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The distribution of Weierstrass points on a tropical curve

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Abstract. The set of (higher) Weierstrass points on a metric graph of genus $g > 1$ is an analogue of the set of N-torsion points on a circle. As N grows, the torsion points "distribute evenly" over a circle. This makes it natural to ask how Weierstrass points distribute on a graph, as the degree of the corresponding divisor grows. We study how Weierstrass points behave on tropical curves (i.e. finite metric graphs) in analogy with complex algebraic curves (i.e. Riemann surfaces), and explain how their distribution can be described in terms of electrical networks. This is a tropical analogue of a result of Neeman, for the distribution of Weierstrass points on a compact Riemann surface, and extends previous work of Zhang and Amini on the non-Archimedean case.

Keywords: tropical curve, metric graph, Weierstrass point, electrical network

1 Introduction

On a circle equipped with a length metric, for any positive integer N there is a natural notion of N-torsion points. This is a set of N points which by definition are "evenly spaced" around the circle. A specific choice of N-torsion points requires the additional data of choosing a basepoint on the underlying circle.

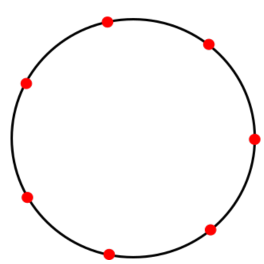

Figure 1: 7-torsion points on a circle.

What if we replace the circle with an arbitrary metric graph, i.e. a graph equipped with edge lengths and the path metric?

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There is no longer an obvious notion of how to choose a finite set of points to be "evenly spaced" on a graph, especially if we want a choice that reflects global combinatorial or topological properties of the underlying graph.

By applying the notions of tropical rank and reduced divisors of Baker–Norine [3] we define the *Weierstrass points* of a divisor on a metric graph, and propose that this should be considered a good analogue of N-torsion points on a metric circle. (The dependence on a divisor corresponds to the dependence on a basepoint of N-torsion on a circle; the parameter N corresponds to the *degree* of the divisor.) This viewpoint follows the suggestion of Mumford [6] for Weierstrass points on an algebraic curve.

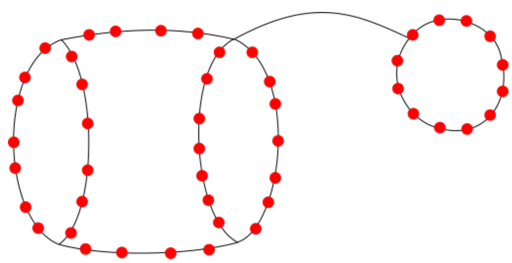


Figure 2: Weierstrass points on a genus 4 metric graph, for a degree 15 divisor.

In the full version of this paper [9] we study the set of Weierstrass points associated to a divisor on an arbitrary metric graph. In particular, we ask

Question 1.1. When is the set of Weierstrass points finite?

and

Question 1.2. How are these points distributed as the degree approaches infinity?

We show that for any metric graph Γ, the Weierstrass locus is finite for a generic divisor class, and we prove that for any degree-increasing sequence of such generic divisors the Weierstrass points become distributed according to the Zhang canonical measure on Γ [10]. This measure can be described via interpreting Γ as an electrical network of resistors, following Baker–Faber [2].

1.1 Previous work

The canonical Weierstrass points on a complex algebraic curve of genus $g \geq 2$ were studied classically. This is a set of $g^3 - g$ points (counted with multiplicity) on X which are intrinsic to X as an abstract curve. They form a useful tool, e.g. for proving that the automorphism group of such a curve is finite. This notion naturally extends to (higher) Weierstrass points, which are a finite set of points on X associated to a choice of divisor D on X. The number of Weierstrass points of D (counted with multiplicity) grows quadratically as a function of the degree of D.

