$\chi_\lambda^+ := \sum_{\mu \in P^+(\lambda)} K_{\lambda,\mu} e^\mu.$$

Lusztig [9] defined a remarkable  $t$ -analogue  $K_{\lambda,\mu}(t)$  of  $K_{\lambda,\mu}$  (a polynomial in  $t$  satisfying  $K_{\lambda,\mu}(1) = K_{\lambda,\mu}$ ) by introducing a variable  $t$  in the Weyl character formula for  $\chi_\lambda$ :

$$\frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda + \rho) - \rho}}{\prod_{\alpha \in R^+} (1 - te^{-\alpha})} = \sum_{\gamma \in P(\lambda)} K_{\lambda, \gamma}(t) e^\gamma; \quad (2.1)$$

here  $\rho$  is the half sum of positive roots. We define the  $t$ -analogue of  $\chi_\lambda^+$  by

$$\chi_\lambda^+(t) := \sum_{\mu \in P^+(\lambda)} K_{\lambda, \mu}(t) e^\mu. \quad (2.2)$$

When  $\gamma = \mu$  is dominant, the polynomial  $K_{\lambda, \mu}(t)$  is known as a *Kostka–Foulkes polynomial*.

This polynomial has remarkable properties, such as being an *affine Kazhdan–Lusztig polynomial*, which implies that it has nonnegative integer coefficients. More precisely,

$$K_{\lambda, \mu}(t) = t^{\langle \lambda - \mu, \rho^\vee \rangle} P_{w_\mu, w_\lambda}(t^{-1}), \quad (2.3)$$

where  $w_\lambda$  denotes the longest element of  $Wt_\lambda W$ , and  $t_\lambda$  is the translation by  $\lambda$  in the extended affine Weyl group [4]; note that  $\langle \lambda - \mu, \rho^\vee \rangle$  is the number of simple roots in the decomposition of  $\lambda - \mu$ , counted with multiplicity. Based on (2.3), we let

$$\tilde{K}_{\lambda, \mu}(t) := t^{\langle \lambda - \mu, \rho^\vee \rangle} K_{\lambda, \mu}(t^{-1}), \quad \text{so } \tilde{K}_{\lambda, \mu}(t) = P_{w_\mu, w_\lambda}(t). \quad (2.4)$$

To each irreducible representation  $V(\lambda)$  is associated an abstract *Kashiwara crystal*  $B(\lambda)$  [3]. This is a colored directed graph which encodes the action of certain modified versions of the Chevalley generators, upon passing to the quantum group of  $\mathfrak{g}$  and letting the quantum parameter go to 0. Denoting by  $\text{wt}(b)$  the weight of the vertex  $b \in B(\lambda)$ , we have

$$\chi_\lambda = \sum_{b \in B(\lambda)} e^{\text{wt}(b)}. \quad (2.5)$$

## 2.2 The definition of the atomic decomposition

For any dominant weight  $\mu$ , define the *layer sum polynomials* by

$$w_\mu := \sum_{\gamma \in P(\mu)} e^\gamma, \quad w_\mu^+ := \sum_{v \in P^+(\mu)} e^v = \sum_{v \leq \mu} e^v, \quad \text{and} \quad (2.6)$$

$$w_\mu^+(t) := \sum_{v \in P^+(\mu)} t^{\langle \mu - v, \rho^\vee \rangle} e^v = \sum_{v \leq \mu} t^{\langle \mu - v, \rho^\vee \rangle} e^v. \quad (2.7)$$

Consider the expansion

$$\chi_\lambda = \sum_{\mu \in P^+(\lambda)} A_{\lambda, \mu} w_\mu, \quad \text{or equivalently } \chi_\lambda^+ = \sum_{\mu \in P^+(\lambda)} A_{\lambda, \mu} w_\mu^+. \quad (2.8)$$

Similarly, consider the polynomials  $A_{\lambda,\mu}(t)$  defined by

$$\chi_\lambda^+(t) = \sum_{\mu \in P^+(\lambda)} A_{\lambda,\mu}(t) w_\mu^+(t). \quad (2.9)$$

We have  $A_{\lambda,\lambda}(t) = 1$ . It is not hard to prove that the expansion (2.9) is equivalent to

$$K_{\lambda,\nu}(t) = \sum_{\nu \leq \mu \leq \lambda} t^{\langle \mu - \nu, \rho^\vee \rangle} A_{\lambda,\mu}(t), \quad \text{for all } \nu \in P^+(\lambda). \quad (2.10)$$

**Definition 2.1.** *The character  $\chi_\lambda$  admits an atomic decomposition if  $A_{\lambda,\mu} \in \mathbb{Z}_{\geq 0}$ . Similarly, we say that  $\chi_\lambda^+(t)$  admits a  $t$ -atomic decomposition if  $A_{\lambda,\mu}(t) \in \mathbb{Z}_{\geq 0}[t]$ . These polynomials are called atomic polynomials.*

