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Alcove random walks and k-Schur functions

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Abstract. We use k-Schur functions to get the minimal boundary of the k-bounded partition poset. This permits to describe the central random walks on affine Grassmannian elements of type A and yields a rational expression for their drift. We also recover Rietsch's parametrization of totally nonnegative unitriangular Toeplitz matrices without using quantum cohomology of flag varieties. All the homeomorphisms we define can moreover be made explicit by using the combinatorics of k-Schur functions and elementary computations based on Perron-Frobenius theorem.

Résumé. Nous obtenons la frontière minimale du graphe des partitions k-bornées en utilisant l'algèbre des fonctions de k-Schur. Nous en déduisons une description des mesures de Gibbs pour les marches sur les Grassmaniennes affines de type A, ainsi qu'une expression rationnelle de la dérive d'une marche aléatoire suivant une telle mesure. Cela donne une preuve alternative et combinatoire de la paramétrisation des matrices de Toeplitz unitriangulaires totalement positives obtenue par Rietsch. Tous les homéomorphismes sont de plus explicites et découlent de la combinatoire des fonctions de k-Schur et de manipulations du théorème de Perron-Frobenius.

Keywords: k-bounded partitions, k-Schur functions, affine Grassmannian, central probability measure

1 Introduction

A real function on the Young graph is harmonic when its value on any Young diagram λ is equal to the sum of its values on the Young diagrams obtained by adding one box to λ. The set of extremal nonnegative such functions (i.e. those that cannot be written as a convex combination) is called the minimal boundary of the Young graph. It is homeomorphic to the Thoma simplex. Kerov and Vershik proved that the extremal nonnegative harmonic functions give the asymptotic characters of the symmetric group. Kerov-Vershik approach of these harmonic functions yields both a simple parametrization of the set of infinite totally nonnegative unitriangular Toeplitz matrices (see [2]) and a characterization of the morphisms from the algebra Λ of symmetric functions to $\mathbb{R}$

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which are nonnegative on the Schur functions. These results were generalized in [9] and [10]. A crucial observation here is the connection between the Pieri rule on Schur functions and the structure of the Young graph (which is then said multiplicative in Kerov-Vershik terminology).

In [14], Rietsch obtained a parametrization for the variety $T_{\geq 0}$ of finite unitriangular $(k+1)\times(k+1)$ totally nonnegative Toeplitz matrices by $\mathbb{R}^k_{\geq 0}$ from the quantum cohomology of partial flag varieties. More precisely, such a matrix is proved to be completely determined by the datum of its k initial minors obtained by considering its southwest corners. On the combinatorial side, there is also an interesting k-analogue $\mathcal{B}_k$ of the Young lattice of partitions whose vertices are the k-bounded partitions (i.e. those with no parts greater than k). Its oriented graph structure is isomorphic to to the Hasse poset on the affine Grassmannian permutations of type A which are minimal length coset representatives in $\widetilde{W}/W$, where $\widetilde{W}$ is the affine type $A_k^{(1)}$ group and W the symmetric group of type A_k. The graph $\mathcal{B}_k$ is also multiplicative but we have then to replace the ordinary Schur functions by the k-Schur functions (see [6] and the references therein) and the algebra Λ by $\Lambda_{(k)} = \mathbb{R}[h_1,\ldots,h_k]$. The k-Schur functions were introduced by Lascoux, Lapointe and Morse [8] as a basis of $\Lambda_{(k)}$. It was established by Lam [4] that their corresponding constant structures (called k-Littlewood-Richardson coefficients) are nonnegative. This was done by interpreting $\Lambda_{(k)}$ in terms of the homology ring of the affine Grassmannian which, by works of Lam and Shimozono, can be conveniently identified with the quantum cohomology ring of partial flag varieties studied by Rietsch. By merging these two geometric approaches one can theoretically deduce that the set of morphisms from $\Lambda_{(k)}$ to $\mathbb{R}$, nonnegative on the k-Schur functions, is also parametrized by $\mathbb{R}^k_{\geq 0}$.

In this note, we shall use another approach to avoid sophisticated geometric notions and make our construction as effective as possible. Our starting point is the combinatorics of k-Schur functions. The latter yield an explicit parametrization of the morphisms φ which are nonnegative on the k-Schur functions, or equivalently of all the minimal t-harmonic functions with $t \geq 0$ on $\mathcal{B}_k$. Both notions are related by the simple equality $t = \varphi(s_{(1)})$. Each such morphism is in fact completely determined by its values $\vec{r} = (r_1,\ldots,r_k) \in \mathbb{R}^k_{\geq 0}$ on the Schur functions indexed by the rectangle partitions $R_a = (k-a+1)^a$. We get a bi-continuous (homeomorphism) parametrization which is moreover effective in the sense one can compute from $\vec{r}$ the values of φ on any k-Schur function from the Perron-Frobenius vector of a matrix Φ encoding the multiplication by $s_{(1)}$ in $\Lambda_{(k)}$. Also, applying the primitive element theorem in the field of fractions of $\Lambda_{(k)}$ permits to prove that for any fixed $t \geq 0$ each $\varphi(s_\lambda^{(k)})$ is a rational functions on $\mathbb{R}^k_{\geq 0}$. It becomes then quite easy to derive Rietsch's parametrization. So, the only place where geometry is needed in this paper is in Lam's proof of the nonnegativity of the k-Schur coefficients. As far as we are aware a complete combinatorial k-Littlewood-Richardson

