Séminaire Lotharingien de Combinatoire **82B** (2019)
Article #56, 12 pp.
Proceedings of the 31st Conference on Formal Power Series and Algebraic Combinatorics (Ljubljana)

Descent Representations of Generalized Coinvariant Algebras

Kyle P. Meyer*

Department of Mathematics, University of California, San Diego

Abstract. The coinvariant algebra R_n is a well-studied $\mathfrak{S}_n$-module that is a graded version of the regular representation of $\mathfrak{S}_n$. Using a straightening algorithm on monomials and the Garsia-Stanton basis, Adin, Brenti, and Roichman(2005) define modules $R_{n,\mu}$, that refine the grading of R_n, and they describe the Frobenius image of $R_{n,\mu}$ in terms of standard Young tableaux with certain descents. Motivated by the Delta Conjecture of Macdonald polynomials, Haglund, Rhoades, and Shimozono (2016) define a module $R_{n,k}$ that extends the coinvariant algebra. Also motivated by the Delta Conjecture, Benkart et al. (2018) defined a crystal structure in terms of the minimaj statistic that up to some twisting has character equal to the Frobenius image of $R_{n,k}$. We generalize the results of Adin, Brenti, and Roichman by defining modules $R_{n,k,\mu}$ that refine $R_{n,k}$ and give a combinatorial description of the Frobenius image. This description not only refines and simplifies some of the results of Haglund, Rhoades, and Shimozono, but also gives a simpler method of obtaining their results. Additionally, these modules give a representation theoretic interpretation for the characters of crystals that Benkart et al. use to build up their minimaj crystal.

Keywords: combinatorics, representations, symmetric group

1 Introduction

The classical coinvariant algebra R_n is constructed as follows: let the symmetric group $\mathfrak{S}_n$ act on the polynomial ring $\mathbb{Q}[x_1, x_2, \ldots, x_n]$ by permutation of the variables $x_1, \ldots, x_n$. The polynomials that are invariant under this action are called symmetric polynomials, and we let I_n be the ideal generated by symmetric polynomials with vanishing constant term. Then R_n is defined as the algebra obtained by quotienting $\mathbb{Q}[x_1, x_2, \ldots, x_n]$ by I_n, that is

$$R_n := \frac{\mathbb{Q}[x_1, x_2, \ldots, x_n]}{I_n}. \tag{1.1}$$

Since symmetric functions in n variables are generated by the elementary symmetric functions e_d, we have that:

$$I_n = \langle e_1, e_2, \ldots, e_n \rangle. \tag{1.2}$$

*kpmeyer@ucsd.edu

Since I_n is homogeneous and invariant under the action of $\mathfrak{S}_n$, the coinvariant algebra is a graded $\mathfrak{S}_n$-module. The irreducible representations of $\mathfrak{S}_n$ are indexed by partitions of n, and we let S^λ denote the irreducible representation corresponding to λ, and we let χ^λ_μ be the character of S^λ evaluated at an element of type μ.

Given a representation V of $\mathfrak{S}_n$, the **Frobenius image** of V, denoted $Frob(V)$ has the following formula

$$Frob(V) = \sum_{\lambda \vdash n} c_\lambda s_\lambda, \tag{1.3}$$

where c_λ is the multiplicity of S^λ in V and s_λ is the Schur function associated to λ. For R_n, Chevalley [3] showed that the Frobenius image of R_n is a sum over standard Young tableaux of shape λ, that is that

$$Frob(R_n) = \sum_{T \in SYT(n)} s_{sh(T)} \tag{1.4}$$

If V is a graded representation of $\mathfrak{S}_n$ with degree d component V_d, then we can also consider the Frobenius image of V_d for all d. This data can be combined into a single function called the **graded Frobenius image**, which is defined as follows:

$$grFrob(V;q) = \sum_{d=0}^{\infty} q^d Frob(V_d). \tag{1.5}$$

Lusztig (unpublished) and Stanley [7] showed that,

$$grFrob(R_n;q) := \sum_{T \in SYT(n)} q^{maj(T)} s_{shape(T)}. \tag{1.6}$$

A further refinement of R_n is given as follows: define

$$P_{\trianglelefteq \mu} := \text{span}\{m \in \mathbb{Q}[x_1, \dots, x_n] : \lambda(m) \trianglelefteq \mu\}, \tag{1.7}$$