We state a property equivalent to the  $t$ -atomic decomposition. To this end, by analogy with (2.2) and using (2.4), we define

$$\tilde{\chi}_\lambda^+(t) := \sum_{\mu \in P^+(\lambda)} \tilde{K}_{\lambda,\mu}(t) e^\mu.$$

Like in (2.9), consider the polynomials  $\tilde{A}_{\lambda,\mu}(t)$  defined by

$$\tilde{\chi}_\lambda^+(t) = \sum_{\mu \in P^+(\lambda)} \tilde{A}_{\lambda,\mu}(t) w_\mu^+, \quad (2.11)$$

where we recall that  $w_\mu^+ := w_\mu^+(1)$ . Like above, we can then prove that the expansion (2.11) is equivalent to

$$\tilde{K}_{\lambda,\nu}(t) = \sum_{\nu \leq \mu \leq \lambda} \tilde{A}_{\lambda,\mu}(t), \quad \text{for all } \nu \in P^+(\lambda). \quad (2.12)$$

**Proposition 2.2.** *The polynomials  $A_{\lambda,\mu}(t)$  and  $\tilde{A}_{\lambda,\mu}(t)$  satisfy*

$$\tilde{A}_{\lambda,\mu}(t) = t^{\langle \lambda - \mu, \rho^\vee \rangle} A_{\lambda,\mu}(t^{-1}).$$

*Thus, the  $t$ -atomic decomposition is equivalent to the fact that  $\tilde{A}_{\lambda,\mu}(t) \in \mathbb{Z}_{\geq 0}[t]$ .*

**Remarks 2.3.** (1) Lascoux [5] stated the atomic decomposition in type  $A$  as in (2.10). However, there is a slight difference in the definition of  $\tilde{K}_{\lambda,\mu}(t)$ , for given partitions  $\lambda, \mu$ ; namely, Lascoux defined  $\tilde{K}_{\lambda,\mu}(t) := t^{n(\mu)} K_{\lambda,\mu}(t^{-1})$ , where  $n(\mu) := \sum_i (i-1)\mu_i$ .

(2) The  $t$ -atomic decomposition, as stated in Definition 2.1, implies the monotonicity of the Kostka–Foulkes polynomials, which holds in the full generality of Kazhdan–Lusztig polynomials for finite and affine Weyl groups [1, Corollary 3.7], cf. (2.4). Indeed, this property says that, for  $x \leq y \leq z$  in such a Weyl group, the difference of Kazhdan–Lusztig polynomials  $P_{x,z}(t) - P_{y,z}(t)$  is in  $\mathbb{Z}_{\geq 0}[t]$ .

(3) We will give a simpler, conceptual proof of the  $t$ -atomic decomposition in type  $A$ , cf. Section 1. However, even the atomic decomposition (i.e., the positivity in (2.8)) might fail beyond type  $A$ ; but this failure seems limited to small ranks, as explained below.

### 2.3 Atomic decomposition of finite crystals

Let  $B(\lambda)^+$  be the subset of  $B(\lambda)$  consisting of vertices with dominant weights.

**Definition 2.4.** An atomic decomposition of the crystal  $B(\lambda)$  is a partition

$$B(\lambda)^+ = \bigsqcup_{h \in H(\lambda)} \mathbb{B}(\lambda, h), \quad (2.13)$$

where  $H(\lambda) \subset B(\lambda)^+$ ,  $h \in \mathbb{B}(\lambda, h)$  is a distinguished vertex, and each component  $\mathbb{B}(\lambda, h)$  consists of exactly one vertex of weight  $\nu$  for each  $\nu \leq \text{wt}(h)$ .

An atomic decomposition of  $B(\lambda)$  clearly gives the atomic decomposition (2.8) of  $\chi_\lambda^+$ , where  $A_{\lambda, \mu}$  is the number of vertices of weight  $\mu$  in  $H(\lambda)$ .