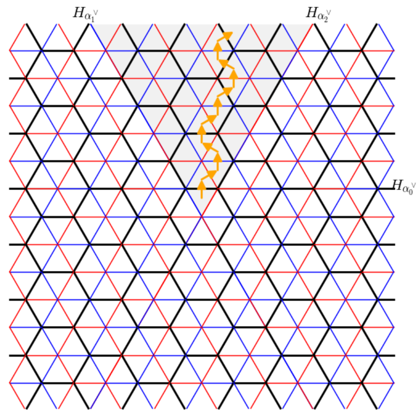

Figure 1: A reduced alcove walk on Grassmannian elements for $k = 2$.

rule is not yet available (see nevertheless [12]).

Random walks on reduced alcove paths have been considered by Lam in [5]. They are random walks on a particular tessellation of $\mathbb{R}^k$ by alcoves supported by hyperplanes, where each hyperplane can be crossed only once. The random walks considered in this paper are central and thus differ from those of [5]. Two trajectories with the same ends will have the same probability. We characterize all the possible laws of these alcove random walks and also get a simple algebraic expression of their drift as a rational function on $\mathbb{R}^k_{\geq 0}$. Our results are more precisely summarized in the following Theorem.

Theorem 1.1.

1. *To each $\vec{r} \in \mathbb{R}^k_{\geq 0}$ corresponds a unique morphism $\varphi : \Lambda_{(k)} \to \mathbb{R}$ nonnegative on the k-Schur functions and such that $\varphi(s_{R_a}) = r_a$ for any $a = 1, \ldots, k$.*

2. *To each $\vec{r} \in \mathbb{R}^k_{\geq 0}$ corresponds a unique matrix M in $T_{\geq 0}$ whose k southwest initial minors are exactly $r_1, \ldots, r_k$.*

3. *Both previous one-to-one correspondences are homeomorphisms, moreover φ and M can be explicitly computed from $\vec{r}$ by using Perron Frobenius theorem.*

4. *The minimal boundary of $\mathcal{B}_k$ is homeomorphic to the simplex $\mathcal{S}_k = \{(r_1, \ldots, r_k) \in \mathbb{R}^k_{\geq 0} \mid r_1 + \cdots + r_k = 1\}$.*

5. *To each $\vec{r} \in \mathcal{S}_k$ corresponds a central random walk $(v_n)_{n \geq 0}$ on affine Grassmannian elements which verifies a law of large numbers. The coordinates of its drift are the image by φ of rational fractions in the k-Schur functions. They are moreover rational on $\mathcal{S}_k$.*

This note is organized as follows. In Section 2, we recall some background on alcoves, partitions and k-Schur functions and prove the first four statements of Theorem 1.1. In

Section 3, we explain briefly the theory of central random walks and establish the fifth statement of Theorem 1.1.

Although we restrict ourselves to type A in this note, we expect to extend our approach to other types in a future work by using notably the results of [7] and [13].

2 Harmonic analysis on the lattice of k-bounded partitions

2.1 The lattice $\mathcal{B}_k$.

In this section, we refer to [6] and [11] for the material which is not defined. Fix $l > 1$ a nonnegative integer and set $k = l - 1$. Let $\widetilde{W}$ be the affine Weyl group of type $A_k^{(1)}$. As a Coxeter group, $\widetilde{W}$ is generated by the reflections $s_0, s_1, \ldots, s_k$ so that its subgroup generated by $s_1, \ldots, s_k$ is isomorphic to the symmetric group S_l. Write ℓ for the length function on $\widetilde{W}$. The group $\widetilde{W}$ determines a Coxeter arrangement by considering the hyperplanes orthogonal to the roots of type $A_k^{(1)}$. The connected components of this hyperplane arrangement yield a tessellation of $\mathbb{R}^k$ by alcoves on which the action of $\widetilde{W}$ is regular. We denote by $A^{(0)}$ the fundamental alcove. Write $\widetilde{R}$ for the set of affine roots of type $A_k^{(1)}$ and R for its subset of classical roots of type A_k. The simple roots are denoted by $\alpha_0, \ldots, \alpha_k$ and P is the weight lattice of type A_k with fundamental weights $\Lambda_1, \ldots, \Lambda_k$.

