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Lattices from graph associahedra

Emily Barnard[1] and Thomas McConville*[2]

[1]*Department of Mathematics, Northeastern University*
[2]*Mathematical Sciences Research Institute*

Abstract. Given a graph G on n vertices, Postnikov defined a graph associahedron P_G as an example of a generalized permutohedron, a polytope whose normal fan coarsens the braid arrangement. Motivated by two general constructions of subalgebras of the Malvenuto-Reutenauer algebra, we consider the poset L_G obtained by orienting the one-skeleton of P_G. Because the normal fan of P_G coarsens the normal fan of the standard permutohedron we obtain a surjection $\Psi_G : \mathfrak{S}_n \to L_G$. We characterize the graphs G for which Ψ_G is a lattice quotient map.

Résumé. À partir d'un graphe G sur les sommets n, Postnikov a défini l'associahedron P_G du graphe comme un exemple de permutohèdre généralisé. Nous définissons un ordre partiel sur les sommets de P_G et étudions sa relation avec l'ordre faible du $\mathfrak{S}_n$.

Keywords: graph associahedra, Hopf algebras, lattices

1 Introduction

In Figure 1, we display the weak order on the symmetric group $\mathfrak{S}_3$ and show how to obtain the corresponding Tamari lattice. These two posets share three important qualities. First, the Hasse diagram for each poset is also the 1-skeleton of a simple polytope, the permutohedron and associahedron respectively. Second, each poset is also a lattice. (Recall that a poset is a lattice if each pair of elements x and y has a unique smallest upper bound $x \vee y$ and a unique largest lower bound $x \wedge y$.) Finally, the normal fan of the associahedron coarsens the normal fan of the standard permutohedron, which is the fan determined by a hyperplane arrangement known as the *braid arrangement*. (We recall the definition of the normal fan in Section 2.1.)

Pictorially, we see that this coarsening induces a canonical surjection Ψ from the vertices of the permutohedron to the vertices of the associahedron. It is well known that Ψ is a *lattice quotient map*. That is, Ψ preserves the meet and the join operations:

$$\Psi(x \vee y) = \Psi(x) \vee \Psi(y) \text{ and } \Psi(x \wedge y) = \Psi(x) \wedge \Psi(y).$$

In this paper, we study the relationship between the weak order on $\mathfrak{S}_n$ and a poset L_G that is analogous to the Tamari Lattice.

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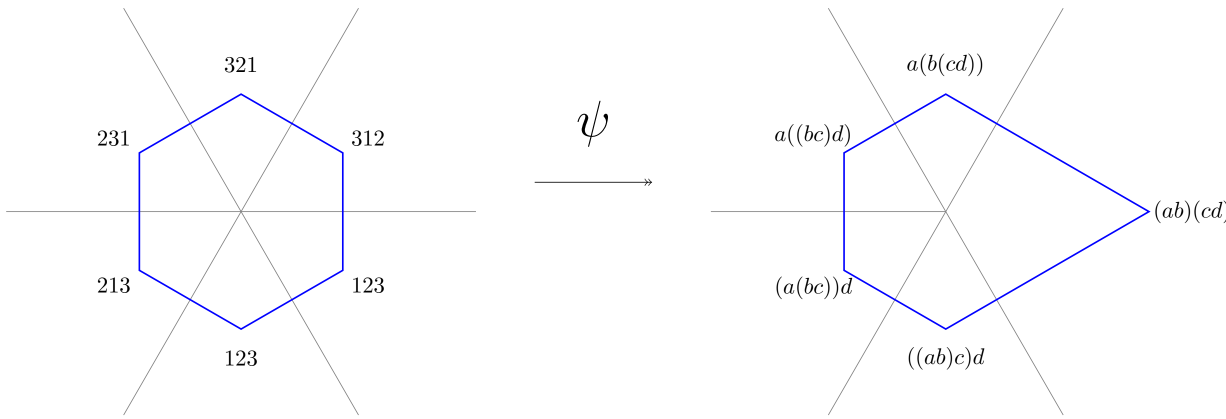

Figure 1: The canonical surjection from the weak order on $\mathfrak{S}_3$ to the Tamari lattice.

Given a graph G, Postnikov defined a graph associahedron P_G as an example of a *generalized permutohedron*, a simple polytope that is a Minkowski summand of the permutohedron [12]. Graph associahedra were also introduced independently in [2] and [3]. Some significant examples of graph associahedra include the associahedron, the cyclohedron, and the permutohedron.