**Definition 2.5.** A  $t$ -atomic decomposition of the crystal  $B(\lambda)$  is an atomic decomposition together with a statistic  $c : H(\lambda) \rightarrow \mathbb{Z}_{\geq 0}$  such that the following polynomials satisfy (2.9):

$$A_{\lambda, \mu}(t) = \sum_{\substack{h \in H(\lambda) \\ \text{wt}(h) = \mu}} t^{c(h)}. \quad (2.14)$$

As we see, the  $t$ -atomic decomposition of  $\chi_\lambda^+(t)$  is part of Definition 2.5. Assuming that  $B(\lambda)$  has a  $t$ -atomic decomposition, one can extend the statistic  $c$  to  $B(\lambda)^+$  by setting

$$c(b) := c(h) + \langle \text{wt}(h) - \text{wt}(b), \rho^\vee \rangle, \quad \text{for any } b \in \mathbb{B}(\lambda, h). \quad (2.15)$$

The  $t$ -analogue of the combinatorial formula (2.5) follows from Definition 2.5:

$$\chi_\lambda^+(t) = \sum_{b \in B(\lambda)^+} t^{c(b)} e^{\text{wt}(b)}. \quad (2.16)$$

Moreover, by comparing (2.16) with (2.2), we obtain the following combinatorial formula for Kostka–Foulkes polynomials:

$$K_{\lambda, \mu}(t) = \sum_{\substack{b \in B(\lambda) \\ \text{wt}(b) = \mu}} t^{c(b)}. \quad (2.17)$$

To summarize, the existence of a  $t$ -atomic decomposition of a crystal is highly desirable because: (i) it gives the  $t$ -atomic decomposition of  $\chi_\lambda^+(t)$  and of  $K_{\lambda, \mu}(t)$ , which are now realized combinatorially; (ii) it leads to combinatorial formulas for both  $K_{\lambda, \mu}(t)$  and the atomic polynomials  $A_{\lambda, \mu}(t)$ , namely (2.17) and (2.14), respectively.

### 3 The partial order on dominant weights

Before we consider the atomic decomposition of finite crystals, we need some information about the dominant weight poset defined in [Section 2.1](#). In full generality, this poset was first studied in [\[13\]](#), so we will recall some results from this paper.

The components of the dominant weight poset are lattices. Each cocover is of the form  $\mu \succ \mu - \alpha$ , where  $\alpha$  is a positive root, so we represent it as a downward edge in the Hasse diagram labeled by  $\alpha$ . The cocovers were completely described in [\[13, Theorem 2.8\]](#). Fixing a dominant weight  $\lambda$ , we will consider the lower order ideal determined by  $\lambda$ . This is an interval  $[\widehat{0}, \lambda]$ , with  $\widehat{0}$  a minimal element of the dominant weight poset.

#### 3.1 Type $A_{n-1}$

Now  $\lambda$  is a partition ( $\lambda_1 \geq \dots \geq \lambda_{n-1} \geq 0$ ), and let  $|\lambda| := \sum_i \lambda_i$ . We denote a partition with  $p$  parts  $a$ ,  $q$  parts  $b$  ( $a \geq b$ ) etc. by  $(a^p b^q \dots)$ . In the interval  $[\widehat{0}, \lambda]$  mentioned above,  $\widehat{0}$  is the partition  $\omega_p = (1^p)$ , where  $p := |\lambda| \bmod n$ . The cocovers  $\mu \succ \mu - \alpha_{ij}$ , where  $\alpha_{ij} = \varepsilon_i - \varepsilon_j$  is a positive root (i.e.,  $i < j$ ), are labeled by  $(i, j)$ . There are two types of cocovers:

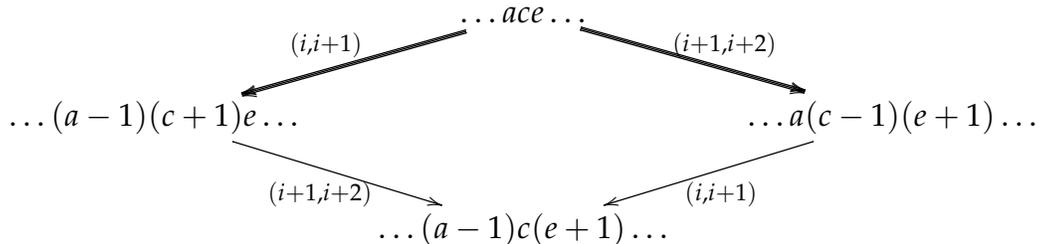
$$(\dots ab \dots) \succ (\dots (a-1)(b+1) \dots), \text{ and } (\dots (a+1)a^p(a-1) \dots) \succ (\dots a^{p+2} \dots). \quad (3.1)$$

where the first cocover is labeled by a simple root. These types are referred to as (\*) and (\*\*), respectively, while a cocover of type (\*\*) which is not of type (\*) is called *proper*.