Mumford suggests in [6] that the Weierstrass points associated to a divisor of degree N form a higher-genus analogue of the set of N-torsion points on an elliptic curve. The fact that N-torsion points on a complex elliptic curve become "evenly distributed" as N grows large leads one to ask whether a similar phenomenon holds for Weierstrass points generally. This was answered in 1984 by Neeman [7] (a student of Mumford): for a complex algebraic curve X of genus $g \geq 2$, the Weierstrass points of a degree N divisor become equidistributed according to the Bergman measure on X as $N \to \infty$.

If the ground field $\mathbb{C}$ is replaced with a non-Archimedean field, one may consider the same question of how Weierstrass points are distributed inside the Berkovich analytification X^{an}, say after retracting to a (compact) skeleton Γ. This was addressed by Amini in [1]: the Weierstrass points are equidistributed according to the "Arakelov–Zhang canonical admissible measure" $\mu = \mu_\Gamma$, constructed by Zhang in [10]; we make use of a definition of μ along more elementary lines following Baker–Faber [2], using the notions of current flow and electric potential in a (1-dimensional) network of resistors.

In his preprint, Amini raises the question of whether the distribution of Weierstrass points is possibly intrinsic to the metric graph Γ, without needing to identify Γ with the skeleton of some Berkovich curve X^{an}. One major obstacle to this idea is that on a

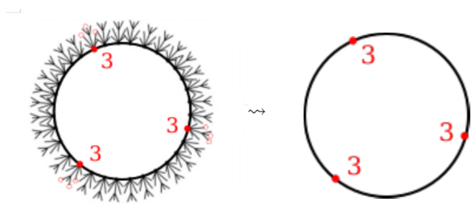


Figure 3: Weierstrass points on a genus 1 Berkovich curve, and retraction to a skeleton (adapted from an illustration of Matt Baker).

metric graph, the Weierstrass locus for a divisor may fail to be a finite set of points. We sidestep this issue by showing that finiteness does hold for a generic choice of divisor class (Theorem A). With this assumption of genericity, we are able to show that the distribution of Weierstrass points in large degree is indeed intrinsic to Γ (Theorem B).

2 Tropical curves

We use the terms "metric graph" and "abstract tropical curve" interchangeably.

A *metric graph* is a compact, connected metric space which comes from assigning positive real edge lengths to a finite connected combinatorial graph. The underlying combinatorial graph $G = (E, V)$ is called a *combinatorial model* for Γ; we allow loops and parallel edges in G. We say e is a *segment* of Γ if it is an edge in some combinatorial model.

The *valence* $\mathrm{val}(x)$ of a point x on a metric graph Γ is the number on connected components of a sufficiently small punctured neighborhood of x. Points in the interior of a segment always have valence 2. All points x with $\mathrm{val}(x) \neq 2$ are contained in the vertex set of any combinatorial model.

The *genus* of a metric graph Γ is its first Betti number as a topological space,

$$g(\Gamma) = b_1(\Gamma) = \dim_{\mathbb{R}} H_1(\Gamma, \mathbb{R}).$$

If G is a combinatorial model for Γ, the genus is equal to $g(\Gamma) = \#E(G) - \#V(G) + 1$.

Example 2.1. The metric graph in Figure 4 has genus 0. A minimal combinatorial model has 8 vertices and 7 edges.

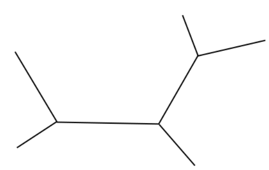


Figure 4: Metric graph of genus 0.

Example 2.2. The metric graph in Figure 5 has genus 2. A minimal combinatorial model has 2 vertices and 3 edges.

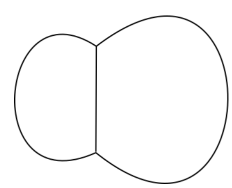


Figure 5: Metric graph of genus 2.