A reduced alcove path is a sequence of alcoves $(A_1, \ldots, A_m)$ such that $A_1 = A^{(0)}$ and for any $i = 1, \ldots, m-1$, the alcoves A_{i+1} and A_i share a common face contained in a hyperplane H_i so that the sequence $H_1, \ldots, H_{m-1}$ is without repetition (each hyperplane can be crossed only once). In the sequel, all the alcove paths we shall consider will be reduced. For any $i = 1, \ldots, m-1$, let w_i be the unique element of $\widetilde{W}$ such that $A_i = w_i(A^{(0)})$. Write $\lhd$ for the weak Bruhat order on $\widetilde{W}$ and $\rightarrow$ for the covering relation $w \rightarrow w'$ if and only if $w \lhd w'$ and $\ell(w') = \ell(w) + 1$. We then have $w_1 \rightarrow w_2 \rightarrow \cdots \rightarrow w_m$. The affine Grassmannian elements are the elements $w \in \widetilde{W}$ whose associated alcoves are exactly those located in the fundamental Weyl chamber (that is, in the Weyl chamber containing the fundamental alcove $A^{(0)}$).

We shall identify a partition and its Young diagram. A k-bounded partition is a partition λ such that $\lambda_1 \leq k$. There is a simple bijection between the k-bounded partitions and the affine Grassmannian elements (see [6] pages 18 and 19). Hence, the weak Bruhat order on $\tilde{W}$ yields a lattice structure on the k-bounded partitions, and the corresponding lattice is denoted by $\mathcal{B}_k$. So reduced alcove paths in the fundamental Weyl chamber, saturated chains of affine Grassmannian elements and paths in $\mathcal{B}_k$ naturally correspond.

2.2 The k-Schur functions

Let Λ be the algebra of symmetric functions in infinitely many variables over $\mathbb{R}$. It is endowed with a scalar product $\langle \cdot, \cdot \rangle$ such that $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$ for any partitions λ and μ. Let $\Lambda_{(k)}$ be the subalgebra of Λ generated by the complete homogeneous functions $h_1, \ldots, h_k$. In particular, $\{h_\lambda \mid \lambda \text{ is } k\text{-bounded}\}$ is a basis of $\Lambda_{(k)}$.

We now define a distinguished basis of $\Lambda_{(k)}$ related to the graph structures of $\mathcal{B}_k$. Consider λ and μ two k-bounded partitions with $\lambda \subset \mu$ and $r \leq k$ a positive integer. Recall the involution ω_k on $\Lambda_{(k)}$ analogue to the conjugation of partitions (see [6]).

Definition 2.1. *We will say that μ/λ is a weak horizontal strip of size r when*

1. *μ/λ is an horizontal strip with r boxes (i.e. the boxes in μ/λ belong to different columns),*
2. *$\mu^{\omega_k}/\lambda^{\omega_k}$ is a vertical strip with r boxes (i.e. the boxes in $\mu^{\omega_k}/\lambda^{\omega_k}$ belong to different rows).*

Definition 2.2 (Pieri rule for k-Schur functions, [6])**.** *The k-Schur functions $s_\kappa^{(k)}, \kappa \in \mathcal{B}_k$ are the unique functions in $\Lambda_{(k)}$ such that for any $r \leq k$, $s_{(r)}^{(k)} = h_r$ and for any $r \leq k$, any $\kappa \in \mathcal{B}_k$ we have*

$$h_r s_\kappa^{(k)} = \sum_{\varkappa \in \mathcal{B}_k} s_\varkappa^{(k)} \tag{2.1}$$

where the sum is over all the k-bounded partitions $\varkappa$ such that $\varkappa/\kappa$ is a weak horizontal strip of size r in $\mathcal{B}_k$.

We write $\kappa \to \mu$ when $\varkappa/\kappa$ is a weak horizontal strip of size 1. Thanks to a geometric interpretation of the k-Schur functions in terms of the homology of affine Grassmannians, Lam showed that the product of two k-Schur functions has nonnegative coefficients on the basis of k-Schur functions:

Theorem 2.3 ([4])**.** *Given κ and δ two k-bounded partitions, we have $s_\kappa^{(k)} s_\delta^{(k)} = \sum_{\nu \in \mathcal{B}_k} c_{\lambda,\delta}^{\nu(k)} s_\nu^{(k)}$ with $c_{\lambda,\delta}^{\nu(k)} \in \mathbb{Z}_{\geq 0}$.*

For any $a = 1, \ldots, k$, let R_a be the rectangle partition $(k - a + 1)^a$.

Proposition 2.4. *(See [6].) For each k-bounded partition λ, there exists a unique irreducible k-bounded partition[1] $\widetilde{\lambda}$ and a unique sequence of nonnegative integers $p_1, \ldots, p_k$ such that*

$$s_\lambda^{(k)} = \prod_{a=1}^{k} s_{R_a}^{p_a} s_{\widetilde{\lambda}}^{(k)}.$$

In particular, the k-Schur functions are completely determined by the k-Schur functions indexed by the irreducible k-bounded partitions and by the ordinary Schur functions $s_{(k-a+1)^a}$, $a = 1, \ldots, k$.