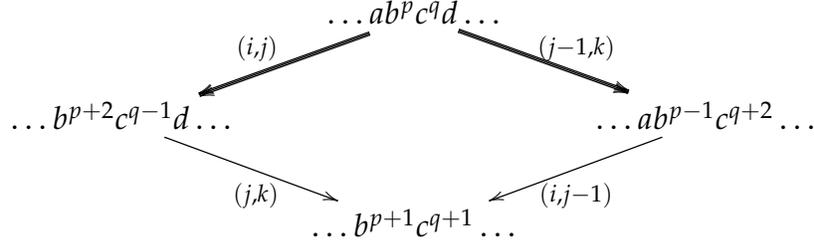
An important result in [\[2\]](#) concerns the structure of “short intervals” in the dominance order. To state it, we need some more definitions. Consider two distinct cocovers  $\mu \succ \nu$  and  $\mu \succ \pi$  of a partition  $\mu$ , which are labeled  $(i, j)$  and  $(k, l)$ , where we assume  $i < k$ . These cocovers can only have one of the following relative positions (in terms of their labels): (i) *nonoverlapping* if  $j < k$ ; (ii) *partially overlapping* if  $j = k$ ; (iii) *fully overlapping* if  $k = j - 1$ . By [\[2, Proposition 3.2\]](#), the interval  $[\nu \wedge \pi, \mu]$  can only have one of the following structures; the two cocovers above are shown in the diagrams below in bold.

**Case A1:** cocovers which are (a) nonoverlapping; (b) partially overlapping and both of type (\*); (c) fully overlapping and both proper of type (\*\*). As subcase (a) is easy, only subcases (b) and (c) are represented in the diagrams below.

In subcase (b), we have  $a \geq c + 2$  and  $c \geq e + 2$ , while  $i$  is the position of  $a$  in  $\mu$ .

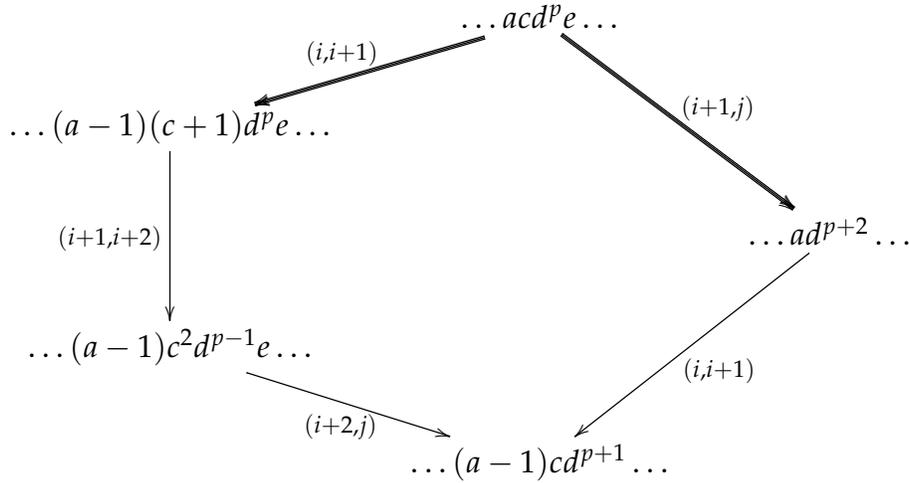


In subcase (c), we have  $b = a - 1$ ,  $c = b - 1$ ,  $d = c - 1$ ,  $p, q \geq 1$ , while  $i$  is the position of  $a$ ,  $j = i + p + 1$  is the position of the first  $c$ , and  $k = j + q$  is the position of  $d$ .



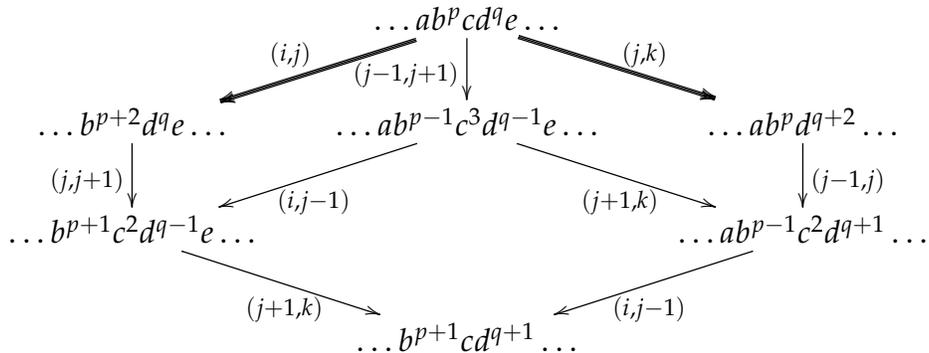
**Case A2:** partially overlapping cocovers, where (a) the first is of type (\*) and the second proper of type (\*\*); (b) vice versa.

In subcase (a), we have  $a \geq c + 2$ ,  $d = c - 1$ ,  $e = d - 1$ ,  $p \geq 1$ , while  $i$  is the position of  $a$  in the partition  $\mu$  and  $j = i + p + 2$  is the position of  $e$ .



In subcase (b), we have a similar pentagon.