A *divisor* on a metric graph Γ is a finite formal sum of points of Γ with integer coefficients. The *degree* of a divisor is the sum of its coefficients; i.e. for the divisor $D = \sum_{x \in \Gamma} a_x x$, we have $\deg(D) = \sum_{x \in \Gamma} a_x$. We let $\mathrm{Div}(\Gamma)$ denote the abelian group of all divisors on Γ, and let $\mathrm{Div}^d(\Gamma)$ denote the divisors of degree d. We say a divisor is *effective* if all its coefficients are nonnegative; we write $D \geq 0$ to indicate that D is effective. We let $\mathrm{Sym}^d(\Gamma)$ denote the set of effective divisors of degree d on Γ. $\mathrm{Sym}^d(\Gamma)$ inherits from Γ the structure of a polyhedral cell complex of dimension d.

We let $\mathrm{Div}_{\mathbb{R}}(\Gamma)$ denote the set of divisors on Γ with coefficients in $\mathbb{R}$. In other words, $\mathrm{Div}_{\mathbb{R}}(\Gamma) = \mathrm{Div}(\Gamma) \otimes_{\mathbb{Z}} \mathbb{R}$.

2.1 Linear equivalence

We define linear equivalence for divisors on metric graphs, following Gathmann–Kerber [5]. This notion is analogous to linear equivalence of divisors on an algebraic curve, where rational functions are replaced with piecewise $\mathbb{Z}$-linear functions.

A *piecewise linear function* on Γ is a continuous function $f : \Gamma \to \mathbb{R}$ such that there is some combinatorial model for Γ such that f restricted to each edge is an affine linear function, i.e. a function of the form

$$f(x) = ax + b, \qquad a, b \in \mathbb{R}$$

where x is a length-preserving parameter on the edge. We let $\mathrm{PL}_{\mathbb{R}}(\Gamma)$ denote the set of all piecewise linear functions on Γ. A *piecewise $\mathbb{Z}$-linear function* on Γ is a piecewise linear function such that all its slopes are integers, i.e. f restricted to each edge (for some combinatorial model) has the form

$$f(x) = ax + b, \qquad a \in \mathbb{Z},\ b \in \mathbb{R}.$$

We let $\mathrm{PL}_{\mathbb{Z}}(\Gamma)$ denote the set of all piecewise $\mathbb{Z}$-linear functions on Γ.

We let $UT_x\Gamma$ denote the unit tangent fan of Γ at x, which is the set of "directions going away from x" on Γ. For $v \in UT_x\Gamma$, the symbol ϵv for sufficiently small $\epsilon \geq 0$ means the point in Γ at distance ϵ away from x in the direction v. For $v \in UT_x\Gamma$ and a function $f : \Gamma \to \mathbb{R}$ we let

$$D_v f(x) = \lim_{\epsilon \to 0^+} \frac{f(x + \epsilon v) - f(x)}{\epsilon}$$

denote the slope of f while travelling away from x in the direction v.

Given $f \in \mathrm{PL}_{\mathbb{Z}}(\Gamma)$, we define the *principal divisor* $\Delta(f) \in \mathrm{Div}^0(\Gamma)$ by

$$\Delta(f) = \sum_{x \in \Gamma} \mathrm{ord}_x(f) x \qquad \text{where} \qquad \mathrm{ord}_x(f) = \sum_{v \in UT_x\Gamma} D_v f(x).$$

In words, the coefficient in $\Delta(f)$ at x is equal to the sum of the outgoing slopes. This divisor is supported on the finite set of points at which f is not linear, called the *break locus* of f.

If $\Delta(f) = D - E$ where D, E are effective divisors with disjoint support, then we call $D = \Delta^+(f)$ the *divisor of zeros of f* and $E = \Delta^-(f)$ the *divisor of poles* of f. We say two divisors D, E are *linearly equivalent*, denoted $D \sim E$, if there exists a piecewise $\mathbb{Z}$-linear function f such that

$$\Delta(f) = D - E.$$

Note that linear equivalence preserves degree. We let $[D]$ denote the linear equivalence class of divisor D, i.e.

$$[D] = \{E \in \mathrm{Div}(\Gamma) : E \sim D\} = \{D + \Delta(f) : f \in \mathrm{PL}_{\mathbb{Z}}(\Gamma)\}.$$

We let $|D|$ denote the *complete linear system* of D, which is the set of effective divisors linearly equivalent to D. Namely, we have

$$|D| = \{E \in \mathrm{Sym}^d(\Gamma) : E \sim D\} = \{D + \Delta(f) : f \in \mathrm{PL}_{\mathbb{Z}}(\Gamma),\, \Delta(f) \geq -D\}.$$

The topology on $|D|$ is induced by the inclusion $|D| \subset \mathrm{Sym}^d(\Gamma)$.

