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Cluster algebras and binary words

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Abstract. We establish a connection between binary subwords and perfect matchings of a snake graph, an important tool in the theory of cluster algebras. Every binary expansion w can be associated to a piecewise-linear poset P and a snake graph G. We describe bijections from the subwords of w to the antichains of P and to the perfect matchings of G. We also construct a tree structure called the antichain trie which is isomorphic to the trie of subwords introduced by Leroy, Rigo, and Stipulanti.

Keywords: snake graph perfect matching, cluster algebra, binary subword, binomial coefficient of words, antichain, order filter

1 Introduction

A planar graph called the *snake graph* appears naturally in the study of cluster algebras [6]. An early version of the snake graph is a bipartite graph which is dual to a polygon triangulation and was studied by Propp et al. along with other equivalent combinatorial models [15]. Musiker, Schiffler, and Williams then used the snake graphs to study positivity and bases of cluster algebras from surfaces [12, 13]. The theory of abstract snake graph was developed further by Çanakçı and Schiffler [3]. The snake graphs are connected to various mathematical objects, including matchings of triangles [1, 15, 8], submodules of a string module [2, 13, 5], T-paths [16, 7], 0-1 sequences called globally compatible sequences (GCSs) [9], matchings of angles and minimal cuts [17], intervals in the weak order determined by a Coxeter element [5], continued fractions [4], and Jones polynomials [10]. We add another item to this list by providing a connection between snake graphs and base-2 expansions of positive integers.

In this paper, let a *binary word* be a finite (possibly empty) sequence of letters on the alphabet $\{0,1\}$ starting with 1. Let a *subword* of a binary word be a "scattered" subsequence which is itself a binary word.

To every nonempty binary word $w = w_1 w_2 \dots w_d$ of length d we associate (the Hasse diagram of) a piecewise-linear partially ordered set (poset) P as follows. The elements of P are labeled $P_1 = 1, \dots, P_d = d$, arranged from left to right in the Hasse diagram of P,

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and there is an edge between P_{i-1} and P_i. For $i \geq 2$, if $w_i = 1$ (respectively, if $w_i = 0$) then the edge between P_{i-1} and P_i is of slope 1 (respectively, -1), so that we have the covering relation $P_{i-1} \lessdot P_i$ (respectively, $P_{i-1} \gtrdot P_i$). See Figure 1.

An *antichain* is a subset $A = \{A_1, A_2, \ldots, A_r\}$ of a poset such that no two distinct elements in A are comparable. For example, the subsets $\{1,3,6\}$, $\{1,4\}$, and $\{2,6\}$ of the poset whose Hasse diagram is given in Figure 1 are antichains, while $\{2,4\}$ is not.

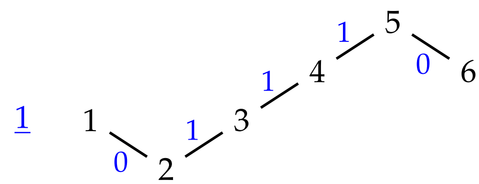

Figure 1: The Hasse diagram associated to the word 101110.

In [11, Sec. 2], Leroy, Rigo, and Stipulanti introduce a specific construction of a prefix tree (called *trie of subwords*) which is a binary tree that is convenient for counting distinct subwords occurring in a given word w. We use this same construction to define a tree we call *antichain trie* to study the antichains of the poset P corresponding to w. We associate each node v of the antichain trie to an antichain $A(v)$ of P in such a way that moving from a node v to its left child replaces P_i in $A(v)$ with P_{i+1} (where i is the largest integer in $A(v)$) and moving from a node v to its right child adds a new element P_i to $A(v)$ (where i is larger than every integer in $A(v)$).

Proposition 1.1 (Proposition 3.2)**.** *The nodes of the antichain trie are distinct antichains.*

Next, we show that this antichain trie contains all antichains by giving a bijection between the subwords and the antichains.

Theorem 1.2 (Theorem 4.3)**.** *Given a nonempty binary word w and its corresponding piecewise-linear poset P, there is a bijection between the subwords of w and the antichains of P.*

It is known that one can associate a binary sequence of length $d-1$ to a snake graph with d tiles (see Definition 5.3). So, given a binary word $w = w_1\, w_2\, \ldots\, w_d$, we associate $(w_2, \ldots, w_d)$ to a snake graph $G(w)$ and present a bijection from the subwords of w to the perfect matchings of $G(w)$.