**Case A3:** partially overlapping cocovers, both proper of type (\*\*). Here  $b = a - 1$ ,  $c = b - 1$ ,  $d = c - 1$ ,  $e = d - 1$ ,  $p, q \geq 1$ , while  $i$  is the position of  $a$  in the partition  $\mu$ ,  $j = i + p + 1$  is the position of  $c$ , and  $k = j + q + 1$  is the position of  $e$ .



Given two distinct covers  $\mu \triangleleft \nu$  and  $\mu \triangleleft \pi$  of  $\mu$ , the isomorphism type of the interval  $[\mu, \nu \vee \pi]$  is always given by one of the above graphs turned upside down.

### 3.2 Types $B_n$ , $C_n$ , and $D_n$

We start with type  $C_n$ . Now  $\lambda$  is a partition ( $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ ). It is easy to see that the minimal element  $\widehat{0}$  mentioned above (i.e., the unique minimal element below  $\lambda$ ) is either 0 or  $\omega_1 = (10^{n-1})$ , depending on  $|\lambda|$  being even or odd, respectively.

**Proposition 3.1.** *If  $n > (|\lambda| + 1)/2$ , then the cocover  $(\dots 1^2 0^k) \succ (\dots 0^{k+2})$  is the only one which can appear in the Hasse diagram of the interval  $[\widehat{0}, \lambda]$  beside the type A cocovers in (3.1).*

We turn to type  $D_n$ . Now  $\lambda$  is a sequence ( $\lambda_1 \geq \dots \geq \lambda_n$ ) with  $\lambda_i \in \frac{1}{2}\mathbb{Z}$ , all congruent mod  $\mathbb{Z}$ , such that  $\lambda_{n-1} + \lambda_n \geq 0$ . We will now assume that  $\lambda_i \in \mathbb{Z}$ . This implies that the interval  $[\widehat{0}, \lambda]$  only contains weights  $\mu = (\mu_1 \geq \dots \geq \mu_n)$  with  $\mu_i \in \mathbb{Z}$ . Note that, in this case, there are the same possibilities for the minimal element  $\widehat{0}$  as in type C.

**Proposition 3.2.** *If  $n > |\lambda|$ , then the cocover  $(\dots 1^2 0^k) \succ (\dots 0^{k+2})$  is the only one which can appear in the Hasse diagram of the interval  $[\widehat{0}, \lambda]$  beside the type A cocovers in (3.1).*

Type  $B_n$  is completely similar.

We will work under the assumptions of [Propositions 3.1](#) and [3.2](#), and we call this the *stable range*. We derived a corresponding classification of the short intervals, consisting of the same cases as in type A, and a few extra ones involving the new cover.

## 4 Modified crystal operators on classical crystals

Now consider a classical Lie algebra, with Dynkin diagram labeled in the standard way.

### 4.1 Definition of the modified crystal operators

Given a positive root  $\alpha \in W\alpha_1$ , consider the shortest length element in the Weyl group  $W$  satisfying  $w(\alpha_1) = \alpha$ . We define the modified crystal operators  $f_\alpha$  and  $e_\alpha$  as the conjugations

$$f_\alpha := w\tilde{f}_1w^{-1}, \quad e_\alpha := w\tilde{e}_1w^{-1} \quad (4.1)$$

of the ordinary crystal operators  $\tilde{f}_1$  and  $\tilde{e}_1$  by the Kashiwara action of  $w$  on  $B(\lambda)$  [3]. This means that  $f_\alpha(b) = 0$  precisely when  $\tilde{f}_1$  applied to  $w^{-1}(b)$  is 0. In addition, for a positive root  $\alpha \in W\alpha_n$  in type  $B_n$ , we define  $f_\alpha$  and  $e_\alpha$  as the appropriate conjugations of  $f_n$  and  $e_n$ . Clearly,  $f_\alpha$  and  $e_\alpha$  are inverses to one another. Moreover, for any  $b \in B(\lambda)$ , we have  $\text{wt}(f_\alpha(b)) = \text{wt}(b) - \alpha$ . We endow the vertices of  $B(\lambda)$  with the structure of a colored directed graph  $\mathbb{B}(\lambda)$  with edges  $b \xrightarrow{\alpha} b'$  when  $b' = f_\alpha(b)$ . The graph  $\mathbb{B}(\lambda)$  is different from the Kashiwara crystal  $B(\lambda)$  and, unlike the latter, it is generally not connected.

## 4.2 Properties of the modified crystal operators

**Lemma 4.1.** For  $b \in \mathbb{B}(\lambda)$  and a positive root  $\alpha \in W\alpha_1$ , if  $\langle \text{wt}(b), \alpha \rangle > 0$ , then  $f_\alpha(b) \neq 0$ .