Given a linear functional λ, we partially order the vertices of P_G by taking the transitive and reflexive closure of the relation $\mathbf{x} \leq \mathbf{y}$ when $[\mathbf{x}, \mathbf{y}]$ is an edge of P_G and $\lambda(\mathbf{x}) \leq \lambda(\mathbf{y})$. We define L_G to be the resulting poset. It turns out that the edge $[\mathbf{x}, \mathbf{y}]$ in P_G is actually a cover relation $\mathbf{x} \lessdot \mathbf{y}$ in L_G. Like the Tamari lattice, the Hasse diagram of L_G is the 1-skeleton of a simple polytope, namely P_G. Like the associahedron, the normal fan of P_G coarsens the normal fan of the permutohedron. Thus we obtain a canonical surjection $\Psi_G : \mathfrak{S}_n \to L_G$. The following theorem is our main result. In the statement, a graph G is *filled* if for each edge $\{i, k\}$ in G, there are edges $\{i, j\}$ and $\{j, k\}$ in G whenever $i < j < k$.

Theorem 1.1. The map Ψ_G is a lattice quotient map if and only if G is filled.

A key element of our proof is a combinatorial description of L_G as certain collections of connected subgraphs of G called *tubings*. We recall these definitions in Section 2. Along the way, we show that each face of the P_G is an interval in the poset L_G. We call this the *non-revisiting chain property*. See Section 3.2.

The genesis for Theorem 1.1 came from comparing two different Hopf algebra constructions. In [14], Ronco defined a binary operation on a vector space generated by the tubings of an "admissible" family of graphs $\mathcal{G}$, which gives this space the structure of an associative algebra. We call this algebra a *tubing algebra*. In particular, when $\mathcal{G}$ is the set of complete graphs K_n or path graphs P_n, the tubing algebra is isomorphic to either the Malvenuto-Reutenauer algebra on permutations [8] or the Loday-Ronco algebra on binary trees [7], respectively. Reading introduced a general technique to construct subalgebras of the Malvenuto-Reutenauer algebra using lattice quotients of the weak order on permutations in [13]. We use Theorem 1.1 to show that these two constructions substantially overlap.

Most statements in this abstract are made without proofs. Complete proofs, additional results and examples can be found in [1].

2 Posets of maximal tubings

In the following sections we recall the necessary background for our main result. We begin by defining the simple polytope P_G. Then, we define the poset L_G, and we recall the canonical surjection $\Psi_G : \mathfrak{S}_n \to L_G$. Finally, we describe a combinatorial realization of L_G in terms of certain connected subgraphs of G that will be useful when we discuss the proof of Theorem 1.1.

2.1 The normal fan of a polytope

Before defining the graph associahedron P_G, we recall the definition of the normal fan of a polytope.

A *(polyhedral) fan* $\mathcal{N}$ is a set of cones in $\mathbb{R}^n$ such that for any two elements $C, C' \in \mathcal{N}$, their intersection $C \cap C'$ is in $\mathcal{N}$ and it is a face of both C and C'. It is *complete* if $\bigcup_{C \in \mathcal{N}} C = \mathbb{R}^n$ and *pointed* if $\{0\} \in \mathcal{N}$. A pointed fan $\mathcal{N}$ is *simplicial* if the number of extreme rays of each $C \in \mathcal{N}$ is equal to its dimension. We consider a simplicial fan to be a type of "realization" of a simplicial complex; more accurately, it is a cone over a geometric realization.

For a polytope $P \subseteq \mathbb{R}^n$ and $f \in (\mathbb{R}^n)^*$ in the dual space, we let P^f be the subset of P at which f achieves its maximum value. We consider an equivalence relation on $(\mathbb{R}^n)^*$ where $f \sim g$ if $P^f = P^g$. It is not hard to show that each equivalence class is a relatively open polyhedral cone. The *normal fan* of P is the set of closures of these cones, which forms a complete polyhedral fan. A polytope is simple if and only if its normal fan is simplicial.

For polytopes $P, Q \subseteq \mathbb{R}^n$, their *Minkowski sum* $P + Q$ is the polytope

$$P + Q = \{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in P,\ \mathbf{y} \in Q\}.$$

Recall that the normal fan of $P + Q$ is the coarsest common refinement of the normal fans of P and Q [16, Proposition 7.12].

2.2 Graph associahedra

Let $G = (V, E)$ be a simple graph with vertex set $V = [n] = \{1, \dots, n\}$. If $I \subseteq V$, we let $G|_I$ denote the induced subgraph of G with vertex set I. A *tube* is a nonempty subset I