Remark 2.3 (Linear interpolation along f)**.** Given a function $f \in \mathrm{PL}_{\mathbb{Z}}(\Gamma)$, we may associate to f a 1-parameter family of effective divisors which "linearly interpolate" between the zeros $\Delta^+(f)$ and poles $\Delta^-(f)$. (We can think of this construction as specifying a unique "geodesic path" between any two points in the complete linear system $|D|$.)

Namely, for $\lambda \in \mathbb{R}$ we let $\lambda \in \mathrm{PL}_{\mathbb{Z}}(\Gamma)$ also denote the constant function on Γ by abuse of notation, and we define the effective divisor $f_\Delta^{-1}(\lambda)$ by

$$f_\Delta^{-1}(\lambda) = \Delta^-(f) + \Delta(\max\{f, \lambda\}).$$

See Figure 6 for an illustration.

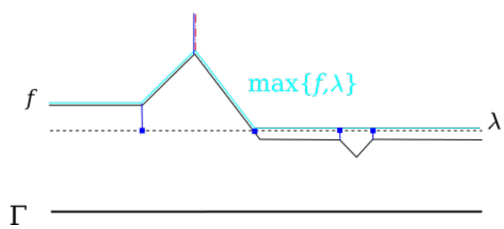


Figure 6: Linear interpolation showing the divisor $f_\Delta^{-1}(\lambda)$ in dark blue.

Note that according to this definition, $f_\Delta^{-1}(\lambda) = \Delta^-(f)$ for λ sufficiently large and $f_\Delta^{-1}(\lambda) = \Delta^+(f)$ for λ sufficiently small. It is clear from definition that for any λ, $f_\Delta^{-1}(\lambda)$ is linearly equivalent to $\Delta^+(f)$ and to $\Delta^-(f)$.

2.2 Reduced divisors

A divisor class $[D]$ is typically very large, so it is convenient to have a method of choosing a somewhat-canonical representative divisor inside $[D]$. We can do so after fixing a basepoint q on our metric graph Γ. The *reduced divisor* $\mathrm{red}_q[D]$ is the unique divisor in $[D]$ which is effective away from q, and which minimizes a certain "energy function" among such representatives [4, Appendix A]. Intuitively, $\mathrm{red}_q[D]$ is the divisor in $[D]$ whose effective part is "as close as possible" to the basepoint q. We state these important properties of the reduced divisor:

(RD1) $[D] \geq 0$ if and only if $\mathrm{red}_q[D] \geq 0$

(RD2) the degree of $\mathrm{red}_q[D]$ away from q is at most g

(RD3) for a fixed divisor D, the map $\Gamma \to |D|$ sending $q \mapsto \mathrm{red}_q[D]$ is continuous

3 Weierstrass points

On an algebraic curve X of genus g, the canonical Weierstrass points are defined as follows. The canonical divisor K on X determines a map to projective space $\varphi_K : X \to \mathbb{P}^{g-1}$. Generically a point on $\varphi_K(X)$ will have an osculating hyperplane in $\mathbb{P}^{g-1}$ which intersects $\varphi_K(X)$ with multiplicity $g-1$. For finitely many "exceptional" points on $\varphi_K(X)$, the osculating hyperplane will intersect the curve with higher multiplicity; the preimages of these exceptional points are the *canonical Weierstrass points* of X. (These are also known as the *flex points* of the embedded curve $\varphi_K(X) \subset \mathbb{P}^{g-1}$)

This notion is generalized by replacing K with an arbitrary divisor. Given a (basepoint-free) divisor D on X, there is an associated map to projective space $\varphi_D : X \to \mathbb{P}^r$. If the degree of D is at least $2g+1$ then $r = \deg D - g$ and this map is an embedding. The set of flex points of the embedded curve $\varphi_D(X)$, where the osculating hyperplane intersects the curve with multiplicity greater than r, are the *(higher) Weierstrass points* associated to the divisor D.