**Theorem 4.2.** Consider two positive roots  $\alpha$  and  $\beta$  in  $W\alpha_1$  and a vertex  $b$  in  $\mathbb{B}(\lambda)$  such that  $\langle \text{wt}(b), \alpha \rangle > 0$  and  $\langle \text{wt}(b), \beta \rangle > 0$ .

1. Assume that the pair  $(\alpha, \beta)$  satisfies: (i) it is  $(\varepsilon_i - \varepsilon_j, \varepsilon_j \pm \varepsilon_k)$  or  $(\varepsilon_j \pm \varepsilon_k, \varepsilon_i - \varepsilon_j)$ , for  $i < j < k$ ; (ii) it is  $(\varepsilon_{j-1} + \varepsilon_j, \varepsilon_i - \varepsilon_j)$  for  $i < j - 1$ , and  $\langle \text{wt}(b) - \beta, \varepsilon_{j-1} - \varepsilon_j \rangle = 0$ . Then we have  $f_\alpha f_\beta(b) = f_{\alpha+\beta}(b) \neq 0$ .
2. Assume that the pair  $(\alpha, \beta)$  is in the  $W$ -orbit of  $(\alpha_1, \alpha_3)$ . Then  $f_\alpha f_\beta(b) = f_\beta f_\alpha(b) \neq 0$ .

**Theorem 4.3.** Consider two positive roots  $\alpha$  and  $\beta$  in  $W\alpha_1$  and a vertex  $b$  in  $\mathbb{B}(\lambda)$  such that  $\langle \text{wt}(b), \alpha \rangle \geq 0$  and  $\langle \text{wt}(b), \beta \rangle \geq 0$ . Assume also that  $e_\alpha(b) \neq 0$  and  $e_\beta(b) \neq 0$ .

1. Assume that the pair  $(\alpha, \beta)$  satisfies: (i) it is  $(\varepsilon_i - \varepsilon_j, \varepsilon_j \pm \varepsilon_k)$  or  $(\varepsilon_j \pm \varepsilon_k, \varepsilon_i - \varepsilon_j)$ , for  $i < j < k$ ; (ii) it is  $(\varepsilon_i - \varepsilon_j, \varepsilon_{j-1} + \varepsilon_j)$  for  $i < j - 1$ , and  $\langle \text{wt}(b), \varepsilon_{j-1} - \varepsilon_j \rangle = 0$ . Then we have  $e_\alpha e_\beta(b) = e_{\alpha+\beta}(b) \neq 0$ .
2. Assume that the pair  $(\alpha, \beta)$  is in the  $W$ -orbit of  $(\alpha_1, \alpha_3)$ , and  $w$  is a shortest length element satisfying  $w(\alpha_1, \alpha_3) = (\alpha, \beta)$ . Let  $\gamma := w(\alpha_2)$ , and also assume that  $\langle \text{wt}(b), \gamma \rangle > 0$ . Then  $e_\alpha e_\beta(b) = e_\beta e_\alpha(b) \neq 0$ .

## 5 The main results

Fix a dominant weight  $\lambda$  for a classical Lie algebra. Consider the subgraph of  $\mathbb{B}(\lambda)$  consisting of the vertices of dominant weight, and the edges  $b \xrightarrow{\alpha} f_\alpha(b)$  for which  $\text{wt}(b) \succ \text{wt}(f_\alpha(b))$  is a cocover in the dominant weight poset. This new colored directed graph on the vertices of  $B(\lambda)^+$  will be denoted by  $\mathbb{B}(\lambda)^+$ . It can also be viewed as a poset (with cocovers given by the above edges), and the weight function is a poset projection to the interval  $[0, \lambda]$  in the dominant weight poset. The two points of view will be used interchangeably.

The main goal is to identify situations in which the components of the poset  $\mathbb{B}(\lambda)^+$  define an atomic, respectively  $t$ -atomic decomposition, cf. [Definitions 2.4](#) and [2.5](#).

### 5.1 Type $A_{n-1}$

**Lemma 5.1.** 1. Consider two distinct edges  $b \dashrightarrow b'$  and  $b \dashrightarrow b''$  in  $\mathbb{B}(\lambda)^+$ . The vertices  $b'$  and  $b''$  have a lower bound in this poset.

2. Consider two distinct edges  $b' \dashrightarrow b$  and  $b'' \dashrightarrow b$  in  $\mathbb{B}(\lambda)^+$ . The vertices  $b'$  and  $b''$  have an upper bound in this poset.