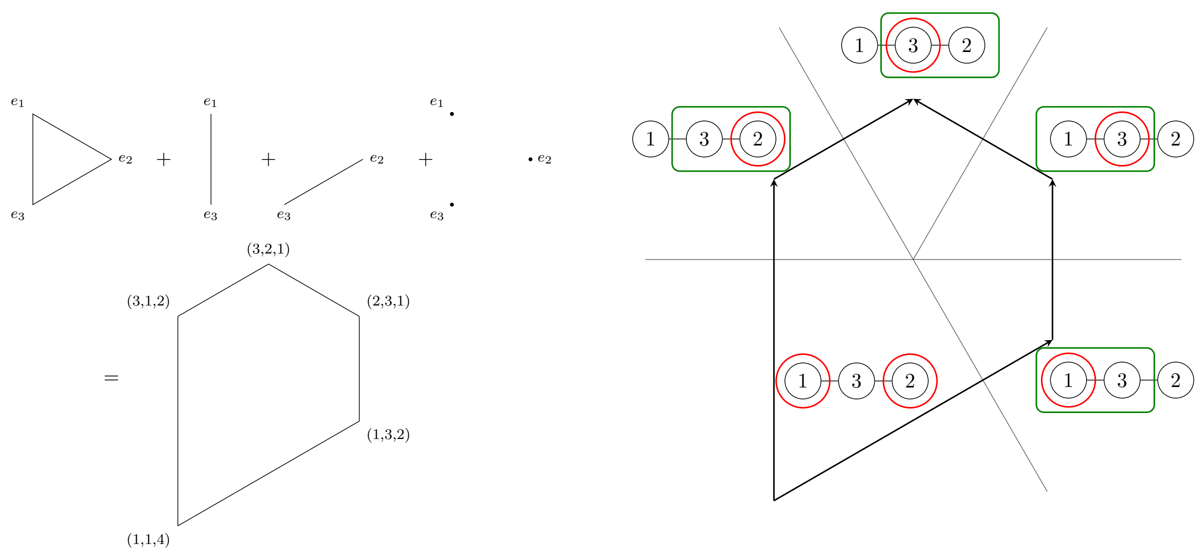

Figure 2: The graph associahedron for the graph with edge set $E = \{\{1,3\},\{3,2\}\}$ and the corresponding poset of maximal tubings L_G.

of vertices such that the induced subgraph $G|_I$ is connected. Any tube not equal to V is called a *proper tube*. We let $\mathcal{I}(G)$ be the set of all tubes of G.

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis vectors in $\mathbb{R}^n$. Given $I \subseteq [n]$, let Δ_I be the simplex with vertices $\{\mathbf{e}_i \mid i \in I\}$. The *graph associahedron* P_G is the Minkowski sum of simplices Δ_I over all tubes I of G; that is,

$$P_G = \sum \Delta_I = \left\{ \sum \mathbf{x}_I \mid (\mathbf{x}_I \in \Delta_I : \; I \text{ is a tube}) \right\}.$$

On the left-hand of Figure 2, we depict the Minkowski sum construction for P_G where G is the path graph with edges $\{1,3\}$ and $\{3,2\}$.

Fix $\lambda : \mathbb{R}^n \to \mathbb{R}$ such that $\lambda(x_1, x_2, \ldots, x_n) = nx_1 + (n-1)x_2 + \cdots + x_n$. We define the *poset of maximal tubings* L_G to be the poset whose partial order is the reflexive and transitive closure of the relation $\mathbf{x} \leq \mathbf{y}$ when $[\mathbf{x}, \mathbf{y}]$ is an edge of P_G and $\lambda(\mathbf{x}) \leq \lambda(\mathbf{y})$.

When G is a complete graph, the polytope P_G is the “standard” permutohedron, and its normal fan $\mathcal{N}_G$ is the set of cones defined by the braid arrangement. The poset L_G is isomorphic to the weak order on $\mathfrak{S}_n$. When G is a path graph, P_G is the associahedron, and L_G is the Tamari lattice. For a general graph G, the polytope P_G is a Minkowski summand of the standard permutohedron, so its normal fan is coarser than that defined by the braid arrangement. Thus, for each graph G, we obtain a canonical surjection $\Psi_G : \mathfrak{S}_n \to L_G$ analogous to the canonical surjection depicted in Figure 1.

2.3 Tubings and G-trees

To describe the proof of Theorem 1.1, we will need a combinatorial realization of L_G in terms of maximal tubings and G-trees. Two tubes I, J are said to be *compatible* if either

- they are *nested*: $I \subseteq J$ or $J \subseteq I$, or
- they are *separated*: $I \cup J$ is not a tube.

A *tubing* $\mathcal{X}$ of G is any collection of pairwise compatible tubes. A collection $\mathcal{X}$ is said to be a *maximal tubing* if it is maximal by inclusion. We let $\mathrm{MTub}(G)$ be the set of maximal tubings of the graph G.

Any maximal tubing $\mathcal{X}$ contains exactly n tubes. In the next lemma, $\mathcal{X}|_I$ is the set of all tubes $J \in \mathcal{X}$ such that $J \subseteq I$.