Theorem 1.3 (Theorem 5.4)**.** *The subwords of a binary word w are in bijection with the perfect matchings of its corresponding snake graph $G(w)$.*

The paper is organized as follows. In Section 2, we recall the construction of the trie of subwords given by Leroy, Rigo, and Stipulanti. In Section 3, we introduce the antichain trie. We define a map from the antichains to the subwords and prove that it is a bijection in Section 4. In Section 5, we give the necessary snake graph theory background and present a bijection between subwords and perfect matchings.

2 Trie of subwords

Let $w = w_1 \dots w_d$ be a nonempty binary word and consider the *trie of (distinct) subwords* of w, denoted by $\mathcal{T}$. It is a tree with the root denoted by ϵ. If u and ua are two subwords of w with a being a one-letter subword, then ua is a child of u. This trie is also called a *prefix tree* because all successors of a node have a common prefix. Note that, since w is a binary word, the trie is a binary tree. For the rest of the section, we describe the specific construction of $\mathcal{T}$ which is given in [11, Sec. 2].

Factor w into consecutive maximal blocks of 1's and blocks of 0's such that

$$w = \underbrace{1^{n_1}}_{u_1}\underbrace{0^{n_2}}_{u_2}\underbrace{1^{n_3}}_{u_3}\underbrace{0^{n_4}}_{u_4}\cdots\underbrace{1^{n_{2j-1}}}_{u_{2j-1}}\underbrace{0^{n_{2j}}}_{u_{2j}}$$

with $j \geq 1$, $n_1, \dots, n_{2j-1} \geq 1$ and $n_{2j} \geq 0$. Let M be such that $w = u_1 u_2 \dots u_M$ where u_M is the last non-empty block of zeroes or ones.

To construct the trie $\mathcal{T}$, begin with a vertical linear tree T_w with nodes $v_0, \dots, v_d$. Let T_w be rooted at $\epsilon = v_0$ and let node v_i be the left child of node v_{i-1} for all $i = 1, \dots, d$. Label the edges of T_w with the letters of w such that the edge between nodes v_{i-1} and v_i is labeled w_i. We identify each node v by the path of edge labels from ϵ to v.

Starting from the bottom of the vertical linear tree T_w, we define a tree T_l for every $l \in \{M-1, \dots, 2, 1\}$. Each tree is rooted at the node $u_1 \dots u_l 1$ if l is even and $u_1 \dots u_l 0$ if l is odd. First, let T_{M-1} be the (linear) subtree of T_w consisting of the last n_M nodes.

We then attach a copy of T_{M-1} to each node (on the vertical tree T_w) of the form

$$\begin{cases} u_1 u_2 \dots u_{M-2} 1^j, & \text{if } u_{M-1} \text{ is a block of 1s} \\ u_1 u_2 \dots u_{M-2} 0^j, & \text{if } u_{M-1} \text{ is a block of 0s} \end{cases} \qquad \text{for } j \in \{0, 1 \dots, n_{M-1} - 1\}.$$

Let the root of each copy of T_{M-1} be the right child of the node of T_w that this root is attached to. This results in a (non-linear) tree T'_w that is larger than T_w.

Let T_{M-2} be the subtree of this larger tree T'_w such that its root is $u_1 \dots u_{M-2} 1$ if $M-2$ is even and $u_1 \dots u_{M-2} 0$ if $M-2$ is odd and T_{M-2} contains all the descendants of this root. Then attach a copy of T_{M-2} to each node of the form

$$\begin{cases} u_1 u_2 \dots u_{M-3} 1^j, & \text{if } u_{M-2} \text{ is a block of 1s} \\ u_1 u_2 \dots u_{M-3} 0^j, & \text{if } u_{M-2} \text{ is a block of 0s} \end{cases} \qquad \text{for } j \in \{0, 1, ..., n_{M-2} - 1\}.$$

Again, let the root of each copy of T_{M-2} be the right child of the node of T_w that this root is attached to.

Let T_{M-3} be the subtree of this larger tree such that it is rooted at $u_1 \dots u_{M-3} 1$ (resp., $u_1 \dots u_{M-3} 0$) if $M-3$ is even (resp., odd) and T_{M-3} contains all descendants of this root.

Continue as such until after we attach a copy of T_2. If $n_1 = 1$ then no copy of T_1 is added (as in Figures 2 and 4 (left)). If $n_1 > 1$ then a copy of T_1 is added to each node of the form $1^j, j \in \{0, 1, \dots, n_1 - 1\}$ (as in [11, Example 8]).

When T_l is copied, keep its respective edge labels. The new edge connecting a copy of T_l to the original vertical linear tree T_w has the same label as the edge (of T_w) above the root of the original copy of T_l.