3.1 Tropical Weierstrass points

We define the Weierstrass points of a divisor on a metric graph Γ, using the Baker–Norine rank function and the notion of reduced divisors [3, 4].

For a divisor on a metric graph, the *Baker–Norine rank* $r(D)$ is defined by

$$r(D) = \max\{r \geq 0 : |D - E| \neq \emptyset \text{ for all } E \in \mathrm{Sym}^r(\Gamma)\}.$$

The rank satisfies Riemann's inequality $r(D) \geq \deg(D) - g$. When the degree is sufficiently large, which is the case of interest in Theorem B, this is an equality;

$$r(D) = \deg(D) - g \quad \text{when} \quad \deg(D) \geq 2g - 1.$$

Definition 3.1. Let D be a divisor on a metric graph Γ, with rank $r = r(D)$. A point $x \in \Gamma$ is a *Weierstrass point for D* if

$$\mathrm{red}_x[D] \geq (r+1)x.$$

The *Weierstrass locus* $W(D) \subset \Gamma$ of D is the set of its Weierstrass points.

Note that the Weierstrass locus of D depends only on the divisor class $[D]$.

Example 3.2. Suppose Γ is a genus 1 graph and D is a divisor of degree 6, indicated by the black dots in Figure 7 with multiplicities. This divisor has rank $r = 5$ since it is in the "non-special range" of Riemann-Roch. The Weierstrass locus of D consists of 6 points evenly spaced around Γ, indicated in red.

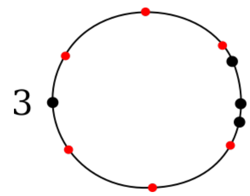


Figure 7: Weierstrass points of a degree 6 divisor on a genus 1 metric graph.

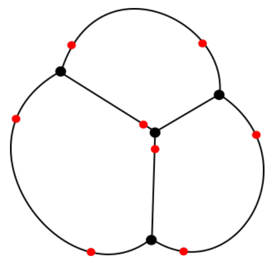
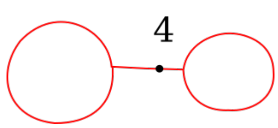


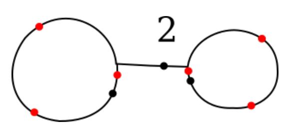
Figure 8: Metric graph with Weierstrass locus $W(K)$ finite.

Example 3.3. Suppose Γ is the genus 3 metric graph in Figure 8. The canonical divisor K on Γ is supported on the four trivalent vertices. This divisor has rank $r = 2$. Assuming generic edge lengths, the Weierstrass locus of K consists of 8 distinct points on Γ.

Example 3.4 (Failure of $W(D)$ to be finite)**.** Suppose Γ is the genus 2 graph shown in Figure 9, and D is a degree 4 divisor supported on the bridge edge in Figure 9a. This divisor has rank $r = 2$. Every point of Γ is in the Weierstrass locus of D.



(a) Nongeneric divisor class $[D]$



(b) Generic divisor class $[D']$

Figure 9: Metric graph with $W(D) = \Gamma$ and a nearby divisor with $W(D')$ finite.

When $[D]$ is deformed to a nearby divisor class $[D']$, the Weierstrass locus becomes a finite set of 6 points as shown in Figure 9b. The rank does not change between D and D'.

Our first result addresses when the Weierstrass locus is finite. Here "generic" means on a dense open subset of the space of divisor classes.