Lemma 2.1. If $\mathcal{X}$ is a maximal tubing, then each tube I contains a unique element $\mathrm{top}_{\mathcal{X}}(I) \in [n]$ not contained in any proper tube of $\mathcal{X}|_I$. Furthermore, the function $\mathrm{top}_{\mathcal{X}}$ is a bijection from the tubes in $\mathcal{X}$ to the vertex set $[n]$.

The set of all tubings of G has the structure of a flag simplicial complex called the *nested set complex*, denoted Δ_G. The nested set complex may be realized as a simplicial fan that is isomorphic to the normal fan of P_G [2, Theorem 2.6], [4, Theorem 3.14], [12, Theorem 7.4]. Thus the face lattice of P_G is dual to the face lattice of Δ_G. So, for example, each maximal tubing of G corresponds bijectively to a vertex of P_G; see [12, Proposition 7.9]. In the lemma below, we interpret $i_\downarrow$ as the smallest tube in $\mathcal{X}$ that contains the element i. (This notation will be explained by the connection to G-trees given later in this section.)

Lemma 2.2. If $\mathcal{X}$ is any maximal tubing, the point $\mathbf{v}^{\mathcal{X}} = (v_1, \ldots, v_n)$ is a vertex of P_G where v_i is the number of tubes $I \in \mathcal{I}(G)$ (not necessarily contained in $\mathcal{X}$) such that $i \in I$ and $I \subseteq i_\downarrow$. Conversely, every vertex of P_G comes from a maximal tubing in this way.

We now explain why L_G is called the poset of maximal tubings. Suppose that I is a non-maximal tube in $\mathcal{X}$. Because the face lattice of P_G is dual to the face lattice of Δ_G, there exists a unique tube J distinct from I such that $\mathcal{Y} = \mathcal{X} \setminus \{I\} \cup \{J\}$ is a maximal tubing of G. Define a *flip* as the relation $\mathcal{X} \to \mathcal{Y}$ if $\mathrm{top}_{\mathcal{X}}(I) < \mathrm{top}_{\mathcal{Y}}(J)$. We say $\mathcal{X} \leq \mathcal{Y}$ holds if there exists a sequence of flips of maximal tubings of the form $\mathcal{X} \to \cdots \to \mathcal{Y}$. The relation $(\mathrm{MTub}(G), \leq)$ was independently introduced by Forcey [5] and Ronco [14].

Lemma 2.3. The poset L_G is isomorphic to $(\mathrm{MTub}(G), \leq)$.

Proof sketch. The edges of the graph associahedron P_G take the following form. Let $\mathcal{X}$ and $\mathcal{Y}$ be maximal tubings of G such that $\mathcal{Y} = \mathcal{X} \setminus \{I\} \cup \{J\}$ for some distinct tubes I, J. Set $i = \mathrm{top}_{\mathcal{X}}(I)$ and $j = \mathrm{top}_{\mathcal{Y}}(J)$. Then the vertices $\mathbf{v}^{\mathcal{X}}$ and $\mathbf{v}^{\mathcal{Y}}$ agree on every coordinate

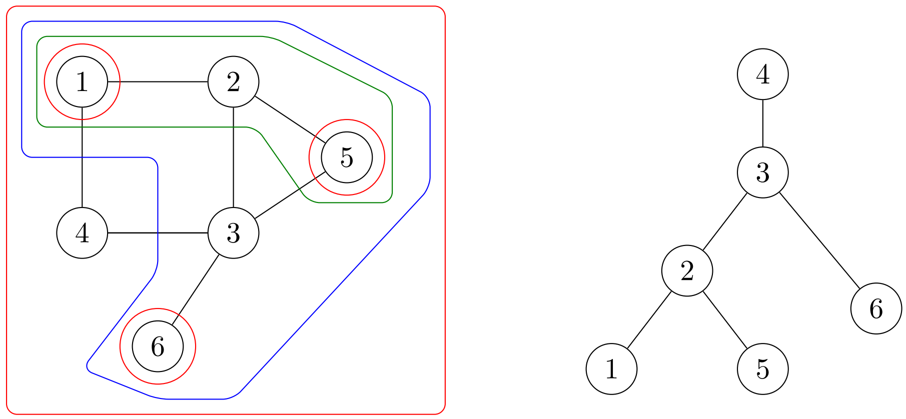

Figure 3: (left) A maximal tubing. (right) Its associated G-tree.

except the i^{th} and j^{th} coordinates. Indeed, $\mathbf{v}^{\mathcal{Y}} - \mathbf{v}^{\mathcal{X}} = r(\mathbf{e}_i - \mathbf{e}_j)$ where r is equal to the number of tubes of G contained in $I \cup J$ that contain both i and j.