[1] A k-bounded partition is irreducible when it contains less than a parts equal to $k - a$ for any $a = 0, \ldots, k - 1$.

2.3 Harmonic functions and minimal boundary of $\mathcal{B}_k$

Definition 2.5. *A function $f : \mathcal{B}_k \to \mathbb{R}$ is harmonic when $f(\lambda) = \sum_{\lambda \to \mu} f(\mu)$ for any $\lambda \in \mathcal{B}_k$. We denote by $\mathcal{H}(\mathcal{B}_k)$ the set of harmonic functions on $\mathcal{B}_k$.*

The harmonic functions on $\mathcal{B}_k$ correspond to the right eigenvectors associated to the eigenvalue 1 of $\mathcal{M}$, the infinite adjacency matrix of the graph $\mathcal{B}_k$. One can also consider t-harmonic functions which correspond to the right eigenvectors for $\mathcal{M}$ associated to the eigenvalue t. Clearly $\mathcal{H}(\mathcal{B}_k)$ is a vector space over $\mathbb{R}$. In fact, we mostly restrict ourselves to the set $\mathcal{H}^+(\mathcal{B}_k)$ of positive harmonic functions for which f takes values in $\mathbb{R}_{\geq 0}$. Then, $\mathcal{H}^+(\mathcal{B}_k)$ is a cone since it is stable by addition and multiplication by a positive real. To study $\mathcal{H}^+(\mathcal{B}_k)$, we only have to consider its subset $\mathcal{H}_1^+(\mathcal{B}_k)$ of normalized harmonic functions such that $f(1) = 1$. In fact, $\mathcal{H}_1^+(\mathcal{B}_k)$ is a convex set and its structure is controlled by its extremal subset $\partial\mathcal{H}^+(\mathcal{B}_k)$.
We aim to characterize the extremal positive harmonic functions defined on $\mathcal{B}_k$ and obtain a simple parametrization of $\partial\mathcal{H}^+(\mathcal{B}_k)$. By applying the Pieri rule on k-Schur functions, we get

$$s_\lambda s_{(1)} = \sum_{\lambda \to \mu} s_\mu$$

for any k-bounded partitions λ and μ. This means that $\mathcal{B}_k$ is a so-called multiplicative graph with associated algebra $\Lambda_{(k)}$. Moreover, if we denote by K the positive cone spanned by the set of k-Schur functions, we can apply the ring theorem of Kerov and Vershik (see for example [10, Section 8.4]) which characterizes the subset of extreme points $\partial\mathcal{H}^+(\mathcal{B}_k)$. Denote by $\mathrm{Mult}^+(\Lambda_{(k)}) \subset \Lambda_{(k)}^*$ the set of multiplicative functions on $\Lambda_{(k)}$ which are nonnegative on K and equal to 1 on s_1. So a linear form $f \in \Lambda_{(k)}^*$ belongs to $\mathrm{Mult}^+(\Lambda_{(k)})$ when $f(K) \subset \mathbb{R}^+$, $f(s_1) = 1$ and it satisfies $f(uv) = f(u)f(v)$ for any $u, v \in \Lambda_{(k)}$. Note that $i : \mathcal{B}_k \longrightarrow \Lambda_{(k)}$ such that $i(\lambda) = s_\lambda^{(k)}$ induces a map $i^* : \Lambda_{(k)}^* \longrightarrow F(\mathcal{B}_k, \mathbb{R})$. Then, the ring theorem yields the following algebraic characterization of $\partial\mathcal{H}^+(\mathcal{B}_k)$.

Proposition 2.6. *The map i^* yields an homeomorphism between $\mathrm{Mult}^+(\Lambda_{(k)})$ and $\partial\mathcal{H}^+(\mathcal{B}_k)$.*

Since $i(\mathcal{B}_k)$ is a basis of $\Lambda_{(k)}$, this means that $\partial\mathcal{H}(\mathcal{B}_k)$ is completely determined by the $\mathbb{R}$-algebra morphisms $\varphi : \Lambda_{(k)} \to \mathbb{R}$ such that $\varphi(s_1) = 1$ and $\varphi(s_\lambda^{(k)}) \geq 0$ for any k-bounded partition λ. Each function $f \in \partial\mathcal{H}^+(\mathcal{B}_k)$ can then be written $f = \varphi \circ i$.