$$P_{\triangleleft \mu} := \text{span}\{m \in \mathbb{Q}[x_1, \dots, x_n] : \lambda(m) \triangleleft \mu\} \tag{1.8}$$

where m are monomials, $\lambda(m)$ is the exponent partition of m, and $\triangleleft$ is the dominance order on partitions. Then let $Q_{\trianglelefteq \mu}$ and $Q_{\triangleleft \mu}$ be the projections of $P_{\trianglelefteq \mu}$ and $P_{\triangleleft \mu}$ onto R_n respectively. Next define

$$R_{n,\mu} := Q_{\trianglelefteq \mu} / Q_{\triangleleft \mu}. \tag{1.9}$$

This is a refinement of the grading since the degree d component of R_n is equal to

$$\bigoplus_{\mu \vdash d} R_{n,\mu}. \tag{1.10}$$

Adin, Brenti, and Roichman [1] show that $R_{n,\mu}$ is zero unless μ is a partition with at most $n-1$ parts such that the differences between consecutive parts are at most 1. They also show that in the case that $R_{n,\mu}$ is not zero, the multiplicity of S^λ in $R_{n,\mu}$ is given by the number of standard Young tableaux of shape λ with descent set equal to the descent set of μ. Where a descent of a partition μ is a value i such that $\mu_i > \mu_{i+1}$.

The Delta Conjecture, introduced by Haglund, Remmel and Wilson [5] as a generalization of the Shuffle Theorem (at the time still a conjecture), is the conjectural equality of three families of formal power series of symmetric functions in two indeterminates q and t. One of the three families is defined in terms of certain Macdonald polynomial eigenoperators and the other two are defined in terms of certain statistics on labeled Dyck paths. By its definition, the Delta Conjecture is intimately related to the combinatorics of labeled Dyck paths, but it is also related to a number of different combinatorial objects such as parking functions and ordered set partitions, making it a rich area of research.

One shortcoming of the Delta Conjecture is that, as defined, the quantities involved are all just formal power series without any algebraic or geometric interpretations that could be used to study them. With a view towards remedying this, Haglund, Rhoades, and Shimozono [6] gave a generalize of the coinvariant ideal by defining the ideal

$$I_{n,k} := \langle x_1^k, x_2^k, \dots x_n^k, e_n, e_{n-1}, \dots, e_{n-k+1} \rangle, \tag{1.11}$$

for a positive integer $k \leq n$. They then define a generalized coinvariant algebra as

$$R_{n,k} := \frac{\mathbb{Q}[x_1, \dots, x_n]}{I_{n,k}}. \tag{1.12}$$

This is connected to the Delta Conjecture because they show that

$$(rev_q \circ \omega)\mathrm{grFrob}(R_{n,k}; q) \tag{1.13}$$

is equal to one of the four expressions $\mathrm{Rise}_{n,k}(x;q,0), \mathrm{Rise}_{n,k}(x;0,q)$, $\mathrm{Val}_{n,k}(x;q,0)$, or $\mathrm{Val}_{n,k}(x;0,q)$, where $\mathrm{Rise}_{n,k}$ and $\mathrm{Val}_{n,k}$ are the quantities appearing in the Delta Conjecture that are defined in terms of labeled Dyck Paths, and ω is the standard involution on symmetric functions. They thus gave an algebraic interpretation of a special case of the Delta Conjecture.

The expression in (1.13) also appears in the work of Benkart, Colmenarejo, Harris, Orellana, Panova, Schilling, and Yip [2] as the character a crystal defined in terms of a minimaj statistic on ordered multiset partitions.

As before, $R_{n,k}$ is a graded $\mathfrak{S}_n$-module and we can refine the grading as follows.

Definition 1.1. *Let μ be a partition with at most n parts. Next define $S_{\trianglelefteq\mu}$ and $S_{\triangleleft\mu}$ to be the* *projections of $P_{\trianglelefteq\mu}$ and $P_{\triangleleft\mu}$ onto $R_{n,k}$. We then define*

$$R_{n,k,\mu} := S_{\trianglelefteq\mu}/S_{\triangleleft\mu}. \tag{1.14}$$

We determine the multiplicities of S^λ in $R_{n,k,\mu}$, thus extending the results of Adin, Brenti, and Roichman on $R_{n,\mu}$ to $R_{n,k,\mu}$ and refining the results of Haglund, Rhoades and Shimozono.