Recall that λ is the linear functional $\lambda(x_1, \ldots, x_n) = nx_1 + (n-1)x_2 + \cdots + x_n$. If $\mathcal{X}$ and $\mathcal{Y}$ are as above and $i < j$, then $\lambda(\mathbf{v}^{\mathcal{Y}} - \mathbf{v}^{\mathcal{X}}) > 0$. Hence, $\mathbf{v}^{\mathcal{Y}} \gtrdot \mathbf{v}^{\mathcal{X}}$. □

An example of the poset L_G is given in Figure 2, where G is the path graph with edge set $E = \{\{1,3\},\{3,2\}\}$. The figure demonstrates that the relation $(\mathrm{MTub}(G), \leq)$ defined above is indeed the transitive and reflexive closure of an orientation of the 1-skeleton of P_G.

It will be convenient to encode a maximal tubing in terms of a certain poset on $[n]$. Let T be a forest with vertex set $[n]$. The *forest poset* associated with T is defined by the relation $i <_T k$ whenever i and k belong to the same connected component of T, and the unique path from i to the root of this component passes through k. We usually denote this forest poset by T as well.

Let $i_\downarrow$ denote the principal order ideal generated by i in T. We say that T is a *G-forest*, or *G-tree* when T is connected, if it satisfies both of the following conditions (see also [11, Definition 8.1]):

- for each $i \in [n]$, the set $i_\downarrow$ is a tube of G;
- if i and k are incomparable in T, then $i_\downarrow \cup k_\downarrow$ is not tube of G.

Given a G-forest T, observe that the collection $\chi(T) = \{i_\downarrow : i \in [n]\}$ is a maximal tubing on G. An example of this correspondence is shown in Figure 3. The following theorem is essentially a specialization of [11, Proposition 8.2].

Theorem 2.4. Let G be a graph with vertex set $[n]$. Then the map $\chi : T \mapsto \{i_\downarrow : i \in [n]\}$ is a bijection from the set of G-forests to the set of maximal tubings of G.

3 Main results

3.1 Covering relations of the poset of maximal tubings

We are now prepared to outline the key steps in the proof of Theorem 1.1. We begin by building some intuition coming from the cover relations in L_G. In terms of maximal tubings, recall that $\mathcal{X} \lessdot \mathcal{Y}$ provided that $\mathcal{Y} = \mathcal{X} \setminus \{I\} \cup \{J\}$ for some distinct tubes I, J, and $i = \mathrm{top}_{\mathcal{X}}(I) < \mathrm{top}_{\mathcal{Y}}(J) = j$. Because each cover relation "swaps" a pair of integers i and j, one might naïvely guess that the size of any maximal chain in L_G is bounded above by $\binom{n}{2}$. For comparison, each maximal chain in the weak order on $\mathfrak{S}_n$ has size equal to $\binom{n}{2}$. Surprisingly, this guess is false in general. The reader can check in Figure 4 that L_G has a maximal chain of size 7. Indeed, the poset L_G in this example is not a lattice. (The two indicated atoms have two minimal upper bounds.) In this example, there is no hope that the canonical surjection $\Psi_G : \mathfrak{S}_n \to L_G$ is a lattice quotient map.

When G is a filled graph, our naïve guess is true. The size of each maximal chain in L_G is bounded by $\binom{n}{2}$. To prove one direction of Theorem 1.1, assume that G is filled and let T be a G-forest. We say that a permutation $\sigma \in \mathfrak{S}_n$ is a *G-permutation* provided that it is the lexicographically minimal linear extension of T. (See [11] for an equivalent definition.) We note that the fiber $\Psi_G^{-1}(T)$ is precisely the set of all linear extensions of T.

For G connected, the associated G-permutation can be constructed recursively as follows. First, remove the root x of T. Let $C_1, \ldots, C_r$ be the connected components of $T \setminus \{x\}$. We index the connected components so that each element of C_i is less than each element of C_j (as integers) whenever $i < j$. (The components of T can be indexed in this way because G is filled.) Next, we apply the construction to each component to obtain a word $\sigma(C_i) = v_{C_{i1}} \ldots v_{C_{is}}$ for $i \in [r]$. Finally, we concatenate the words $\sigma(C_1) \ldots \sigma(C_r)$, ending with the root x. For example, G-permutation for the G-tree shown Figure 3 is 152634. When G is the path graph with vertices labeled $1, 2, \ldots, n$ from left to right, the set of G-permutations is equal to the set of 312 -avoiding permutations of $\mathfrak{S}_n$.