Example 2.1. *Figure 2 (left) shows the complete trie of subwords for the word* 101110*. Since* $w = \underbrace{1^1}_{u_1}\underbrace{0^1}_{u_2}\underbrace{1^3}_{u_3}\underbrace{0^1}_{u_4}$*, we have* $M = 4$*. The subtree* T_3 *is the sole diamond node on* T_w *because* T_3 *is rooted at the node* $u_1u_2u_30$*. We then attach a copy of* T_3 *to the nodes* $u_1u_21^j, j \in \{0,1,2\}$*. The root of* T_2 *is the node* u_1u_21 *(the square node) and we attach a copy of* T_2 *to the node* $u_10^j, j \in \{0\}$*. Lastly, because* $n_1 = 1$*, no copy of* T_1 *is added. See also Figure 4 (left) and [11, Figs. 3-4].*

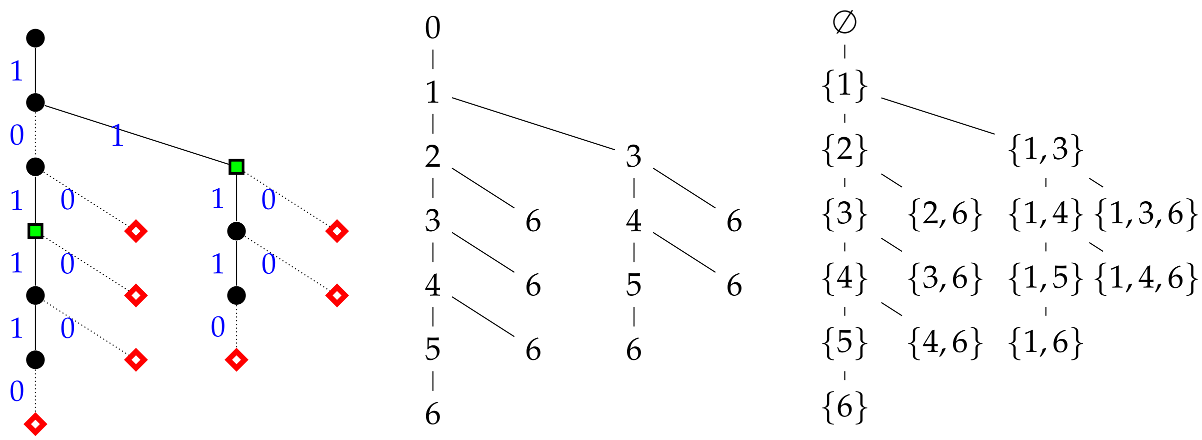

Figure 2: Trie of subwords of 101110 (left); antichain trie corresponding to 101110 (center) and its corresponding antichains (right)

3 Antichain trie

We define an antichain analog of the [11] trie of subwords, call it the *antichain trie*, and assign a distinct antichain to each of its nodes.

Definition 3.1 (Antichain trie)**.** *Given a binary word* w *of length* d*, construct the trie of subwords as above, but remove all edge labels (of* 1*s and* 0*s). We label each node* v *with a non-distinct label* $L(v)$ *as follows. First, we label the* $d+1$ *nodes of the leftmost vertical linear tree* T_w*. Starting from the top left node (*ϵ*) and moving down, label* ϵ *as* 0*, then its descendants as* $1, 2, \ldots, d$*. When we attach copies of* $T_{M-1}, \ldots, T_1$*, we keep these original node labels (so that two different nodes may have the same label* k*). See Figure 2 (center).*

To each node v, we associate the path $\epsilon, v_1, \ldots, v_\ell$ along the vertices from ϵ to v. Let $p(v)$ be the sequence of labels $L(\epsilon) = 0, L(v_1), \ldots, L(v_\ell)$ of these vertices. Note that $p(v)$ is a unique ordered subsequence of $(0, 1, \ldots, d)$. Let $A(\epsilon) = \emptyset$, and for the rest of the nodes v, let

$$A(v) = \Big\{ j \in \{1, 2, \ldots, d\} \mid j \text{ is the largest number in a block of consecutive integers in } p(v) \Big\}.$$

Proposition 3.2. *Let P be the poset corresponding to a binary word w. Then for each node v of the antichain trie of w, the set $A(v)$ is a distinct antichain of P.*

Example 3.3. *In Figure 2 (right), we have labeled every node v with $A(v)$. For example, if v is the node obtained by the path $p(v) = (0, 1, 2, 3, 6)$ of Figure 2 (center), then $A(v) = \{3, 6\}$. For the node v obtained by walking along $p(v) = (0, 1, 3, 4, 6)$, we have the antichain $A(v) = \{1, 4, 6\}$.*