2.4 The matrix Φ

By Proposition 2.4, each morphism $\varphi : \Lambda_{(k)} \to \mathbb{R}$ is uniquely determined by its values on the rectangle Schur functions $s_{R_a}, 1 \leq a \leq k$ and on each $s_{\tilde{\lambda}}^{(k)}$ where $\tilde{\lambda}$ is an irreducible

k-bounded partition. Set $r_a = \varphi(s_{R_a}), a = 1, \ldots, k$ and $\vec{r} = (r_1, \ldots, r_k)$. Denote by $\mathcal{P}_{\mathrm{irr}}$ the set of irreducible k-bounded partitions (including the empty partition). Then, for $\lambda \in \mathcal{P}_{\mathrm{irr}}$,

$$\varphi(s_\lambda^{(k)})\varphi(s_{(1)}) = \sum_{\lambda \to \mu} \varphi(s_\mu^{(k)}). \tag{2.2}$$

By Proposition 2.4, for each k-bounded partition μ there exists a sequence $\{p_1^\mu, p_2^\mu \ldots, p_k^\mu\}$ of integers and an irreducible partition $\widetilde{\mu}$ such that

$$s_\mu^{(k)} = \prod_{a=1}^{k} s_{R_a}^{p_a^\mu} s_{\widetilde{\mu}}^{(k)} \text{ and thus } \varphi(s_\mu^{(k)}) = \prod_{a=1}^{k} r_a^{p_a^\mu} \varphi(s_{\widetilde{\mu}}^{(k)}). \tag{2.3}$$

Hence by setting

$$\varphi_{\lambda\nu} = \sum_{\substack{\lambda \to \mu \\ \widetilde{\mu} = \nu}} \prod_{1 \le a \le k} r_a^{p_a^\mu}$$

we get

$$\varphi(s_\lambda^{(k)}) = \sum_{\nu \in \mathcal{P}_{\mathrm{irr}}} \varphi_{\lambda\nu}\varphi(s_\nu^{(k)}).$$

Let $\Phi_{(r_1,\ldots,r_k)} := (\varphi_{\nu\lambda})_{\lambda,\nu \in \mathcal{P}_{\mathrm{irr}}}$ [2] and define $f \in \mathbb{R}^{\mathcal{P}_{\mathrm{irr}}}$ as the vector $(\varphi(s_\lambda^{(k)}))_{\lambda \in \mathcal{P}_{\mathrm{irr}}}$. When there is no risk of confusion, we simply write Φ instead of $\Phi_{(r_1,\ldots,r_k)}$. The vector f is a left eigenvector of Φ for the eigenvalue $\varphi(s_1)$ with positive entries having value 1 on $\emptyset$ and $\varphi(s_1)$ on s_1.

Recall that a matrix $M \in M_n(\mathbb{R})$ with nonnegative entries is irreducible if and only if for each $1 \le i, j \le n$ there exists $n \ge 1$ such that $(M^n)_{ij} > 0$. In view of using the Perron-Frobenius theorem, we have the following result on the irreducibility of Φ.

Proposition 2.7. *Consider $\vec{r} = (r_1, \ldots, r_k) \in \mathbb{R}^k_{\ge 0}$. Then, the matrices $\Phi_{(r_1,\ldots,r_k)}$ and $\Phi^t_{(r_1,\ldots,r_k)}$ associated to φ are irreducible if and only if for all $1 \le a \le k-1$, r_a or r_{a+1} is positive.*

2.5 Field extensions and k-Schur functions

Recall that $\Lambda_{(k)} = \mathbb{R}[h_1, \ldots, h_k]$. Since $h_1, \ldots, h_k$ are algebraically independent over $\mathbb{R}$, we can consider the fraction field $\mathbb{L} = \mathbb{R}(h_1, \ldots, h_k)$. Write $\mathbb{A} = \mathbb{R}[s_{R_1}, \ldots s_{R_k}]$ the subalgebra of $\Lambda_{(k)}$ generated by the rectangle Schur functions $s_{R_a}, a = 1, \ldots, k$. By standard arguments one can prove that $s_{R_1}, \ldots s_{R_k}$ are algebraically independent over $\mathbb{R}$, so that we can consider the fraction field $\mathbb{K} = \mathbb{R}(s_{R_1}, \ldots s_{R_k})$ of the algebra $\mathbb{A}$. By classical results from [3] and Proposition 2.4, $s_1 = h_1$ is algebraic over $\mathbb{K}$. The following result is a main step in the proof of Theorem 1.1.

[2]Observe we have defined $\Phi_{(r_1,\ldots,r_k)}$ as the transpose of the matrix $(\varphi_{\lambda\mu})_{\lambda,\nu \in \mathcal{P}_{\mathrm{irr}}}$ to make it compatible with the multiplication by $s_{(1)}$ used in Section 2.5.