Theorem 1.2. *The module $R_{n,k,\rho}$ is zero unless ρ fits in an $(n-1)\times k$ rectangle and $\rho_i - \rho_{i+1} \leq 1$ for $i > n-k$. In the case that $R_{n,k,\rho}$ is not zero, the multiplicity of S^λ in $R_{n,k,\rho}$ is given by*

$$|\{T \in SYT(\lambda) : Des_{n-k+1,n}(\rho) \subseteq Des(T) \subseteq Des(\rho)\}|. \tag{1.15}$$

This result can be used to recover the graded Frobenius image of $R_{n,k}$ as described by Haglund, Rhoades, and Shimozono. Furthermore, the methods that we use are much simpler than the methods they use.

Also motivated by the Delta Conjecture, Benkart, Colmenarejo, Harris, Orellana, Panova, Schilling, and Yip [2] define an alternative algebraic interpretation of the quantities $\mathrm{Rise}_{n,k}(x;q,0), \mathrm{Rise}_{n,k}(x;0,q)$, $\mathrm{Val}_{n,k}(x;q,0)$, and $\mathrm{Val}_{n,k}(x;0,q)$ as the character of a certain crystal structure that is built up from smaller crystals. The way that these smaller crystals break up the whole crystal corresponds exactly to how the modules $R_{n,k,\mu}$ decompose $R_{n,k}$. More precisely, for the correct choice of μ each of these smaller crystals has characters equal to

$$(rev_q \circ \omega)\mathrm{F}rob(R_{n,k,\mu}). \tag{1.16}$$

Thus $R_{n,k,\mu}$ give a module theoretic interpretation of Benkart et al.'s crystal structure.

2 Definitions and Background

2.1 Descents and Monomials

An important component of the results of [1] on R_n is the use of a certain monomial basis for R_n. We will recall this basis and the generalization of this basis given in [6] for $R_{n,k}$. This basis for R_n will be indexed by permutations, and will be defined in terms of the descents of the corresponding permutation.

Given a permutation $\sigma \in \mathfrak{S}_n$, i is a descent of σ if $\sigma(i) > \sigma(i+1)$. We denote by Des(σ) the set of descents of σ. We denote by $d_i(\sigma)$, the number of descents of σ that are at least as large as i, that is

$$d_i(\sigma) := |\{i, i+1, \ldots, n\} \cap Des(\sigma)|. \tag{2.1}$$

Finally for two integers i, j such that $1 \le i \le j \le n$ we let $Des_{i,j}(\sigma)$ denote the set of descents of σ that are between i and j inclusively, that is

$$Des_{i,j}(\sigma) := Des(\sigma) \cap \{i, i+1, \ldots, j-1, j\}. \tag{2.2}$$

For example if $\sigma = 31427865 \in \mathfrak{S}_8$, then

$$Des(\sigma) = \{1,3,6,7\}, \tag{2.3}$$
$$(d_1(\sigma), \ldots, d_8(\sigma)) = (4,3,3,2,2,2,1,0), \text{and} \tag{2.4}$$
$$Des_{2,6}(\sigma) = \{3,6\}. \tag{2.5}$$

Descents are used to define a set of monomials which descend to a basis for R_n, see [4].

Definition 2.1. *Given a permutation $\sigma \in \mathfrak{S}_n$, the* **Garsia-Stanton monomial** *or simply* **descent monomial** *associated to σ is*

$$gs_\sigma := \prod_{i=1}^{n} x_{\sigma(i)}^{d_i(\sigma)}. \tag{2.6}$$

These monomials descend to a basis for R_n.

For example, if $\sigma = 31427865 \in \mathfrak{S}_8$, then $gs_\sigma = x_3^4 x_1^3 x_4^3 x_2^2 x_7^2 x_8^2 x_6^1$.

These monomials are generalized by Haglund, Rhodes, and Shimozono in [6] to (n,k)-descent monomials that are indexed by ordered set partitions of n with k blocks. Alternatively they can be indexed by pairs (π, I) consisting of a permutation $\pi \in \mathfrak{S}_n$ and a sequence $i_1, \ldots, i_{n-k}$ such that

$$k - des(\pi) > i_1 \ge i_2 \ge \ldots \ge i_{n-k} \ge 0. \tag{2.7}$$

This is done as follows:

Definition 2.2. *Given a permutation $\pi \in \mathfrak{S}_n$ and a sequence $I = (i_1, i_2, \ldots, i_{n-k})$ such that*

$$k - des(\pi) > i_1 \ge i_2 \ge \ldots \ge i_{n-k} \ge 0, \tag{2.8}$$

*the (n,k)-**descent monomial** associated to (π, I) is*

$$gs_{\pi,I} := gs_\pi x_{\pi(1)}^{i_1} x_{\pi(2)}^{i_2} \ldots x_{\pi(n-k)}^{i_{n-k}} \tag{2.9}$$

These monomials descend to a basis for $R_{n,k}$.