Example 3.4. *Let $w = 10010111$. Figure 3 shows the Hasse diagram of the 8-element poset P corresponding to w. Figure 4 (left) depicts the trie of subwords for w. Write $w = \underbrace{1^1}_{u_1}\underbrace{0^2}_{u_2}\underbrace{1^1}_{u_3}\underbrace{0^1}_{u_4}\underbrace{1^3}_{u_5}$, so $M = 5$. The root of T_4 is $u_1u_2u_3u_4 1$ (the diamond node), the root of T_3 is $u_1u_2u_3 0$ (the square node), and the root of T_2 is $u_1u_2 1$ (the star node). Figure 4 (right) shows the antichain trie for w. Lastly, Figure 5 shows the 32 antichains of P which we assign to the 32 nodes of the antichain trie.*

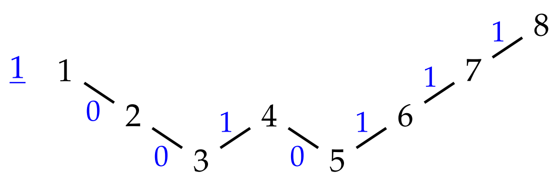

Figure 3: The Hasse diagram of the poset corresponding to the word 10010111.

Remark 3.5. *Given an antichain trie, let a* vertical branch *be a linear subtree with nodes $v_1, \ldots, v_{r+1}$ ($r \geq 1$) so that v_1 is either the root ϵ or is the right child of its parent (in particular, it is not a left child), v_{i+1} is the left child of v_i for $i = 1, \ldots, r$, and v_{r+1} is not a parent. For example, in Figure 2 (center), the original left-most vertical tree and the subtree labeled $\{3, 4, 5, 6\}$ are the only vertical branches while Figure 4 (right) has 7 vertical branches. Note that, by construction of the antichain trie, a vertical move (downward) from a node v to its left child v' removes the label $L(v)$ from $A(v)$ and replaces it with the label $L(v') = L(v) + 1$.*

Similarly, let a horizontal branch *be a linear subtree with nodes $v_1, \ldots, v_{r+1}$ ($r \geq 1$) so that v_1 is the left child of some node (in particular, v_1 cannot be a right child), v_{i+1} is the right child of v_i for all $i = 1, \ldots, r$, and v_{r+1} does not have a right child. For example, in Figure 2 (center), the subtrees labeled $\{1, 3, 6\}$, $\{2, 6\}$, $\{3, 6\}$, and $\{4, 6\}$ are the horizontal branches, with 2 different horizontal branches both labeled by $\{4, 6\}$. The trie in Figure 4 (right) has exactly 4 horizontal*