Theorem 2.8. *We have $\mathbb{L} = \mathbb{K}(s_{(1)})$, that is $s_{(1)}$ is a primitive element for $\mathbb{L}$ regarded as an extension of $\mathbb{K}$. In particular, there exist $\Delta \in \mathbb{A}$ and for each irreducible k-bounded partition κ a polynomial $P_\kappa \in \mathbb{A}[T]$ such that*

$$s_\kappa^{(k)} = \frac{1}{\Delta} P_\kappa(s_{(1)}).$$

In particular, for any morphism $\varphi : \Lambda_{(k)} \to \mathbb{R}$ such that $\varphi(\Delta) \neq 0$ we have

$$\varphi(s_\kappa^{(k)}) = \frac{1}{\varphi(\Delta)} \varphi(P_\kappa)\varphi(s_{(1)}).$$

2.6 Minimal boundary of $\mathcal{B}_k$

We give here a sketch of the proof of the first part of Theorem 1.1. Set $\mathcal{R} = \{s_{R_i}, 1 \leq i \leq k\}$. Recall that by Proposition 2.6, we have $\partial\mathcal{H}^+(\mathcal{B}_k) \simeq \mathrm{Mult}^+(\Lambda_{(k)})$. Using the particular structure of morphisms on $\Lambda_{(k)}$ given by (2.3) and the Perron-Frobenius Theorem, one can prove the following crucial step:

Theorem 2.9.

1. *Let $\varphi : \mathbb{A} \to \mathbb{R}$ be a morphism, nonnegative on $\mathcal{R}$ and such that its associated matrix Φ is irreducible. Then there exists a unique morphism $\tilde{\varphi} : \Lambda_{(k)} \longrightarrow \mathbb{R}_{\geq 0}$ extending φ, nonnegative on the k-Schur functions and positive on $\mathcal{I} = \{s_\lambda \mid \lambda \in \mathcal{P}_{\mathrm{irr}}\}$.*

2. *A positive morphism $\varphi : \Lambda_{(k)} \longrightarrow \mathbb{R}$ is uniquely determined by its values on $\mathcal{R}$.*

Remark that this solves almost all the cases thanks to Proposition 2.7. To conclude the proof of the first part of Theorem 1.1, we need to extend the previous result to the case when Φ is not irreducible. This extension is proven by classical continuity arguments using the algebraic structure of $\Lambda_{(k)}$ given in Section 2.5. Since morphisms on $\mathbb{A}$ which are nonnegative on $\mathcal{R}$ are homeomorphic to $\mathbb{R}^k_{\geq 0}$, simple computations yield the following result.

Proposition 2.10. $\mathrm{Mult}^+(\Lambda_{(k)})$ *is homeomorphic to $\mathbb{R}^k_{\geq 0}$ and $\partial\mathcal{H}^+(\mathcal{B}_k)$ is homeomorphic to $\mathcal{S}_k = \{(r_1, \ldots, r_k) \in \mathbb{R}^k_{\geq 0} \mid r_1 + \cdots + r_k = 1\}$.*

2.7 Rietsch parametrization of Toeplitz matrices

Consider the variety $T_{\geq 0} \subset \mathbb{R}^k_{>0}$ of totally nonnegative unitriangular Toeplitz $(k+1) \times (k+1)$ matrices

$$M = \begin{bmatrix} 1 & & & & \\ h_1 & 1 & & & \\ \vdots & h_1 & \ddots & & \\ h_{k-1} & \vdots & \vdots & \ddots & \\ h_k & h_{k-1} & \cdots & h_1 & 1 \end{bmatrix}.$$

The set $T_{>0}$ of totally positive unitriangular Toeplitz $(k+1) \times (k+1)$ matrices is defined as the subset of $T_{\geq 0}$ of matrices M whose minors with no row and no column in the upper part of M are positive. By Theorem 3.2.1 in [1], M is totally positive if and only if for $a = 1, \ldots, k$, the $a \times a$ initial minors obtained by selecting a rows of M arbitrary and then the first a columns of M are positive.

Lemma 2.11.

1. *The previous initial minors are equal to Schur functions s_λ, where the maximal hook of the partition λ has length less or equal to k.*

2. *We have $\overline{T}_{>0} = T_{\geq 0}$ that is, each totally nonnegative unitriangular Toeplitz matrix is the limit of a sequence of totally positive unitriangular Toeplitz matrices.*

Observe in particular that for any $a = 1, \ldots, k$, the initial minor $\Delta_{[k-a+1,k]}$ gives the value r_a of the rectangle Schur function s_{R_a} evaluated in $(h_1, \ldots, h_k)$. In [14], Rietsch obtained the parametrization of $T_{\geq 0}$ by using the quantum cohomology of partial flag varieties. Our approach permits to reprove this theorem and makes both the associated homeomorphism and its converse explicit.

Theorem 2.12. *The map $g : T_{\geq 0} \to \mathbb{R}^k_{\geq 0} : (h_1, \ldots, h_k) \longmapsto (r_1, \ldots, r_k)$ is a homeomorphism. Moreover, g^{-1} can be made explicit from the Perron-Frobenius theorem applied on Φ and Theorem 2.8.*

So our Theorem 2.12 permits in fact to compute the nonnegative Toeplitz matrix associated to any point of $\mathbb{R}^k_{\geq 0}$ (i.e. to reconstruct M from the datum of the minors $(r_1, \ldots, r_k)$). When Φ is irreducible this is achieved directly by applying the Perron-Frobenius theorem which gives the $\varphi(s_\kappa)$'s with κ any irreducible k-bounded partition of Theorem 2.8. In particular, one so gets $\varphi(s_{(a)}) = h_a$ for $a = 1, \ldots, k$. In general we show that the formulas of Theorem 2.8 can be extended by continuity when $\Delta = 0$.