As an example if $\sigma = 31427865 \in \mathfrak{S}_8$, $k = 6$, and $I = (1,0)$, then

$$gs_{\sigma,I} = gs_\sigma \cdot x_3^1 x_1^0 = x_3^5 x_1^3 x_4^3 x_2^2 x_7^2 x_8^2 x_6^1. \tag{2.10}$$

2.2 Permutation and Partitions

The way that Adin, Brenti, and Roichman[1] make use of the classical descent monomial basis is by using a basis for $\mathbb{Q}[x_1, \ldots, x_n]$ given by Garsia in [4]. This basis is the set $\{gs_\pi e_\mu\}_{\pi \in \mathfrak{S}_n, \mu \vdash n}$, where

$$e_\mu = e_{\mu_1} e_{\mu_2} \ldots e_{\mu_{\ell(\mu)}}. \tag{2.11}$$

In making use of this basis it is necessary to associate certain permutations and partitions to monomials. Our results also use these, so we recall them here.

The **index permutation** of a monomial $m = \prod_{i=1}^n x_i^{p_i}$ is the permutation that rearranges the indices of the monomial to have decreasing exponents, that is it is the unique permutation π, such that the following hold:

1. $p_{\pi(i)} \geq p_{\pi(i+1)}$
2. $p_{\pi(i)} = p_{\pi(i+1)} \implies \pi(i) < \pi(i+1)$

We denote the index permutation of m as $\pi(m)$.

Next, the **exponent partition** of a monomial is the partition consisting of the individual exponents of the monomial written in decreasing order. We denote the exponent partition of m as $\lambda(m)$.

If λ is the exponent partition of a descent monomial, then $\lambda_n = 0$ and $\lambda_i - \lambda_{i+1} \leq 1$. We call a partition that satisfies these conditions a **descent partition**. If λ is the exponent partition of an (n,k)-descent monomial, then λ has at most n parts, and it has parts of size less than k. We call such partitions (n,k)**-partitions**. The **complementary partition** of a monomial m is the partition that is conjugate to $(\lambda_i - d_i(\pi))_{i=1}^n$, where $\pi = \pi(m)$ and $\lambda = \lambda(m)$. We denote the complementary partition of m as $\mu(m)$.

To clarify these definitions we present an example.

Example 2.3. *Let $n = 8$, $k = 5$, $I = (2,1,1)$ and let*

$$m = x_1^6 x_2 x_3 x_4^2 x_6^4 x_7 x_8^2 = x_1^6 x_6^4 x_4^2 x_8^2 x_2 x_3 x_7, \tag{2.12}$$

then

$$\pi(m) = 16482375, \tag{2.13}$$
$$\lambda(m) = (6,4,2,2,1,1,1,0), \tag{2.14}$$
$$Des(\pi(m)) = \{2,4,7\}, \tag{2.15}$$
$$gs_{\pi(m)} = x_1^3 x_6^3 x_4^2 x_8^2 x_2 x_3 x_7, \tag{2.16}$$
$$\mu(m)' = (3,1), \tag{2.17}$$
$$\mu(m) = (2,1,1), and \tag{2.18}$$
$$gs_{\pi(m),I} = x_1^5 x_6^4 x_4^3 x_8^2 x_2 x_3 x_7. \tag{2.19}$$

The final key component is a partial ordering on monomials of a given degree together with a result on how multiplying monomials by elementary symmetric functions interacts with this partial order. For a proof of Proposition 2.5 we refer the reader to [1].

Definition 2.4. *For m_1, m_2 monomials of the same total degree, $m_1 \prec m_2$ if one of the following holds:*

1. $\lambda(m_1) \lhd \lambda(m_2)$
2. $\lambda(m_1) = \lambda(m_2)$ *and* $inv(\pi(m_1)) > inv(\pi(m_2))$

Where $\lhd$ is the strict dominance order on partitions.