Theorem A. *Let Γ be a compact, connected metric graph. For a generic divisor class $[D]$ on Γ, the associated Weierstrass locus $W(D)$ is finite.*

The next theorem describes the distribution of tropical Weierstrass points; this is the main result of our paper.

Theorem B. *Let Γ be a metric graph of genus g, and let $\{[D_N] : N \geq 1\}$ be a sequence of generic divisor classes on Γ with $\deg D_N = N$. Let W_N denote the Weierstrass locus of the divisor D_N, and let*

$$\delta_N = \frac{1}{N} \sum_{x \in W_N} \delta_x$$

denote the normalized discrete measure on Γ associated to W_N, where δ_x is the Dirac measure at x. Then as $N \to \infty$, the measures δ_N converge weakly to the Zhang canonical measure μ on Γ.

In other words, for any segment e of Γ, as $N \to \infty$ we have

$$\frac{\#(W_N \cap e)}{N} \to \mu(e).$$

We also obtain a quantitative version of this result which specifies a bound on the rate of convergence.

Theorem C. *Let Γ be a metric graph of genus g, let $[D_N]$ be a generic degree N divisor class, and let W_N denote the Weierstrass locus of D_N. Let μ denote the Zhang canonical measure on Γ.*

(a) For any segment e in Γ,

$$N\mu(e) - 3g - 1 \leq \#(W_N \cap e) \leq N\mu(e) + g + 2.$$

(b) For a fixed continuous function $f : \Gamma \to \mathbb{R}$,

$$\frac{1}{N} \sum_{x \in W_N} f(x) = \int_\Gamma f(x)\mu(dx) + O\left(\frac{1}{N}\right).$$

(The big-O constant may depend on f.)

(c) If e is a segment of Γ with $\mu(e) > \frac{3g+1}{N}$, then e contains at least one D_N-Weierstrass point.

4 Canonical measure

In this section we describe the limiting measure of Weierstrass points on a metric graph (defined by Zhang [10]) via the perspective of resistor networks following Baker–Faber [2]. We may view this construction as a one-dimensional analogue of Gaussian curvature on a closed two-dimensional surface.

4.1 Electrical networks

We view a metric graph Γ as a resistor network by interpreting an edge of length L as a resistor of resistance L.

On a resistor network we may send current from one point to another. By Ohm's law, the voltage drop across a segment is equal to its resistance (i.e. length) multiplied by the amount of current passing through. Under an externally-applied current, the flow of current within the network is determined by Kirchoff's circuit laws: the current law says that the sum of directed currents out of any point is equal to zero (accounting for external currents), and the voltage law says that the sum of directed voltage differences around any closed loop is equal to zero. It is a well-known empirical fact that Kirchoff's circuit laws can be solved uniquely for any externally-applied current flow which satisfies conservation of current. (To some, it is a well-known mathematical result.)

Our convention is that current flows from higher voltage to lower voltage.

Example 4.1. We illustrate in Figure 10 the induced current flow from sending 1 unit of current from y to z, where the metric graph has unit edge-lengths. Assuming Γ is grounded at z, the voltages at the trivalent vertices are 0, $\frac{2}{12}$, $\frac{3}{12}$, $\frac{3}{12}$, and $\frac{5}{12}$.

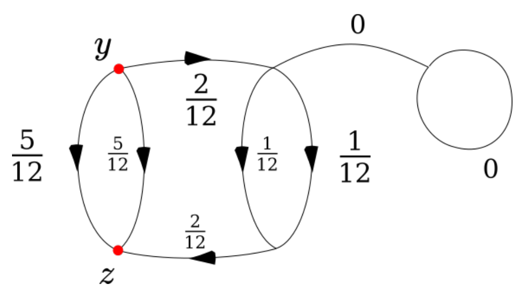


Figure 10: Current flow induced by external current from y to z.