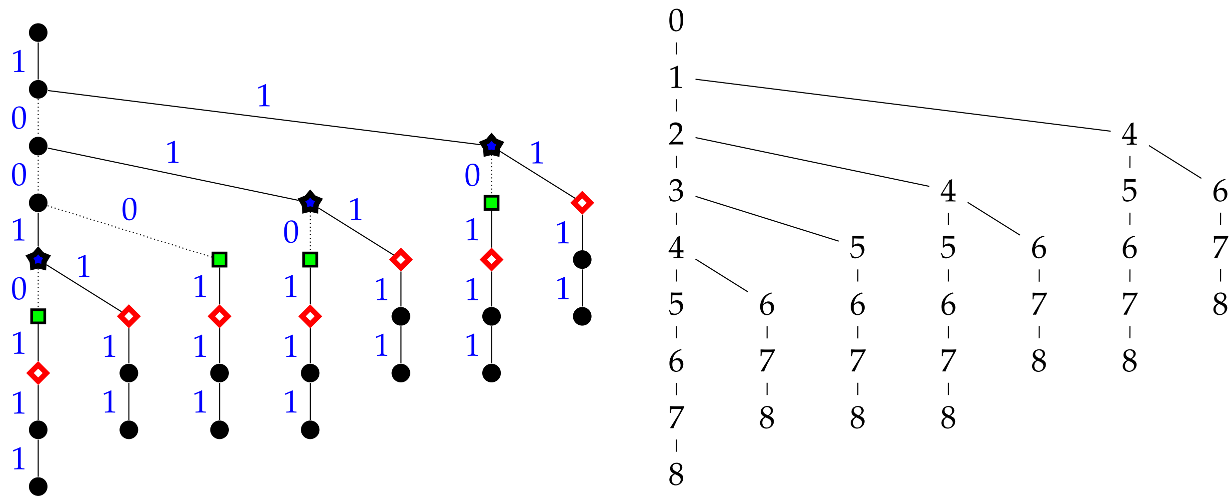

Figure 4: The trie of subwords (left) and the antichain trie (right) of 10010111

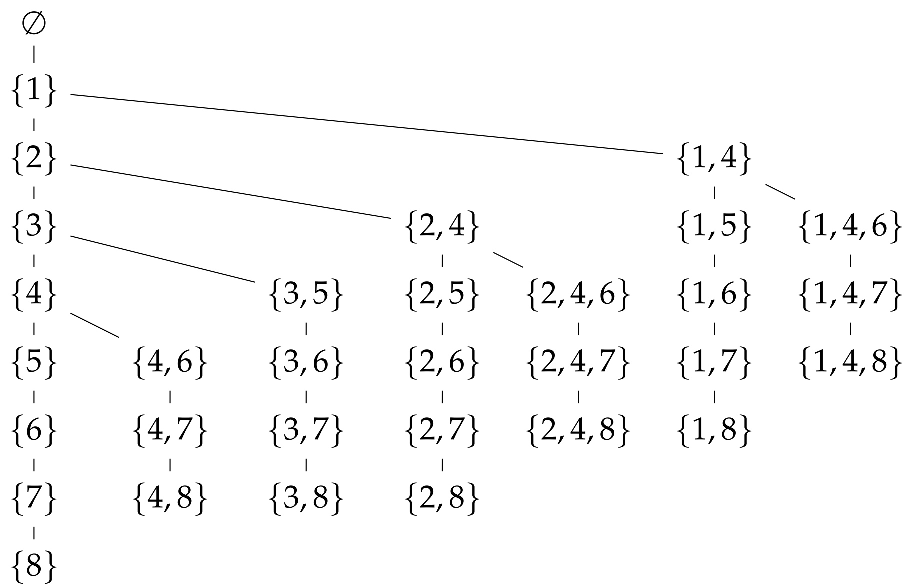

Figure 5: Antichains corresponding to nodes of Figure 4 (right)

branches, labeled by $\{1,4,6\}$, $\{2,4,6\}$, $\{3,5\}$, and $\{4,6\}$. A horizontal move (to the right) from a node v to its right child v' adds to the antichain $A(v)$ the (positive integer) label $L(v')$ of v' which is greater than $L(v)+1$.

4 Bijection between antichains and subwords

Let $w = w_1 \dots w_d$ be a binary word of length d. Let $P = \{P_1 = 1, \dots, P_d = d\}$ be the corresponding piecewise-linear poset whose Hasse diagram H has edges labeled by $w_2, \dots, w_d$. We now define a map f from the antichains in P to the subwords of w.

Definition 4.1. *Let $f(\emptyset)$ be the empty subword. If $A = \{A_1, A_2 \dots, A_r\}$ is a nonempty antichain in P, let $f(A)$ be the subword of w which is constructed as follows: The first letter is 1. The next letters are the (possibly empty) sequence of edge labels of H between P_1 and A_1. If A contains one element, we are done. If A contains more than one element, jump to the first minimal or maximal element M_1 appearing after A_1. Record the labels of edges between M_1 and A_2. Next, jump to the first minimal or maximal element M_2 appearing after A_2. Record the labels of edges between M_2 and A_3. Continue as such until we finish recording the edge labels between M_{r-1} and A_r.*

Example 4.2. *In Figure 6, the antichains A=$\{A_1$=4, A_2=10$\}$ and A=$\{A_1$=1, A_2=3, A_3=7, A_4=9$\}$ are mapped to the subwords s=1$\boxed{011}\boxed{01100}$ and s=1$\boxed{}\boxed{1}\boxed{01}\boxed{0}$ of* $w = 10010111$.

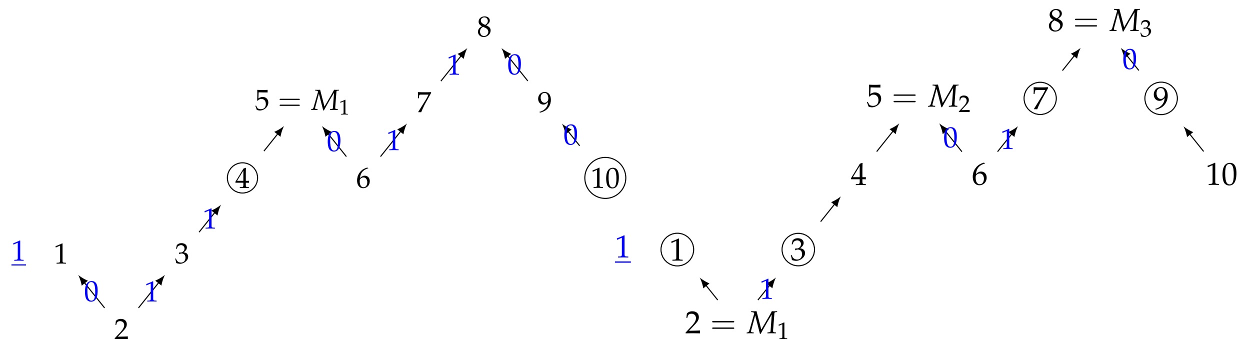

Figure 6: The antichains mapped to s=1$\boxed{011}\boxed{01100}$ (left) and s=1$\boxed{}\boxed{1}\boxed{01}\boxed{0}$ (right) of $w = 10010111$.