This partial order is useful because of how it interacts with multiplication of monomials and elementary symmetric functions. This interaction is encapsulated in the following proposition:

Proposition 2.5 (Adin-Brenti-Roichman [1]). *Let m be a monomial equal to $x_1^{p_1} \dots x_n^{p_n}$, then among the monomials appearing in $m \cdot e_\mu$, the monomial*

$$\prod_{i=1}^{n} x_{\pi(i)}^{p_{(\pi(i))}+u'_i} \tag{2.20}$$

is the maximum with respect to $\prec$, where π is the index permutation of m.

2.3 Standard Young Tableaux

Our main results come in the form of counting certain standard Young tableaux.

A **Ferrers diagram** is a collection of unit boxes which, since we are using English notation, are justified to the left and up. The lengths of the rows of a Ferrers diagram form a partition which we call the **shape** of the Ferrers diagram. A **standard Young tableau** of size n is a Ferrers diagram containing n boxes where the boxes are assigned the values $1, 2, \dots, n$ such that the values increase along rows and down columns. We denote the set of standard Young tableaux of size n by $SYT(n)$. For a partition μ, we let $SYT(\mu)$ denote the set of all standard Young tableaux of shape μ.

An integer i is a **descent** of a standard Young tableaux T if the box containing $i+1$ is strictly below the box containing i. We denote by $Des(T)$ the set of all descents of T. Furthermore given two integers $1 \le i \le j \le n$ we define $Des_{i,j}$ to be the set of descents of T that are between i and j inclusively.

As examples, consider the following Young tableaux:

$T_1 =$
<table><tr><td>1</td><td>4</td><td>6</td><td>7</td></tr><tr><td>2</td><td>5</td><td>8</td><td></td></tr><tr><td>3</td><td></td><td></td><td></td></tr></table>

$T_2 =$
<table><tr><td>1</td><td>3</td><td>4</td><td>7</td></tr><tr><td>2</td><td>5</td><td>6</td><td>8</td></tr></table>

$T_3 =$
<table><tr><td>1</td><td>2</td><td>4</td><td>7</td><td>8</td></tr><tr><td>3</td><td>5</td><td>6</td><td></td><td></td></tr></table>

T_1, T_2, T_3 are standard Young tableaux. The shape of T_1 and S_1 is $(4,3,1)$, the shape of T_2 and S_2 is $(4,4)$, and the shape of (T_3) and S_3 is $(5,3)$. The descent sets of these tableaux are as follows: $Des(T_1) = \{1,2,4,7\}, Des(T_2) = \{1,4,7\}$, and $Des(T_3) = \{2,4\}$.

Next, $Des_{5,7}(T_1) = Des_{5,7}(T_2) = \{7\}$, and $Des_{5,7}(T_3) = \emptyset$

3 Descent representations of $R_{n,k}$

In the case of the classical coinvariant algebra, Adin, Brenti, and Roichman determine the isomorphism type of $R_{n,\rho}$ by comparing the graded traces of the actions of $\mathfrak{S}_n$ on $\mathbb{Q}[x_1, \dots, x_n]$ and on R_n. We will follow a similar path, but instead of considering the action of $\mathfrak{S}_n$ on $\mathbb{Q}[x_1, \dots, x_n]$, we will consider its action on the space of rational polynomials in n variables with individual powers at most $k-1$, that is

$$P_{n,k} := \text{span}_{\mathbb{Q}}\{x_1^{p_1} x_2^{p_2} \dots x_n^{p_n} : p_1, p_2, \dots, p_n < k\}. \tag{3.1}$$

We begin by giving a straightening Lemma:

Lemma 3.1. *If $m = \prod_{i=1}^n x_i^{p_i}$ is a monomial in $P_{n,k}$ (that is $p_i < k$ for all i), then*

$$m = gs_{\pi,I} e_\nu + \sum. \tag{3.2}$$

Where $\pi = \pi(m)$; $\sum$ is a sum of monomials $m' \prec m$; I is the length $n-k$ sequence defined by $i_\ell = \mu'_\ell - \mu'_{n-k+1}$, where μ is the complementary partition of m; and ν is specified by:

1. *$\nu'_\ell = \mu'_\ell$ for $\ell > n-k$*
2. *$\nu'_\ell = \mu'_{n-k+1}$ for $\ell \le n-k$*

Furthermore ν consists of parts of size at least $n-k+1$.