Remark 4.2. We may interpret any piecewise linear function $f \in \mathrm{PL}_{\mathbb{R}}(\Gamma)$ as a voltage function on Γ, which results from the externally applied current $\Delta(f) \in \mathrm{Div}_{\mathbb{R}}(\Gamma)$. In other words, f gives a voltage resulting from sending current from $\Delta^{-}(f)$ to $\Delta^{+}(f)$ in Γ.

Definition 4.3. Let $r : \Gamma \times \Gamma \to \mathbb{R}$ denote the resistance function on the metric graph Γ. Namely, viewing Γ as a resistor network

$$\begin{aligned} r(x,y) &= \text{effective resistance between } x \text{ and } y \\ &= \text{total voltage drop when sending 1 unit of current from } x \text{ to } y \end{aligned}$$

It is straightforward to verify that the resistance function satisfies the following properties: (1) $r(x,x) = 0$, (2) $r(x,y) > 0$ when $x \neq y$, (3) $r(x,y)$ is continuous with respect to x and y, and (4) $r(x,y) = r(y,x)$.

Example 4.4. Let Γ be a circle of circumference L. By choosing a basepoint which we denote as 0, we may parametrize Γ with the interval $[0, L]$. Identifying points in this way, we have

$$\begin{aligned} r(x,0) &= \text{parallel combination of resistances } x \text{ and } L - x \\ &= \frac{x(L-x)}{x+(L-x)} = x - \frac{1}{L}x^2. \end{aligned}$$

The effective resistance is maximized when $x = \frac{1}{2}L$, with maximum value $\frac{1}{4}L$. The effective resistance is minimized when $x = 0$ or $x = L$, with effective resistance 0.

Definition 4.5. The *canonical measure* $\mu = \mu_\Gamma$ on a metric graph Γ is the continuous measure defined by

$$\mu = -\frac{1}{2}\frac{d^2}{dx^2}r(x, y_0)\,dx.$$

where x is a length-preserving parameter on a segment, dx is the Lebesgue measure, and y_0 is a fixed point in Γ. This defines μ on the open dense subset of Γ where the second derivative exists; otherwise we let $\mu_\Gamma = 0$.

Example 4.4 shows that on a circle of circumference L the canonical measure is $\frac{1}{L}dx$. For an arbitrary metric graph, the canonical measure on an open segment is a constant multiple of the Lebesgue measure. The following characterization of the canonical measure is used to prove Theorem B on the distribution of Weierstrass points.

Proposition 4.6 (Baker–Faber [2])**.** *Let e be a segment on a metric graph with endpoints e^-, e^+ and length $R(e)$. Then the canonical measure on e is*

$$\begin{aligned} \mu(e) &= 1 - (\textit{current through } e \textit{ when } 1 \textit{ unit is sent from } e^- \textit{ to } e^+) \\ &= 1 - \frac{r(e^-, e^+)}{R(e)}. \end{aligned}$$

Example 4.7. In Figure 11 we show the canonical measure of each edge on a metric graph with unit edge-lengths. The value $\mu(e) = \frac{7}{12}$ follows from the current shown in Figure 10 and Proposition 4.6.

On a metric graph with unit edge-lengths, the canonical measure has the following combinatorial interpretation.

Proposition 4.8 (Classical [8])**.** *Let G be a metric graph with unit edge-lengths, and let μ denote the canonical measure on G. Then for each (unit-length) edge e,*

$$\mu(e) = \frac{\kappa(G \backslash e)}{\kappa(G)}$$

where $\kappa(G)$ denotes the number of spanning trees and $G\backslash e$ denotes the edge-deleted subgraph.

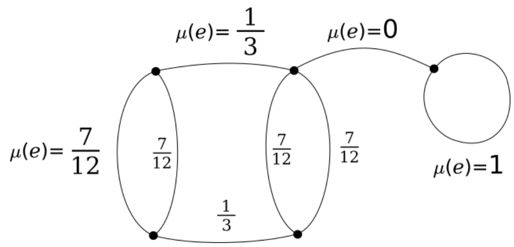


Figure 11: Canonical measure on edges of a unit-length metric graph.

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