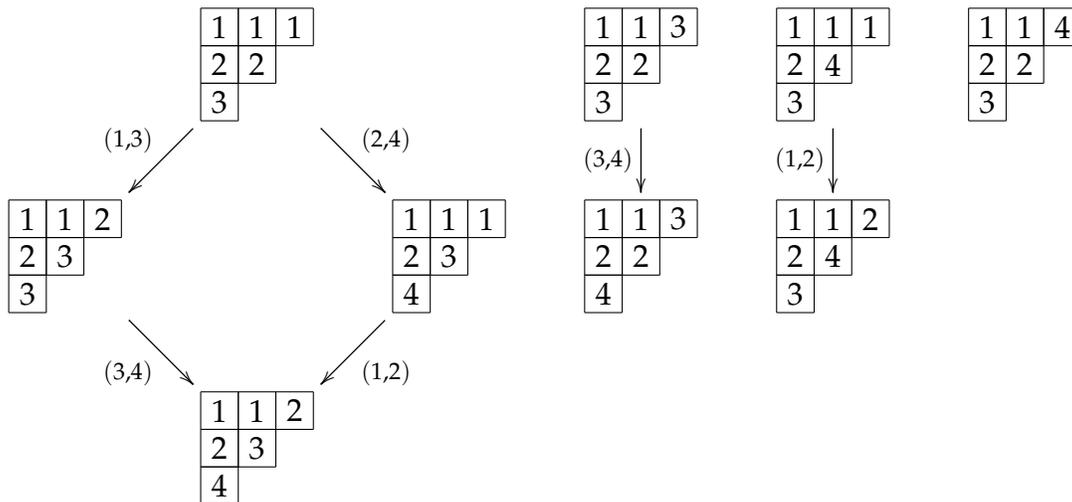


Figure 1: The modified crystal graph  $\mathbb{B}(\lambda)^+$  in Example 5.3.

The proof of this lemma relies on the structure of the short intervals in the dominant weight poset, discussed in Section 3.1. More precisely, we consider one by one all the types of short intervals, and for each of them, we show that we obtain the same structure in the poset  $\mathbb{B}(\lambda)^+$ . This is achieved by using the commutation relations between the modified crystal operators discussed in Section 4.1, namely Theorems 4.2 and 4.3.

**Theorem 5.2.** *The components of  $\mathbb{B}(\lambda)^+$  define a  $t$ -atomic decomposition. These components are isomorphic to intervals of the form  $[\widehat{0}, \mu]$  in the dominant weight poset via the weight projection, and the distinguished vertex  $h \in H(\lambda)$  in each of them is chosen to be the respective maximum.*

The proof has two parts. First, the atomic decomposition is proved (i.e., the  $t = 1$  case), by using Lemma 5.1 to derive the existence of a maximum and a minimum in each interval. Using the realization of  $B(\lambda)$  in terms of semistandard tableaux, we show that we can choose the statistic  $c(\cdot)$  in Definition 2.5 to be the Lascoux–Schützenberger charge [6], which expresses the type  $A$  Kostka–Foulkes polynomials combinatorially; only some basic properties of charge are needed. See [7] for the details.

**Example 5.3.** Consider  $\lambda = (3,2,1)$  in type  $A_3$ . The modified crystal graph  $\mathbb{B}(\lambda)^+$  is shown in Figure 1. Its vertices are labeled by semistandard Young tableaux whose content is a partition, and its edges are labeled as above. In particular, this graph gives the following atomic decomposition of the character:

$$\chi_\lambda = w_{(3,2,1)} + w_{(2,2,2)} + w_{(3,1,1,1)} + w_{(2,2,1,1)}.$$

## 5.2 Types $B_n, C_n,$ and $D_n$

We study the graph/poset  $\mathbb{B}(\lambda)^+$  in types  $B, C, D$  via a similar approach to that in type  $A$ .

**Theorem 5.4.** *The components of  $\mathbb{B}(\lambda)^+$  define an atomic decomposition. These components are isomorphic to intervals of the form  $[\widehat{0}, \mu]$  in the dominant weight poset via the weight projection.*

## 6 Additional facts and perspectives

We conjecture that the atomic decompositions in type  $B, C, D$ , as stated in [Theorem 5.4](#), are  $t$ -atomic decompositions, for appropriate choices of the statistic  $c(\cdot)$  in [Definition 2.5](#). Let us describe a possible such choice in type  $C$ , for partitions  $\lambda$  of even size. In [\[8\]](#), we gave a combinatorial formula for  $K_{\lambda,0}(t)$  based on a new statistic on *King tableaux* of zero weight (indexing the crystal vertices of zero weight). We can translate this statistic via the *Sheats bijection* [\[11\]](#) between King and Kashiwara–Nakashima tableaux (the crystal structure is only known on the latter), and then we can extend it recursively on  $\mathbb{B}(\lambda)^+$  via [\(2.15\)](#). By [\(2.17\)](#), such a statistic would give a combinatorial formula for all  $K_{\lambda,\mu}(t)$ .

