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# Restricting Schubert classes to symplectic Grassmannians using self-dual puzzles

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**Abstract.** Given a Schubert class on $Gr(k, V)$ where $V$ is a symplectic vector space of dimension $2n$, we consider its restriction to the symplectic Grassmannian $SpGr(k, V)$ of isotropic subspaces. Pragacz gave tableau formulæ for positively computing the expansion of these $H^*(Gr(k, V))$ classes into Schubert classes of the target when $k = n$, which corresponds to expanding Schur polynomials into $Q$-Schur polynomials. Coşkun described an algorithm for their expansion when $k \le n$. We give a puzzle-based formula for these expansions, while extending them to equivariant cohomology. We make use of a new observation that usual Grassmannian puzzle pieces are already enough to do some 2-step Schubert calculus, and apply techniques from quantum integrable systems ("scattering diagrams").

**Keywords:** Schubert calculus, puzzles, Grassmannian, symplectic Grassmannian

# 1 Introduction

## 1.1 Grassmannian duality of puzzles

The Littlewood–Richardson coefficients $c^{\nu}_{\lambda\mu}$, where $\lambda, \mu, \nu$ are (for now) partitions, satisfy a number of symmetries, one of which is $c^{\nu}_{\lambda\mu} = c^{\nu^T}_{\mu^T\lambda^T}$. One origin of L-R coefficients is as structure constants in the product in $H^*(Gr(k, V))$ of Schubert classes on the Grassmannian of $k$-planes in $V$. In that formulation, the **Grassmannian duality** homeomorphism $Gr(k, V) \cong Gr((\dim V) - k, V^*)$, $(U \le V) \mapsto (U^\perp \le V^*)$ induces an isomorphism of cohomology rings and a correspondence of Schubert bases, giving the symmetry above. This symmetry is not at all manifest in tableau-based computations of the $\{c^{\nu}_{\lambda\mu}\}$, but it is in the "puzzle" rule of [6], which replaces partitions by binary strings and is based on

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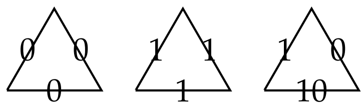
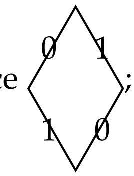
the puzzle pieces




, their rotations, and the equivariant piece




;

we recall and generalize this rule in Theorem 2 below. Specifically, the **dual** of a puzzle is made by flipping it left-right while exchanging all $0 \leftrightarrow 1$ (in particular, 10-labels again become 10s). The duals of the puzzles counted by $c^{\nu}_{\lambda\mu}$ are exactly those counted by $c^{\nu^T}_{\mu^T\lambda^T}$.

This prompts the question: what do *self-dual* puzzles count? One might expect it is something related to an isomorphism $V \cong V^*$ i.e. a bilinear form, and indeed our main theorems 1A, 1B, and 1C interpret self-dual puzzles as computing the restrictions of Schubert classes on $Gr(k, 2n)$ to the *symplectic Grassmannian* $SpGr(k, 2n)$. (We will address elsewhere the minimal modifications necessary to handle the orthogonal case.) For $k = n$, there was already a tableau-based formula for these restrictions[1] in [8] which is less simple to state than Theorem 1A; see also [4]. This is perhaps another effect of tableaux being less suited to Grassmannian duality than puzzles are.

## 1.2 Restriction from $Gr(n, 2n)$

Let $V$ be a vector space over $\mathbb{C}$ equipped with a symplectic form, so the Grassmannian $Gr(k, V)$ of $k$-planes contains the subscheme

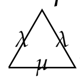
$$SpGr(k, V) := \{L \leq V \ : \ \dim L = k, \ L \leq L^{\perp}\}$$

where $\perp$ means perpendicular with respect to the symplectic form. Then the inclusion $\iota : \ SpGr(k, V) \hookrightarrow Gr(k, V)$ induces a pullback $\iota^* : \ H^*(Gr(k, V)) \to H^*(SpGr(k, V))$ in cohomology. As both cohomology rings possess bases consisting of *Schubert classes* $\{S_\lambda\}$, one can ask about expanding $\iota^*(S_\lambda)$ in the basis of $SpGr(k, V)$'s Schubert classes $\{S_\nu\}$.