This Lemma gives rise to a basis for $P_{n,k}$ which will be key to relating how $\mathfrak{S}_n$ acts on $P_{n,k}$ to how it acts on $R_{n,k}$.

Proposition 3.2. *The set $D_{n,k}$ consisting of products $gs_{\pi,I}e_\nu$ where ν is a partition with parts of size at least $n-k+1$ and $(\lambda(gs_{\pi,I}) + \nu')_1 < k$ form a basis for $P_{n,k}$.*

Proposition 3.3. *Let p be the projection from $\mathbb{Q}[x_1, \dots, x_n]$ to $R_{n,k}$ and let m be a monomial in $P_{n,k}$. Then*

$$p(m) = \sum_{\pi,I} \alpha_{\pi,I} gs_{\pi,I} \tag{3.3}$$

where $\alpha_{\pi,I}$ are some constants, and the sum is over pairs (π, I) such that $\lambda(gs_{\pi,I}) \trianglelefteq \lambda(m)$.

This Proposition gives the following Corollary:

Corollary 3.4. *$R_{n,k,\rho}$ is zero unless ρ is the exponent partition of an (n,k)-descent monomial, which occurs precisely when ρ is an (n,k) partition such that the last k parts form a descent partition.*

This basis allows us to express the action of $\tau \in \mathfrak{S}_n$ on $P_{n,k}$ in terms of its action on $R_{n,k}$ with the basis of (n,k)-Garsia-Stanton monomials. That is, for some constants $\alpha_{\phi,J}$

$$\tau(gs_{\pi,I}) = \sum_{\phi,J} \alpha_{\phi,J} gs_{\phi,J}, \implies \tau(gs_{\pi,I} e_\nu) = \sum_{\phi,J} \alpha_{\phi,J} gs_{\phi,J} e_\nu. \tag{3.4}$$

We now move to the Lemmas that will allow us to prove our main result.

Lemma 3.5. *Given an (n,k)-partition μ and an (n,k)-descent partition ν there exists a (n,k)-partition ρ such that $\mu = \nu + \rho$ if and only if $Des(\nu) \subseteq Des(\mu)$. If it exists, ρ is unique.*

Example 3.6. *As an example of the Lemma 3.5, let $n = 8, k = 6$ then let*

$$\mu = (5,5,3,3,1,1,1,0), \tag{3.5}$$

$$\nu_1 = (2,2,1,1,0,0,0,0), \tag{3.6}$$

and

$$\nu_2 = (3,3,2,2,1,1,0,0). \tag{3.7}$$

Then $Des(\mu) = \{2,4,7\}$, $Des(\nu_1) = \{2,4\}$, and $Des(\nu_2) = \{2,4,6,8\}$.

We then have that $Des(\nu_1) \subseteq Des(\mu)$, and that $\mu - \nu_1 = (3,3,2,2,1,1,1,0)$ is a partition. On the other hand, $Des(\nu_2) \not\subseteq Des(\mu)$, and $\mu - \nu_2 = (2,2,1,1,0,0,1,0)$ is not a partition.

Lemma 3.7. *Given an (n,k)-partition μ and a set $S \subseteq Des_{n-k+1,n}(\mu)$, there is a unique pair (ν,ρ) such that $\mu = \nu + \rho$ and ν is the exponent partition of an (n,k)-descent monomial with $Des_{n-k+1,n}(\nu) = S$, and ρ is an (n,k)-partition with $\rho_1 = \rho_2 = \ldots = \rho_{n-k+1}$.*

We give an example of how Lemma 3.7 works.

Example 3.8. *Let $n = 8, k = 6$, $S = \{4\}$, and let*

$$\mu = (5,5,3,3,1,1,1,0), \tag{3.8}$$

then

$$\nu = (4,4,2,2,1,1,1,0), \tag{3.9}$$

and

$$\rho = (1,1,1,1,0,0,0,0). \tag{3.10}$$

We now give a proof of Theorem 1.2

Proof of Theorem 1.2. The determination of when $R_{n,k,\rho}$ is zero is from Corollary 3.4.