There is a natural surjection from the weak order on $\mathfrak{S}_n$ to the subposet of the weak order induced by the set of G-permutations of $[n]$. As a first step in our proof of Theorem 1.1, we show that this surjection, which only involves the combinatorics of $\mathfrak{S}_n$, is a lattice quotient map. The second (and more technical) step of the proof is showing that L_G is isomorphic to this subposet of G-permutations. Recall that the inversion set of a permutation σ is the set of pairs (i, j) where $i < j$ and j precedes i in the one-line notation for σ. By analogy, define a pair of integers (i, j) to be an *inversion* of a G-tree T if $i < j$ and $j <_T i$. It follows from our recursive construction that the inversion set of T is equal to the inversion set of the G-permutation $\sigma(T)$. In the weak order, $\sigma < \tau$ if and only if $\mathrm{inv}(\sigma) \subset \mathrm{inv}(\tau)$. To complete the proof, we show that two G-trees are ordered $T < T'$ in L_G if and only if $\mathrm{inv}(T) \subset \mathrm{inv}(T')$. Characterizing the cover relations in L_G was a key element of this argument. (See [1, Proposition 2.24 and Lemma 4.12].)

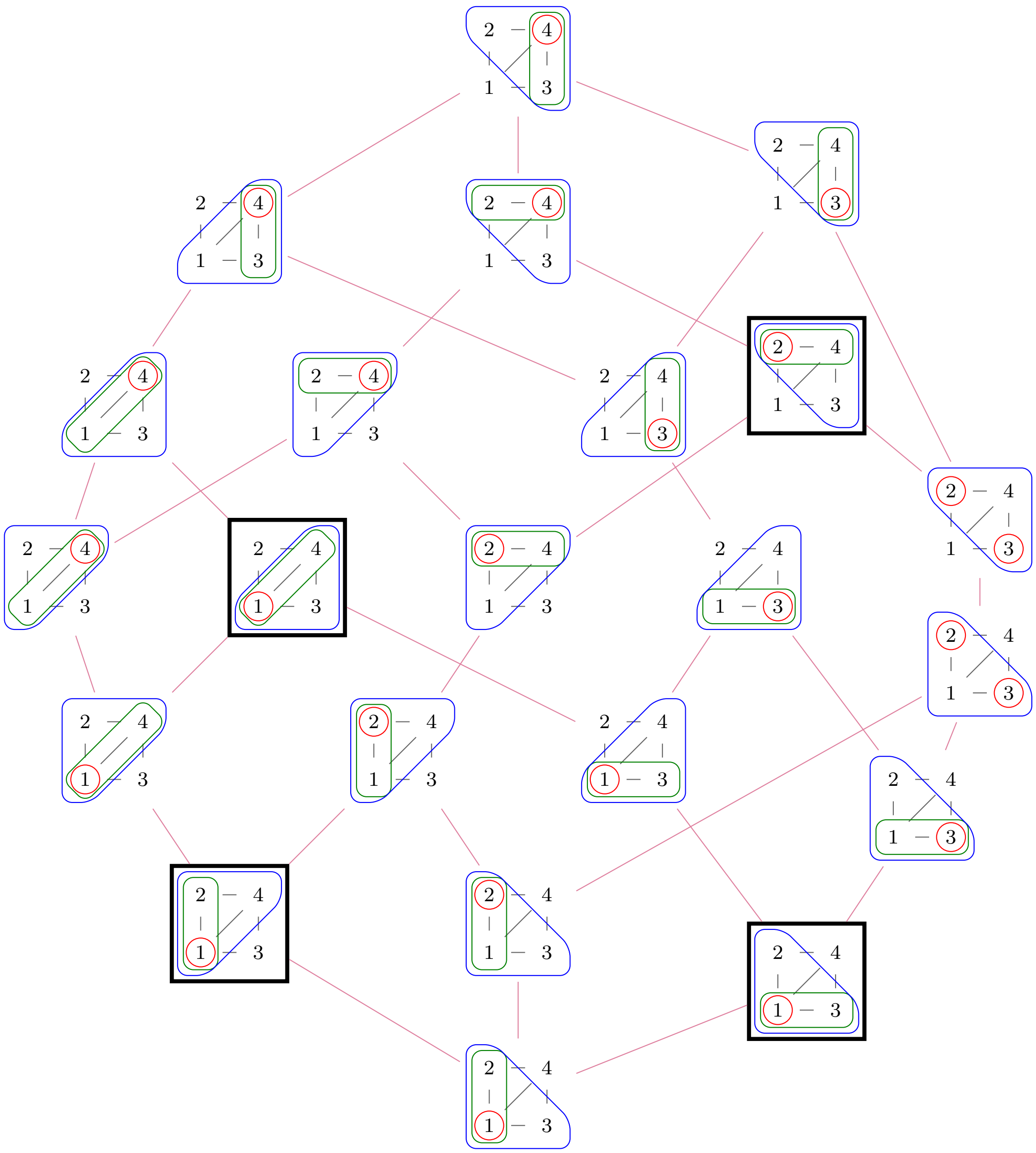

Figure 4: A poset of maximal tubings that is not a lattice.