Let $\dim V = 2n$ (necessarily even, since $V$ is symplectic), and for the simplest version of the theorem assume $k = n$. Then the Schubert classes on $Gr(n, V)$ are indexed by the $\binom{2n}{n}$ binary strings with $n$ 0s and $n$ 1s, whereas the Schubert classes on $SpGr(n, V)$ are indexed by the $2^n$ binary strings of length $n$ (with more detail on this indexing in Section 2).

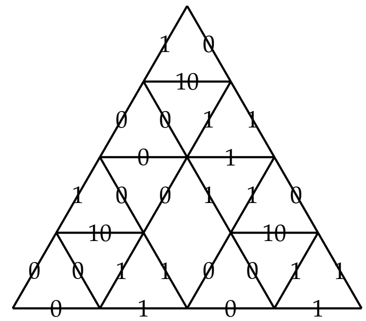
**Theorem 1A.** *Let $S_\lambda$ be a Schubert class on $Gr(n, 2n)$, indexed by a string $\lambda$ with content in $0^n 1^n$, and $S_\nu$ a Schubert class on $SpGr(n, 2n)$, indexed by a length $n$ binary string. Then the coefficient of $S_\nu$ in $\iota^*(S_\lambda)$ is the number of self-dual puzzles with $\lambda$ on the Northwest side, $\nu$ on the left half of the South side (both $\lambda$ and $\nu$ read left to right), and equivariant pieces only allowed along the axis of reflection.*

[1] In particular, [8] provides a cohomological interpretation of algebraic results of Stembridge [10] about expanding Schur functions into Schur $P$- and $Q$-functions.

**Example 1.** *For $\lambda = 0101$, a self-dual puzzle with $\lambda$ on the Northwest side has to be of the form* 

*for some $\mu$.*

*So, it will appear in the usual calculation of $S_{0101}^2 \in H_T^*(Gr(2,4))$, which involves three puzzles. Only one of these puzzles is self-dual, and its only equivariant piece is on the centerline. From this we compute $\iota^*(S_{0101}) = S_{01}$ in $H^*(SpGr(2,4))$.*



A surprising aspect of Theorem 1A is that equivariant pieces appear in this nonequivariant calculation, albeit only down the centerline.[2] If we allow them elsewhere (self-dually occurring in pairs), the puzzles compute the generalization of Theorem 1A to the map $\iota^* : H_T^*(Gr(n,2n)) \to H_T^*(SpGr(n,2n))$ in *(torus-)equivariant* cohomology, whose coefficients now live in the polynomial ring $H_T^*(pt) \cong \mathbb{Z}[y_1,\ldots,y_n]$. We leave this statement until Theorem 1C in Section 5 because it requires some precision about the locations of the symplectic Schubert varieties.

## 1.3 Interlude: puzzles with 10s on the South side

To generalize Theorem 1A to $SpGr(k,2n)$, not just $k = n$, we need strings that index its $\binom{n}{k}2^k$ many Schubert classes. We do this using the third edge label, 10: consider strings $\nu$ of length $n$ with $(n-k)$ 10s, the rest a mix of 1s and 0s.

Before considering *self-dual* puzzles with Southside 10s, we mention a heretofore unobserved capacity of the puzzle pieces from [6], available once we allow for Southside 10s. It turns out they are already sufficient to compute certain products[3] in the $T$-equivariant cohomology of 2-step flag manifolds! The only necessary new idea is to allow the previously internal label 10 to appear on the South side.