3 Markov chains on alcoves

3.1 Central Markov chains on alcoves from harmonic functions

Recall the notation of Section 2.1 for the notion of reduced alcove paths. A probability distribution on reduced alcove paths is said central when the probability p_π of the path

$\pi = (A_1 = A^{(0)}, A_2, \dots, A_m)$ only depends on m, A_1 and A_m, that is only on its length and its alcoves ends. In the situation we consider, central random reduced alcove paths in the Weyl chamber correspond to central random paths on $\mathcal{B}_k$. They are determined by the positive harmonic functions on $\mathcal{B}_k$ (see [2]).

More precisely any central probability distribution on the affine Grassmannian alcove paths can be written

$$p_\pi = \frac{h(\mu)}{h(\lambda)}$$

where $h \in \mathcal{H}^+(\mathcal{B}_k)$ is positive and for any path $\pi = (A_1, \dots, A_m)$, μ and λ are the k-bounded partitions associated to A_1 and A_m. Also we then get a Markov chain on $\mathcal{B}_k$ (or equivalently on the affine Grassmannian elements) with transition matrix

$$\Pi(\lambda, \mu) = \frac{h(\mu)}{h(\lambda)}.$$

When the harmonic function h is extremal, it corresponds to a morphism φ on $\Lambda_{(k)}$ with $\varphi(s_{(1)}) = 1$, nonnegative on the k-Schur functions. We get an extremal central distribution on the trajectories starting at $A^{(0)}$ verifying $p_\pi = \frac{\varphi(s_\mu^{(k)})}{\varphi(s_\lambda^{(k)})}$. The associated Markov chain has then the transition matrix $\Pi(\lambda, \mu) = \frac{\varphi(s_\mu^{(k)})}{\varphi(s_\lambda^{(k)})}$.

Involutions on the reduced walk

Let $\mathcal{A}_k$ be the set of alcoves in the dominant Weyl chamber. We describe here two important involutions on the set of irreducible k-bounded partitions. The first symmetry is due to the action of ω_k on $\Lambda_{(k)}$ which sends s_{R_a} to $s_{R_{k-a}}$ for any $a = 1, \dots, k$. By bijection between alcoves in the Weyl chamber and k-bounded partitions, this symmetry yields an involution Ω on $\mathcal{A}_k$, which corresponds to the involution on the Dynkin diagram of affine type $A_k^{(1)}$ fixing the node 0 and sending each node $i \in \{1, \dots, k\}$ to $i^* = k + 1 - i$.

For the second symmetry, we need some basic facts about the affine Coxeter arrangement of type $A_k^{(1)}$. For any root α and any integer, let $H_{\alpha,r}$ be the affine hyperplane

$$H_{\alpha,r} = \{v \in \mathbb{R}^k, \langle v, \alpha \rangle = r\}.$$

We denote by $s_{\alpha,r}$ the reflection with respect to this hyperplane and for β in the weight lattice P, we write t_β for the translation by β. We have then $s_{\alpha,r} = t_{r\alpha} s_{\alpha,0}$. Affine Grassmannian elements are in bijection with alcoves in the dominant Weyl chamber through a map $w \mapsto A_w$ such that $w \to w'$ (that is we have a covering relation for the weak order from w to w') if and only if there is a hyperplane $H_{\alpha,r}$ such that $A_{w'} = s_{\alpha,r}(A_w)$. In this case, we write $w \xrightarrow{\alpha,r} w'$.

Write v_w for the center of the alcove A_w (defined as the mean of the its extreme weights). Any alcove A_w is completely determined by its center v_w, and with this notation $w \xrightarrow{\alpha,r} w'$ if and only if $v_{w'} = s_{\alpha,r}(v_w)$ and $r < \langle \alpha, v_{w'} \rangle < r+1$.

Let B be the set of alcoves which are included in the fundamental parallelepiped $\{v \in \mathbb{R}^k | \langle v, \alpha_i \rangle \in]0,1[\text{ for } i = 1,\dots,k\}$. This set is in bijection with the set of irreducible k-bounded partitions. For example, after identifying an alcove with its center, we have $\mathsf{B} = \{\Lambda_1/3 + \Lambda_2/3, 2\Lambda_1/3 + 2\Lambda_2/3\}$ for $k = 2$. Consider now the involution $I : \mathbb{R}^k \to \mathbb{R}^k$ defined by

$$I = t_\rho \circ w_0,$$

where $\rho = \sum_{i=1}^k \Lambda_i$ and w_0 is the longest element of W. Observe we indeed get an involution because $w_0 \circ t_\rho \circ w_0 = t_{-\rho}$. The involution I has the simple expression $I(l_1\Lambda_1 + \cdots + l_k\Lambda_k) = (1 - l_k)\Lambda_1 + \cdots + (1 - l_1)\Lambda_k$ for $l_1, \dots, l_k \in \mathbb{R}$.