Theorem 4.3. *The map f given in Definition 4.1 is a bijection from the antichains in P to the subwords of w.*

Proof. To show that f is surjective, let s be a subword of $w = w_1 w_2 \dots w_d$. If s is nonempty, write $s = w_{i_1} w_{i_2} \dots w_{i_\ell}$ in such a way that each index i_k is as small as possible (see Example 4.4). Note that $w_{i_1} = w_1 = 1$ per our definition of subwords. Partition $w_{i_1}, w_{i_2}, \dots, w_{i_\ell}$ into a set $\Sigma = \Sigma_s$ of (at least one) maximal blocks of subsequences of w such that each subsequence is a consecutive subsequence.

Let
$$A = A_\Sigma = \{j \in P | (w_i, w_{i+1} \dots, w_j) \in \Sigma\}.$$
In other words, $(w_1) \in \Sigma$ if and only if $1 \in A$; if $2 \le n \le d$, then $n \in A$ if and only if the node n in the Hasse diagram H is immediately to the right of a block in Σ.

We claim that A is an antichain. If Σ only contains one block, then A consists of one element, and hence A is an antichain in P. Otherwise, let $(w_i, \dots, w_j)$, where $3 \le i \le d$, be a second block in Σ. If $w_i = 0$, then $w_{i-1} = 1$ since the indices i_k's for the w_{i_k}'s were chosen to be as small as possible. Likewise, if $w_i = 1$, then $w_{i-1} = 0$. This means that the node $i - 1$ (which is not in A) between w_{i-1} and w_i is either a minimal or maximal element of P. Hence no node to the left of $i - 1$ is related to the node j. Similarly, if there is another block $(w_{i'}, \dots, w_{j'})$ of Σ which appears after $(w_i, \dots, w_j)$, the node j is not related to the node j'. This shows that j is not related to any other element in A.

To show that the map is injective, assume $f(A) = f(A')$. Then $f(A) = s = f(A')$ for some subword $s = w_{i_1} \dots w_{i_\ell}$. Let Σ_s be the set of maximal blocks of $w_{i_1}, \dots, w_{i_\ell}$ as defined on the first paragraph of this proof. But both A and A' are defined by the same set Σ_s of maximal blocks, so $A = A'$. □

Example 4.4. *Consider the word $w = w_1 \dots w_{10} = 1011101100$. Identify $w_2, \dots, w_{10}$ with the edges of the Hasse diagram H of P, see Figure 7. We write the subword s=11010 as $s = w_1 w_3 w_6 w_7 w_9$ so that the index of each letter w_{i_k} is as small as possible. Breaking $w_1 w_3 w_6 w_7 w_9$ into maximal blocks of consecutive subsequences of w gives four blocks (w_1), (w_3), (w_6, w_7), and (w_9). We build an antichain as follows. Since (w_1) is a block, we take the left-most node of H, node 1. We take the nodes of H to the right of the other three blocks, nodes 3, 7, and 9. Therefore, the subword 11010 corresponds to the antichain $\{①, ③, ⑦, ⑨\}$.*

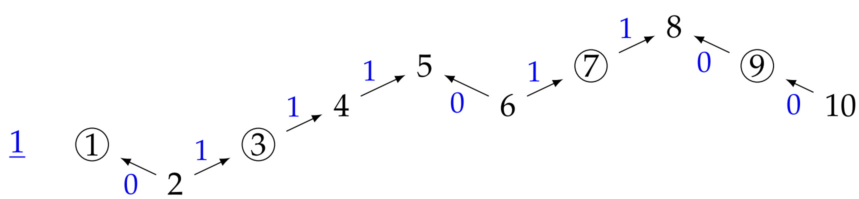

Figure 7: Hasse diagram representing 1011101100; the antichain corresponding to the subword 1 $\boxed{1}$ $\boxed{01}$ $\boxed{0}$ is $\{1, 3, 7, 9\}$.