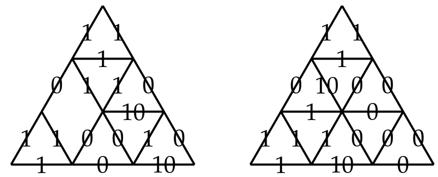
**Theorem 2.** *Let $0 \le j \le k \le n$, and let $\lambda$, $\mu$ be $0,1$-strings with content $0^j1^{n-j}$, $0^k1^{n-k}$ respectively, defining equivariant Schubert classes $S_\lambda, S_\mu$ on $Gr(j,\mathbb{C}^n)$, $Gr(k,\mathbb{C}^n)$ respectively. Let $\pi_j, \pi_k$ be the respective projections of the 2-step flag manifold $Fl(j,k;\ \mathbb{C}^n)$ to those Grassmannians. Let $\nu$ be a string in the ordered alphabet $0, 10, 1$ with content $0^j(10)^{k-j}1^{n-k}$, defining a Schubert class $S_\nu$ in $H_T^*(Fl(j,k;\ \mathbb{C}^n))$. We emphasize that the alphabet order is 0, 10, 1!*

*Then as in [6], the coefficient of $S_\nu$ in the product $\pi_i^*(S_\lambda)\pi_j^*(S_\mu) \in H_T^*(Fl(j,k;\ \mathbb{C}^n))$ is the sum over puzzles $P$ with boundary labels $\lambda$, $\mu$, $\nu$, made from the puzzle pieces in Section 1.1, of the "fugacities" $fug(P) := \prod_{\text{equivariant pieces } \diamondsuit \text{ in } P} (y_{\text{NE–SW diagonal of } \diamondsuit} - y_{\text{NW–SE diagonal of } \diamondsuit})$.*

[2]The number of equivariant pieces down the centerline is in fact fixed and equal to the number of 1s in $\nu$, by a weight conservation argument.

[3]Note that the general 2-step problem has received a puzzle formula [3], but using many more puzzle pieces than we use here. The problem of multiplying classes from different Grassmannians was studied already in [9].

**Example 3.** *If $\lambda = 101$, $\mu = 100$, then their pullbacks give $\pi_1^*(S_{101}) = S_{10,0,1}$, $\pi_2^*(S_{100}) = S_{1,0,10}$, with product $(y_1 - y_2)S_{1,0,10} + S_{1,10,0}$ (note: to compare strings to permutations requires inversion, as in Section 4).*

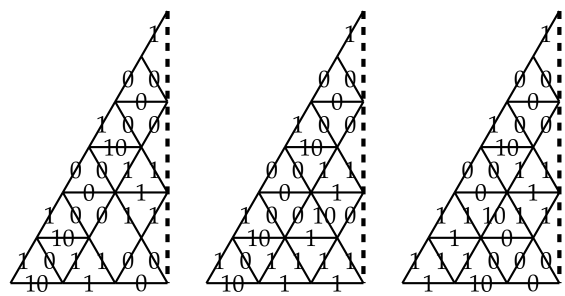


## 1.4 Restriction from $Gr(k, 2n)$, $k < n$

**Theorem 1B.** *Let $\lambda$ be a string with content $0^k 1^{2n-k}$, whereas $\nu$ is of length $n$ with $(n-k)$ 10s, the rest a mix of 1s and 0s. Consider the puzzles from Theorem 2, where we allow 10 labels to appear on the South side.*

*Then as before, in $H^*(SpGr(k, 2n))$, the coefficient of $S_\nu$ in $\iota^*(S_\lambda)$ is the number of self-dual puzzles with $\lambda$ on the Northwest side, $\nu$ on the left half of the South side (both $\lambda$ and $\nu$ read left to right), and equivariant pieces only allowed along the axis of reflection.*

**Example 4.** *In the remainder of the paper we work with the left halves ◺ of self-dual puzzles, since the centerline and right half can be inferred. The half-puzzles pictured here (really for equivariant Theorem 1C to come) show $\iota^*(S_{110101}) = (y_2 - y_3)S_{10,1,0} + S_{10,1,1} + S_{1,10,0}$.*



The proof is based on the "quantum integrability" of $R$-matrices, and closely follows that of [7] (see also [11]); in particular, following the quantum integrable literature, we use graph-dual pictures (scattering diagrams) which are more amenable than puzzles to topological manipulations. The principal new feature is the appearance of $K$-matrices. The "reflection equation" $RKRK = KRKR$ (more precisely, eq. (3.4) in Lemma 7) is standard; however since the approach of [7] requires not just $R$-matrices but the trivalent $U$-matrix, we need here the possibly novel "$K$-fusion equation" (3.5) in Lemma 7.