To prove the remaining direction of Theorem 1.1, we reduce the problem to a certain subgraph of G. We show that if $\Psi_G : \mathfrak{S}_n \to L_G$ is a lattice quotient map, then restricting to any subset I of vertices also produces a lattice quotient map from the weak order to $L_{G|_I}$. If G is not filled then there exists some edge $\{i,k\}$ in G such that either $\{i,j\}$ or $\{j,k\}$ is not an edge, for some $j \in [i+1, k-1]$. To complete the proof, it is enough to show that we do not have a lattice quotient map from the weak order to $L_{G|_I}$ where $I = \{i,j,k\}$.

The graph associahedron $P_{G|_I}$ is a face of P_G. In the next section, we show that each face of P_G is actually an interval in the poset L_G. This is equivalent to the statement that for any tubing $\mathcal{X}$, the set of maximal tubings containing $\mathcal{X}$ is an interval of L_G.

3.2 The non-revisiting chain property

In this section, we prove that graph associahedra have the non-revisiting chain property, defined below.

Given a polytope P, we will say a linear functional $\lambda : \mathbb{R}^n \to \mathbb{R}$ is *generic* if it is not constant on any edge of P. When λ is generic, we let $L(P,\lambda)$ be the poset on the vertices of P where $v \leq w$ if there exists a sequence of vertices $v = v_0, v_1, \ldots, v_l = w$ such that $\lambda(v_0) < \lambda(v_1) < \cdots < \lambda(v_l)$ and $[v_{i-1}, v_i]$ is an edge for all $i \in \{1, \ldots, l\}$.

The following properties of $L(P,\lambda)$ are immediate.

Proposition 3.1. Let P be a polytope with a generic linear functional λ.

1. The dual poset $L(P,\lambda)^*$ is isomorphic to $L(P,-\lambda)$.
2. If F is a face of P, then the inclusion $L(F,\lambda) \hookrightarrow L(P,\lambda)$ is order-preserving.
3. $L(P,\lambda)$ has a unique minimum $v_{\hat{0}}$ and a unique maximum $v_{\hat{1}}$.

The pair (P,λ) is said to have the *non-revisiting chain (NRC) property* if whenever $\mathbf{x} < \mathbf{y} < \mathbf{z}$ in $L(P,\lambda)$ such that $\mathbf{x}$ and $\mathbf{z}$ lie in a common face F, then $\mathbf{y}$ is also in F. The name comes from the fact that if P has the NRC property, then any sequence of vertices following edges monotonically in the direction of λ does not return to a face after leaving it. By definition, the NRC property means that faces are *order-convex* subsets of $L(P,\lambda)$. (Recall that a subset S of a poset is *order-convex* provided that whenever elements $x, z \in S$ satisfy $x < z$ then the entire interval $[x,z]$ belongs to S.) In light of Proposition 3.1, this is equivalent to the condition that for any face F, the set of vertices of F form an interval of $L(P,\lambda)$ isomorphic to $L(F,\lambda)$.

In contrast to the non-revisiting path property, many low-dimensional polytopes lack the non-revisiting chain property. For example, if P is a simplex of dimension at least 2, then $[\mathbf{v}_{\hat{0}}, \mathbf{v}_{\hat{1}}]$ is an edge of P that is not an interval of $L(P,\lambda)$. However, the property does behave nicely under Minkowski sum.

Proposition 3.2. If (P,λ) and (Q,λ) have the non-revisiting chain property, then so does $(P+Q,\lambda)$.

The proof of Proposition 3.2 relies on Lemma 3.3. For polytopes P and Q, the normal fan of $P+Q$ is the common refinement of $\mathcal{N}(P)$ and $\mathcal{N}(Q)$; that is,

$$\mathcal{N}(P+Q) = \{C \cap C' \mid C \in \mathcal{N}(P),\ C' \in \mathcal{N}(Q)\}.$$

Let $V(P)$ be the set of vertices of P, and let C_v be the normal cone to the vertex v in P. From the description of the normal fan of $P+Q$, there is a canonical injection $\iota : V(P+Q) \hookrightarrow V(P) \times V(Q)$ that assigns a vertex $\mathbf{v} \in P+Q$ to $(\mathbf{u},\mathbf{w})$ if the normal cones satisfy $C_{\mathbf{v}} = C_{\mathbf{u}} \cap C_{\mathbf{w}}$.

Lemma 3.3. The map $\iota : V(P+Q) \hookrightarrow V(P) \times V(Q)$ is an order-preserving function from $L(P+Q,\lambda)$ to $L(P,\lambda) \times L(Q,\lambda)$.