5 Subwords to snake graph matchings

5.1 Background

We review the theory of snake graphs developed in [15, 12, 13, 3].

Definition 5.1. *A* snake graph *is a nonempty connected sequence of square tiles* $\square$*. To build a snake graph* G *with* d *tiles, start with one tile, then glue a new tile to the north or the east of the previous tile. We refer to the southwest-most tile of* G *as the first tile* G_1 *and the northeast-most tile as the last tile* G_d*. Figure 8 (left) illustrates a snake graph with* 10 *tiles.*

Definition 5.2. *A* matching *of a graph* G *is a subset of non-adjacent edges of* G*. A* perfect matching *of* G *is a matching where every vertex of* G *is adjacent to exactly one edge of the matching, see Figure 8. Define the* minimal matching P_{min} *to be the unique perfect matching of* G *which contains the first south edge and only boundary edges, see Figure 8 (center).*

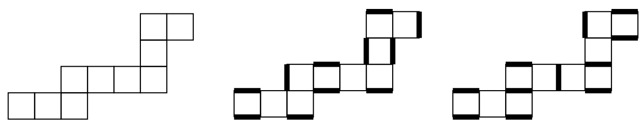

Figure 8: A snake graph (left); the minimal perfect matching (center); another perfect matching of the snake graph (right)

A cluster algebra [6] is a commutative algebra with distinguished generators called *cluster variables* which can be written as Laurent polynomials with positive coefficients. In the case of a family of cluster algebras called *cluster algebras from surfaces*, given such a Laurent polynomial x_γ, it was shown in [12, Thm. 4.17] that x_γ can be associated to a certain snake graph G_γ and that x_γ can be written as a sum over all perfect matchings of G_γ. In particular, the terms of x_γ are in bijection with the perfect matchings of G_γ.

The following allows us to associate a snake graph to a binary word.

Definition 5.3 ([3, Sec. 2.1])**.** *A* sign function *on a snake graph* G *is a map from the set of edges of* G *to* $\{+,-\}$ *such that, for every tile of* G*, the north edge and the west edge have the same sign, the south edge and the east edge have the same sign, and the sign on the north edge is opposite to the sign on the south edge.*

Note that there are exactly two sign functions on every snake graph. We consider only the sign function where the south edge of the first tile has label $-$, see Figure 9 (left). Since we study binary expansions, we replace $+$ with 1 and $-$ and 0, see Figure 9 (center).

Given the sign function of a snake graph G whose west edge of the first tile has sign 1, let the *sign sequence* of G be the sequence $(1, w_2, \ldots, w_d)$ where $w_2, \ldots, w_d$ are the signs of the interior edges of the snake graph, see Figure 9 (right). As this sequence uniquely determines a snake graph, we can associate to each binary word $w = 1w_2 \ldots w_d$ a snake graph $G(w)$.

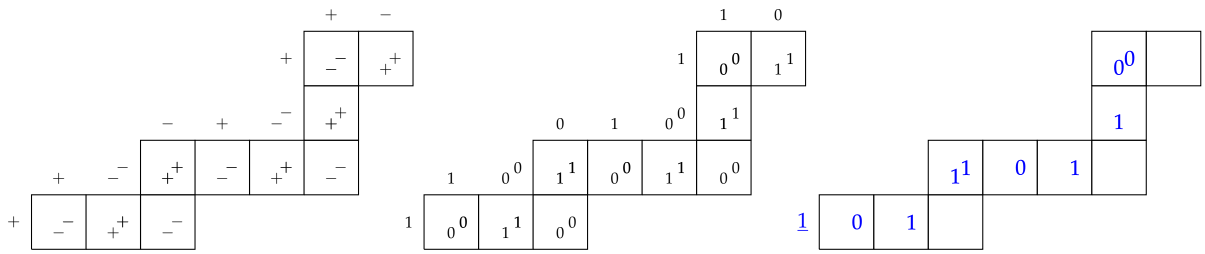

Figure 9: Corresponding sign function (left and center) and sign sequence (right) of the snake graph for the binary expansion 1011101100.

5.2 Bijection from subwords to perfect matchings

An *order filter* is a subset F of P such that if $t \in F$ and $s \geq t$, then $s \in F$. The perfect matchings of a snake graph G is known to form a lattice isomorphic to the lattice of order filters of the piecewise-linear poset corresponding to G [14, Thm. 2], [13, Sec. 5]. It is also known that the map which sends an order filter to its set of minimal elements is a bijection between the order filters and the antichains in a poset. Therefore, by Theorem 4.3, there is a bijection from the subwords of w to the perfect matchings of $G(w)$.