# 2 The groups, flag manifolds, and cohomology rings

We take the Gram matrix of our symplectic form to be antidiagonal; this is so that if $B_\pm$ are the upper/lower triangular Borel subgroups of $GL_{2n}$, then $B_\pm \cap Sp_{2n}$ will be opposed Borel subgroups of $Sp_{2n}$.

Consider $Gr(k, 2n) = \{0 \leq V \leq \mathbb{C}^{2n} \mid \dim V = k\} \cong GL_{2n}/P$ where $k \leq n$ and $P$ is the parabolic subgroup of block type $(k, 2n-k)$ containing $B = B_+$. Then $P_{Sp_{2n}} = P \cap Sp_{2n}$ is a parabolic for the Lie subgroup $Sp_{2n}$ and the symplectic Grassmannian is $SpGr(k, 2n) = \{0 \leq V < \mathbb{C}^{2n} \mid \dim V = k, V \leq V^\perp\} \cong Sp_{2n}/P_{Sp_{2n}}$. Let $T^{2n} := B_+ \cap B_-$ be the diagonal matrices in $GL_{2n}$, and $T^n := Sp_{2n} \cap T^{2n}$. Note that

$(GL_{2n}/P)^{T^n} = (GL_{2n}/P)^{T^{2n}}$ since there exist $x \in T^n$ with no repeated eigenvalues. The following diagram of spaces commutes.

$$\begin{array}{ccccc} \{\nu \ : \ \exists\, a \text{ s.t. content}(\nu) = 0^a(10)^{n-k}1^{k-a}\} & \xrightarrow{\sim} & (SpGr(k,2n))^{T^n} & \hookrightarrow & SpGr(k,2n) \\ \big\downarrow \tilde{\iota} & & \big\downarrow & & \big\downarrow \iota \\ \{\lambda \ : \ \text{content}(\lambda) = 0^k 1^{2n-k}\} & \xrightarrow{coord} & (Gr(k,2n))^{T^n} & \hookrightarrow & Gr(k,2n) \end{array}$$

The map $\tilde{\iota}$ takes a sequence $\nu$ first to its double $\nu\overline{\nu}$ where $\overline{\nu}$ is $\nu$ reflected and its 0s and 1s are switched; after that, all 10s in $\nu\overline{\nu}$ are turned into 1s, e.g.

$$0,10,1,0,10 \quad\mapsto\quad 0,10,1,0,10,\ 10,1,0,10,1 \quad\mapsto\quad 0,1,1,0,1,\ 1,1,0,1,1$$

The bijective map *coord* takes a $0,1$-sequence $\lambda$ to the coordinate $k$-plane that uses the coordinates in the 0 positions of $\lambda$ (so, $1,4,8$ in the above example). Note that $coord \circ \tilde{\iota}(\nu) \in SpGr(k,2n)$ by the antidiagonality we required of the Gram matrix.

The right-hand square, and the inclusion $T^n \hookrightarrow T^{2n}$, induce the ring homomorphisms

$$\begin{array}{ccccc} H^*_{T^{2n}}(Gr(k,2n)) & \xrightarrow{f_1} & H^*_{T^n}(Gr(k,2n)) & \xrightarrow{f_2 = \iota^*} & H^*_{T^n}(SpGr(k,2n)) \\ g_1 \big\downarrow & & \big\downarrow g_2 & & \big\downarrow g_3 \\ H^*_{T^{2n}}(Gr(k,2n)^{T^{2n}}) & \xrightarrow{h_1} & H^*_{T^n}(Gr(k,2n)^{T^n}) & \xrightarrow{h_2} & H^*_{T^n}(SpGr(k,2n)^{T^n}) \end{array}$$

and since each $g_i$ is injective (see e.g. [5]), we can compute along the bottom row, which is the proof technique used in [7] and Section 5. On each of our flag manifolds, we define our Schubert classes as associated to the closures of orbits of $B_-$ or $B_- \cap Sp_{2n}$.