Proof of Proposition 3.2. Every face of $P+Q$ is of the form $F+F'$ where F is a face of P and F' is a face of Q. Suppose $\mathbf{u},\mathbf{v},\mathbf{w}$ are vertices of $P+Q$ such that $\mathbf{u} < \mathbf{v} < \mathbf{w}$ in $L(P+Q,\lambda)$ and $\mathbf{u},\mathbf{w} \in F+F'$. Set $\iota(\mathbf{u}) = (\mathbf{u}_P,\mathbf{u}_Q)$, and analogously for $\iota(\mathbf{v})$ and $\iota(\mathbf{w})$. Then $\mathbf{u}_P \leq \mathbf{v}_P \leq \mathbf{w}_P$ in $L(P,\lambda)$ and $\mathbf{u}_Q \leq \mathbf{v}_Q \leq \mathbf{w}_Q$ in $L(Q,\lambda)$. Since P and Q have the non-revisiting chain property, $\mathbf{v}_P$ is in F and $\mathbf{v}_Q$ is in F'. Hence, $\mathbf{v} = \mathbf{v}_P + \mathbf{v}_Q$ is in $F+F'$, as desired. □

Corollary 3.4 (Proposition 7.2 [6]). Every zonotope has the non-revisiting chain property with respect to any generic linear functional.

We now return to graph associahedra.

Theorem 3.5. The pair (P_G,λ) has the non-revisiting chain property.

Corollary 3.6. For any tubing $\mathcal{Y}$ of G, the set of maximal tubings which contain $\mathcal{Y}$ is an interval in L_G.

Remark 3.7. Another property that a polytope graph may have is the *non-leaving face property*, which is satisfied if for any two vertices $\mathbf{u},\mathbf{v}$ that lie in a common face F of P, every geodesic between $\mathbf{u}$ and $\mathbf{v}$ is completely contained in F. This property holds for all zonotopes, but is quite special for general polytopes. Although ordinary associahedra are known to have the non-leaving face property [15], not all graph associahedra do. We note that the example geodesic in [9, Figure 6] that leaves a particular facet cannot be made into a monotone path, so it does not contradict our Theorem 3.5.

Recall that the Möbius function $\mu = \mu_L : \mathrm{Int}(L) \to \mathbb{Z}$ is the unique function on the intervals of a finite poset L such that for $x \leq y$:

$$\sum_{x \leq z \leq y} \mu(x,z) = \begin{cases} 1 \text{ if } x = y \\ 0 \text{ if } x \neq y \end{cases}.$$

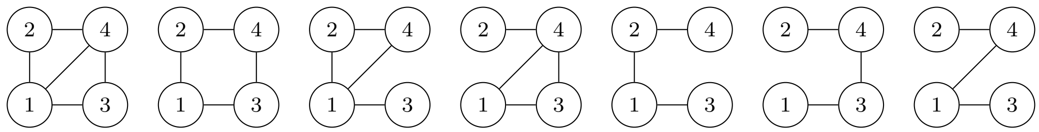

Figure 5: Graphs with four vertices such that L_G is not a lattice.

When $L(P, \lambda)$ is a lattice with the non-revisiting chain property, the Möbius function was determined in [6]. One way to prove this is to show that $L(P, \lambda)$ is a crosscut-simplicial lattice; cf. [10]. In the case of the poset of maximal tubings, we may express the Möbius function as follows. For a tubing $\mathcal{X}$, let $|\mathcal{X}|$ be the number of tubes it contains.

Corollary 3.8. Let G be a graph with vertex set $[n]$ such that L_G is a lattice. Let $\mathcal{X}$ be a tubing that contains every maximal tube. The set of maximal tubings containing $\mathcal{X}$ is an interval $[\mathcal{Y}, \mathcal{Z}]$ of L_G such that $\mu(\mathcal{Y}, \mathcal{Z}) = (-1)^{n-|\mathcal{X}|}$. If $[\mathcal{Y}, \mathcal{Z}]$ is not an interval of this form, then $\mu(\mathcal{Y}, \mathcal{Z}) = 0$.

Based on some small examples, we are inclined to believe that Corollary 3.8 is true even without the assumption that L_G is a lattice.

4 Open problems

A fundamental problem is to characterize all graphs such that L_G is a lattice. To this end, we make the simple observation that an interval L' of a lattice L is a sublattice of L. In particular if G' is any graph obtained by contracting or deleting vertices of G such that $L_{\mathrm{std}(G')}$ is not a lattice, then L_G is not a lattice either. Continuing to borrow from matroid terminology, we say that G' is a *minor* of G if it is the standardization of a sequence of contractions and deletions.

Problem 4.1. Give an explicit list of minors such that L_G is a lattice whenever G does not contain a minor from the list.

By exhaustive search, we found that when G is a connected graph with four vertices, the poset L_G is not a lattice if and only if $\{1, 3\}$ and $\{2, 4\}$ are edges but $\{2, 3\}$ is not an edge in G. These are the seven graphs shown in Figure 5.

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