Next we define an inner product on polynomials by $\langle m_1, m_2 \rangle = \delta_{m_1 m_2}$ for two monomials m_1, m_2, and then extending bilinearly. We then consider the graded trace of the action of $\tau \in \mathfrak{S}_n$ on $P_{n,k}$ defined for the monomial basis by

$$Tr_{P_{n,k}}(\tau) := \sum_m \langle \tau(m), m \rangle \cdot \bar{q}^{\lambda(m)} \tag{3.11}$$

where $\bar{q}^{\lambda} = \prod_{i=1}^n q_i^{\lambda_i}$ for any partition λ. Adin, Brenti, Roichman show that

$$Tr_{\mathbb{Q}[x_1,\ldots,x_n]}(\tau) = \sum_{\lambda \vdash n} \chi_\mu^\lambda \frac{\sum_{T \in SYT(\lambda)} \prod_{i=1}^n q_i^{d_i(T)}}{\prod_{i=1}^n (1 - q_1 q_2 \ldots q_i)} \tag{3.12}$$

(where μ is the cycle type of τ). From this we can recover $Tr_{P_{n,k}}(\tau)$ by restricting to powers of q_1 that are at most $k-1$. Doing this gives

$$\sum_{\lambda \vdash n} \chi_\mu^\lambda \sum_{T \in SYT(\lambda), \nu} \bar{q}^{\lambda_{Des(T)}} \bar{q}^{\nu}. \tag{3.13}$$

Where the ν's are partitions such that $(\lambda_{Des(T)})_1 + \nu_1 < k$, and $\lambda_{Des(T)}$ is the descent partition with descent set T.

Alternatively, we can calculate $Tr_{P_{n,k}}(\tau)$ with the basis from Proposition 3.2, this gives

$$Tr_{P_{n,k}}(\tau) = \sum_{\sigma, I, \nu} \langle \tau(gs_{\sigma,I} e_\nu), gs_{\sigma,I} e_\nu \rangle \bar{q}^{\lambda(gs_{\sigma,I})} \bar{q}^{\nu'} \tag{3.14}$$

$$= \sum_{\sigma, I, \nu} \langle \tau(gs_{\sigma,I}), gs_{\sigma,I} \rangle \bar{q}^{\lambda(gs_{\sigma,I})} \bar{q}^{\nu'} \tag{3.15}$$

$$= \sum_{\lambda, \nu} Tr_{R_{n,k}}(\tau; \bar{q}^{\lambda}) \bar{q}^{\lambda} \bar{q}^{\nu'} \tag{3.16}$$

where the ν's are partitions with parts of size at least $n-k+1$ such that $(\lambda(gs_{\sigma,I}))_1 + (\nu')_1 < k$, and $Tr_{R_{n,k}}(\tau; \bar{q}^{\lambda})$ is the coefficient of $\bar{q}^{\lambda}$ in the graded trace of τ acting on $R_{n,k}$.

We now consider the coefficient of $\bar{q}^{\rho}$ for some partition ρ. Using the first calculation and Lemma 3.5, the inner sum can be reduced to T such that $Des(T) \subseteq Des(\rho)$, so that

$$\sum_{\lambda \vdash n} \chi_\mu^\lambda |\{T \in SYT(\lambda), Des(T) \subseteq Des(\rho)\}|. \tag{3.17}$$

Looking at the second calculation and using Lemma 3.7 gives

$$\sum_{S \subseteq Des_{n-k+1,n}(\rho)} Tr_{R_{n,k}}(\tau; \bar{q}^{\lambda_S}), \tag{3.18}$$

where λ_S is the exponent partition of some (n,k)-descent monomial $gs_{\sigma,I}$ with

$$Des_{n-k+1,n}(\lambda(gs_{\sigma,I})) = S. \tag{3.19}$$

Together this gives that

$$\sum_{\lambda \vdash n} \chi^{\lambda}_{\mu} |\{T \in SYT(\lambda) : Des(T) \subseteq Des(\rho)\}| = \sum_{S \subseteq Des_{n-k+1,n}(\rho)} Tr_{R_{n,k}}(\tau; \bar{q}^{\lambda_S}). \tag{3.20}$$