In [\[7\]](#) we also derive a  $t$ -atomic decomposition for: (1) the crystal  $B(\infty)$  in types  $A - D$  and  $G_2$ ; (2) the crystal of the adjoint representation of  $\mathfrak{g}$  in any type.

## 7 Geometric interpretation

We give an interpretation of the combinatorial atomic decomposition in terms of the *geometric Satake correspondence*. For a reductive group  $G$ , this important theory exhibits a geometric realization of the irreducible representation  $V(\lambda)$  of highest weight  $\lambda$  of the Langlands dual group, as the *intersection cohomology*  $IH^*(\overline{Gr}_\lambda)$  of the *Schubert variety* denoted  $\overline{Gr}_\lambda$  in the *affine Grassmannian*  $Gr_G$  for  $G$ ; there is also a geometric basis of *MV cycles* [\[10\]](#). However, it is hard to give concrete formulas for the MV cycles and the action. We will show how one can understand the combinatorics of the geometric Satake correspondence via our combinatorial atomic decomposition.

The module  $IH^*(\overline{Gr}^\lambda)$  has the *truncation filtration* (or *standard Grothendieck filtration*), which gives the Kostka–Foulkes polynomials when restricted to the weight spaces [\[10\]](#). The degree 0 piece in this filtration is the cohomology of the constant sheaf  $H^*(\overline{Gr}^\lambda)$ , so

$$IH^*(\overline{Gr}^\lambda) \simeq H^*(\overline{Gr}^\lambda) \oplus \text{other summands.} \quad (7.1)$$

On another hand,  $H^*(\overline{Gr}^\lambda)$  has the basis of classes of Schubert varieties inside  $\overline{Gr}^\lambda$ , which are indexed by the weights of  $V(\lambda)$  considered without multiplicity, as recorded by the layer sum polynomials. In this language, the atomic decomposition decomposition [\(2.9\)](#),

cf. [Definition 2.1](#), is expressing the fact that there is a refinement of the truncation filtration (with the  $H^*(Gr_G)$ -action), whose successive quotients are isomorphic to  $H^*(\overline{Gr}^\mu)$  for  $\mu \in P^+(\lambda)$ . These quotients correspond precisely to the blocks of the partition in the combinatorial atomic decomposition, cf. [Definition 2.4](#).

## Acknowledgments

The second author gratefully acknowledges the partial support from the NSF grant DMS-1362627, and the hospitality of IHÉS during July-August 2018.

## References

- [1] T. Braden and R. MacPherson. “From moment graphs to intersection cohomology”. *Math. Ann.* **321.3** (2001), pp. 533–551. [Link](#).
- [2] T. Brylawski. “The lattice of integer partitions”. *Discrete Math.* **6** (1973), pp. 201–219. [Link](#).
- [3] M. Kashiwara. “Crystal bases of modified quantized enveloping algebra”. *Duke Math. J.* **73.2** (1994), pp. 383–413. [Link](#).
- [4] S. Kato. “Spherical functions and a  $q$ -analog of Kostant’s weight multiplicity formula”. *Invent. Math.* **66.3** (1982), pp. 461–468. [Link](#).
- [5] A. Lascoux. “Cyclic permutations on words, tableaux and harmonic polynomials”. *Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989)*. Manoj Prakashan, Madras, 1991, pp. 323–347.
- [6] A. Lascoux and M.-P. Schützenberger. “Sur une conjecture de H. O. Foulkes”. *C. R. Acad. Sci. Paris Sér. I Math.* **288** (1979), pp. 95–98.
- [7] C. Lecouvey and C. Lenart. “Atomic decomposition of characters and crystals”. 2018. [arXiv:1809.01262](#).
- [8] C. Lecouvey and C. Lenart. “Combinatorics of generalized exponents”. *Int. Math. Res. Not.* (2018). In press. [Link](#).
- [9] G. Lusztig. “Singularities, character formulas, and a  $q$ -analog of weight multiplicities”. *Analysis and Topology on Singular Spaces, II, III (Luminy, 1981)*. Astérisque 101. Soc. Math. France, Paris, 1983, pp. 208–229.
- [10] C. Mirković and V. Vilonen. “Geometric Langlands duality and representations of algebraic groups over commutative rings”. *Ann. of Math. (2)* **166.1** (2007), pp. 95–143. [Link](#).
- [11] J. Sheats. “A symplectic jeu de taquin bijection between the tableaux of King and of De Concini”. *Trans. Amer. Math. Soc.* **351.9** (1999), pp. 3569–3607. [Link](#).
- [12] M. Shimozono. “Multi-atoms and monotonicity of generalized Kostka polynomials”. *European J. Combin.* **22.3** (2001), pp. 395–414. [Link](#).
- [13] J. Stembridge. “The partial order of dominant weights”. *Adv. Math.* **136.2** (1998), pp. 340–364. [Link](#).