Lemma 3.1. *The involution I restricts to an involution on the set B, and $A \to A'$ if and only if $I(A') \to I(A)$.*

Hence, I yields a bijection on B which preserves the graph structure coming from $\mathcal{B}_k$.

Drift under harmonic measures

We denote by $\Gamma_f(\mathcal{A}_k)$ the set of reduced finite alcove paths which start at $A^{(0)}$ and remain in the dominant Weyl chamber. For any A in $\mathcal{A}_k$, write $\lambda_A \in \mathcal{B}_k$ its corresponding k-bounded partition. Conversely recall that for any $\lambda \in \mathcal{B}_k$, $A_{w_\lambda} \in \mathcal{A}_k$ is the alcove associated to λ. Let φ be an extremal harmonic measure on $\mathcal{B}_k$ associated to $\vec{r} = (r_1, \dots, r_k) \in \mathbb{R}^k$, and let $(A_n)_{n \geq 1}$ be the central Markov chain on $\mathcal{A}_k$ defined in Section 3.1. By considering for each $n \geq 1$ the center v_n of the alcove A_n, we get a genuine random walk $(v_n)_{n\geq 1}$ on $\mathbb{R}^k$. Our goal is now to prove the law of large numbers for this random walk. For any alcove A, set $\overline{A} = \mathsf{I}\Omega(A)$.

Theorem 3.2. *As n goes to infinity, the normalized random walk $\left(\frac{1}{n} v_n\right)_{n \geq 1}$ converges almost surely to a vector $v_\varphi \in \mathbb{R}^k$, whose coordinate on Λ_i satisfies*

$$v_\varphi(i) = \varphi \left(\frac{s_{R_i}}{\sum_{A \in \mathsf{B}} s^{(k)}_{\lambda_A} s^{(k)}_{\lambda_{\overline{A}}}} \sum_{\substack{e: A \to A' \\ c(e) = i}} s^{(k)}_{\lambda_{\overline{A}}} s^{(k)}_{\lambda_{A'}} \right)$$

which is a rational function on $\mathbb{R}^k$.

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References

[1] A. Berenstein, S. Fomin, and A. Zelevinsky. "Parametrizations of canonical bases and totally positive matrices". *Adv. Math.* **122**.1 (1996), pp. 49–149. Link.

[2] S. V. Kerov. *Asymptotic Representation Theory of the Symmetric Group and its Applications in Analysis*. Translations of Mathematical Monographs 219. American Mathematical Society, Providence, RI, 2003.

[3] M. Kreuzer and L. Robbiano. *Computational Commutative Algebra 2*. Springer-Verlag, Berlin, 2005.

[4] T. Lam. "Schubert polynomials for the affine Grassmannian". *J. Amer. Math. Soc.* **21**.1 (2008), pp. 259–281. Link.

[5] T. Lam. "The shape of a random affine Weyl group element and random core partitions". *Ann. Probab.* **43**.4 (2015), pp. 1643–1662. Link.

[6] T. Lam, L. Lapointe, J. Morse, A. Schilling, M. Shimozono, and M. Zabrocki. *k-Schur Functions and Affine Schubert Calculus*. Fields Institute Monographs 33. Springer, New York, 2014.

[7] T. Lam, A. Schilling, and M. Shimozono. "Schubert polynomials for the affine Grassmannian of the symplectic group". *Math. Z.* **264**.4 (2010), pp. 765–811. Link.

[8] L. Lapointe, A. Lascoux, and J. Morse. "Tableau atoms and a new Macdonald positivity conjecture". *Duke Math. J.* **116**.1 (2003), pp. 103–146. Link.

[9] C. Lecouvey, E. Lesigne, and M. Peigné. "Conditioned random walks from Kac-Moody root systems". *Trans. Amer. Math. Soc.* **368**.5 (2016), pp. 3177–3210. Link.

[10] C. Lecouvey and P. Tarrago. "Harmonic functions on multiplicative graphs and weight polytopes of representations". 2016. arXiv:1609.00138.

[11] I. G. Macdonald. *Symmetric Functions and Hall Polynomials*. 2nd ed. Oxford Classic Texts in the Physical Sciences. The Clarendon Press, Oxford University Press, New York, 2015.

[12] J. Morse and A. Schilling. "Crystal approach to affine Schubert calculus". *Int. Math. Res. Not. IMRN* 8 (2016), pp. 2239–2294. Link.

[13] S. Pon. "Affine Stanley symmetric functions for classical types". *J. Algebraic Combin.* **36**.4 (2012), pp. 595–622. Link.

[14] K. Rietsch. "Totally positive Toeplitz matrices and quantum cohomology of partial flag varieties". *J. Amer. Math. Soc.* **16**.2 (2003), pp. 363–392. Link.