Theorem 5.4. *Given a binary subword w and its corresponding snake graph $G = G(w)$, the following map pm from the subwords of w to the perfect matchings of G is a bijection:*

- *Let s be a subword of w. If s is the empty word, let $pm(s)$ be P_{min}. Otherwise, write $s = w_{i_1}w_{i_2}\ldots w_{i_\ell}$ in such a way that each index i_k is as small as possible (as we do in Section 4) and circle the edges of G corresponding to the sign sequence for s.*
- *For each block L of consecutive circled edges, let $\square_L$ be the tile which is immediately north/east of the last edge in L.*
- *Let $fil(\square_L)$ be the smallest connected sequence of tiles such that $\square_L \in fil(\square_L)$ and the set of edges bounding $fil(\square_L)$ not in P_{min} forms a perfect matching of $fil(\square_L)$.*
- *Let $fil(s) = \bigcup_L fil(\square_L)$, and define $pm(s)$ to be the symmetric difference $\{$edges bounding $fil(s)\} \ominus P_{min}$.*

Example 5.5. *Consider the word $w = 1011101100$. In Figure 10, we circle the edges of G corresponding to $s = 101101100$. The corresponding 2 blocks of shaded tiles are $\left\{\boxed{d}, \boxed{e}\right\}$ and $\left\{\boxed{h}, \boxed{i}, \boxed{j}\right\}$, and $pm(s)$ is the set of thick edges. In Figure 11, we circle the edges of G corre-*

sponding to $s = 11010$*. Note that* $fil\left(\boxed{g}\right) = \left\{\boxed{g}, \boxed{h}, \boxed{i}\right\} = fil\left(\boxed{i}\right)$*. The 3 blocks of shaded tiles are* $\{\boxed{a}\}$, $\left\{\boxed{c}, \boxed{d}, \boxed{e}\right\}$*, and* $\left\{\boxed{g}, \boxed{h}, \boxed{i}\right\}$ *so that* $fil(11010) = \left\{\boxed{a}, \boxed{c}, \boxed{d}, \boxed{e}, \boxed{g}, \boxed{h}, \boxed{i}\right\}$.

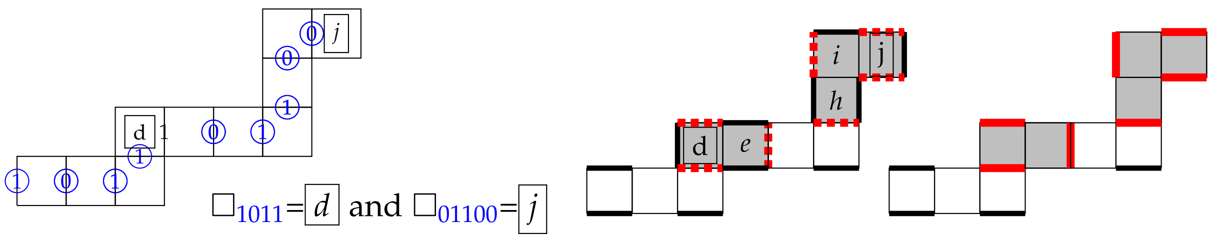

Figure 10: Tiles $\square_L$ associated to blocks L of circled edges (left); the set $fil(s)$ of shaded tiles and the set $pm(s)$ of thick solid edges (right) for the subword s=$\boxed{1011}\boxed{01100}$.

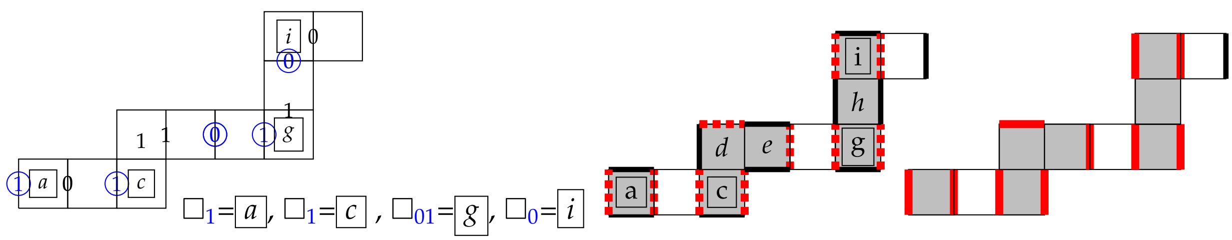

Figure 11: Tiles $\square_L$ associated to blocks L of circled edges (left); the set $fil(s)$ of shaded tiles and the set $pm(s)$ of thick solid edges (right) for the subword s=$\boxed{1}\boxed{1}\boxed{01}\boxed{0}$.

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