We want to further refine this result by showing that

$$\sum_{\lambda \vdash n} \chi^{\lambda}_{\mu} |\{T \in SYT(\lambda) : S \subseteq Des(T) \subseteq Des(\rho)\}| = Tr_{R_{n,k}}(\tau; \bar{q}^{\lambda_S}) \tag{3.21}$$

for any specific S'. We do this by induction on $|\lambda_{S'}|$. The base case is $\lambda_{S'} = \emptyset$ is trivial. If we take $\rho = \lambda_{S'}$, then $\lambda_{S'}$ will appear in the sum since we can take the ν from Lemma 3.7 to be 0, and all other λ_S's will be smaller since the corresponding ν's will be non-empty. Thus by the inductive hypothesis,

$$\sum_{\lambda \vdash n} \chi^{\lambda}_{\mu} |\{T \in SYT(\lambda) : S' \not\subseteq Des(T) \subseteq Des(\rho)\}| = \sum_{S \subsetneq S'} Tr_{R_{n,k}}(\tau; \bar{q}^{\lambda_S}), \tag{3.22}$$

subtracting this from our result gives the desired refinement. This then proves the Theorem since the exponent partition of any (n,k)-descent monomial $gs_{\sigma,I}$ appears when we take $\rho = \lambda(gs_{\sigma,I})$.

□

Example 3.9. *Let $n = 8$, $k = 6$, $\rho = (5,3,2,2,1,1,1)$, $\lambda = (4,3,1)$, then $Des_{1,2}(\rho) = \{1,2\}$, and $Des_{3,8} = \{4,7\}$.*

The standard Young tableaux T of shape λ with $\{4,7\} \subseteq Des(T) \subseteq \{1,2,4,7\}$ are as follows:

1	4	6	7
2	5	8	
3			

1	3	4	7
2	6	8	
5			

1	3	4	7
2	5	6	
8			

1	2	4	7
3	5	6	
8			

1	2	4	7
3	6	8	
5			

1	2	6	7
3	4	8	
5			

1	2	3	4
5	6	7	
8			

Therefore by Theorem 1.2, the coefficient of S^{λ} in $R_{n,k,\rho}$ is 7.

Theorem 1.2 is related to the a crystal structure that was defined by Benkart, Colmenarejo, Harris, Orellana, Panova, Schilling, and Yip [2]. Like $R_{n,k}$, the crystal structure that they define is motivated by the Delta Conjecture, and has graded character equal to $(rev_q \circ \omega)\mathrm{grFrob}(R_{n,k}; q)$. This crystal is built up from crystal structures on ordered multiset partitions in minimaj ordering with specified descents sets, and the characters of these smaller crystals is given in terms of skew ribbon tableaux. The algebras $R_{n,k,\rho}$ play the same role as these smaller crystals.

This connection immediate once we rewrite the result of Theorem 1.2 as the following:

$$\omega(\mathrm{Frob}(R_{n,k,\rho})) = s_{\gamma'} \prod_{i=1}^{p} e_{d_i}. \tag{3.23}$$

where we let d_i be the difference between the ith and $(i-1)$th descents of ρ, with d_1 being the first descent, and γ is the skew ribbon shape with rows of lengths $(n-(d_1+d_2+\ldots+d_{des(\rho)}), d_{des(\rho)}, d_{des(\rho)-1}, \ldots, d_{p+1})$.

4 Acknowledgements

I would like to thank my advisor Brendon Rhoades for his work in directing and advising this work.

References

[1] R. Adin, F. Brenti, and Y. Roichman. “Descent representations and multivariate statistics”. *Trans. Amer. Math. Soc* **357**.8 (2005), pp. 3051–3082. Link.

[2] G. Benkart, L. Colmenarejo, P. E. Harris, R. Orellana, G. Panova, A. Schilling, and M. Yip. “A minimaj-preserving crystal on ordered multiset partitions”. *Adv. Appl. Math.* **95** (2018), pp. 96–115. Link.

[3] C. Chevalley. “Invariants of finite groups generated by reflections”. *Amer. J. Math.* **77**.4 (1955), pp. 778–782. Link.

[4] A. M. Garsia. “Combinatorial methods in the theory of Cohen-Macaulay rings”. *Adv. in Math.* **38**.3 (1980), pp. 229–266. Link.

[5] J. Haglund, J. Remmel, and A. Wilson. “The delta conjecture”. *Trans. Amer. Math. Soc* **370**.6 (2018), pp. 4029–4057. Link.

[6] J. Haglund, B. Rhoades, and M. Shimozono. “Ordered set partitions, generalized coinvariant algebras, and the Delta Conjecture”. *Adv. in Math.* **329** (2018), pp. 851–915. Link.

[7] R. P. Stanley. “Invariants of finite groups and their applications to combinatorics”. *Bull. Amer. Math. Soc. (N.S.)* **1**.3 (1979), pp. 475–511. Link.