## 3 Scattering diagrams and their tensor calculus

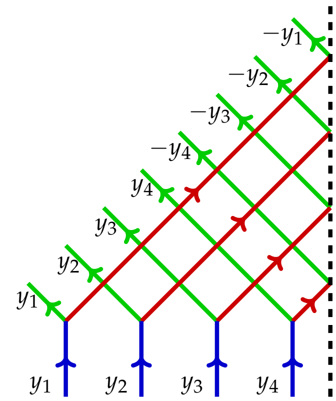
In the statement and proof of Theorem 1B, we work with *half-puzzles*, i.e., labeled half-triangles $2n\measuredangle$ of size $2n$, tiled with the triangle and rhombus puzzle pieces described in Section 1.1, as well as half-rhombus puzzle pieces obtained by cutting the existing self-dual ones vertically in half. As discussed earlier, a half-puzzle can be considered as half of a self-dual puzzle with all three sides of length $2n$. In our notation, a "rhombus" can also be made of a $\Delta$ and a $\nabla$ triangle glued together.

To linearize the puzzle pictures and relate them back to the restriction of cohomology classes, we consider the puzzle labels $\{0,10,1\}$ as indexing bases for three spaces $\mathbb{C}^3_G, \mathbb{C}^3_R, \mathbb{C}^3_B$ (Green, Red, Blue). In our scattering diagrams below, each coloured edge will carry its corresponding vector space.

1. Take an **unlabeled** size $2n$ half-puzzle triangle $2n\measuredangle$ tiled by rhombi, half-rhombi (on the East) and triangles (on the South) as before, with assigned "spectral parameters" $y_1, \ldots, y_n, -y_n, \ldots, -y_1$ on the Northwest side.

2. Consider the dual-graph picture of strands, oriented upwards. Each rhombus corresponds to a crossing of two strands, each half-rhombus to a bounce off the East wall and negates the spectral parameter, and each triangle to a trivalent vertex with all parameters equal.

   We also colour the Northwest-pointing strands green, Northeast-pointing red, and North-pointing blue.




3. We let $a$ and $b$ denote two spectral parameters from Step 1. We assign the following linear maps

   - to each crossing of two strands with left and right parameters $a$ and $b$, and colours $C$ and $D$, a linear map $R_{CD}(a-b) : \mathbb{C}^3_C \otimes \mathbb{C}^3_D \longrightarrow \mathbb{C}^3_D \otimes \mathbb{C}^3_C$;
   - to each wall-bounce of a colour $C$ strand with parameter $a$ bouncing to $-a$, a map $K_C(a) : \mathbb{C}^3 \to \mathbb{C}^3$;
   - to each trivalent vertex with incoming blue strand and outgoing green and red strands, all with parameters $a$, a map $U(a) : \mathbb{C}^3_B \longrightarrow \mathbb{C}^3_G \otimes \mathbb{C}^3_R$.

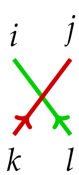
Connecting two strands corresponds to composing the corresponding maps, so the whole $2n\measuredangle$ corresponds to a linear map $\Phi : (\mathbb{C}^3_B)^{\otimes n} \longrightarrow (\mathbb{C}^3_G)^{\otimes 2n}$.

**Definition 5** (The $R$-, $U$-, and $K$-matrices)**.** *In terms of the bases of $\mathbb{C}^3_G, \mathbb{C}^3_R, \mathbb{C}^3_B$ indexed by $\{0, 10, 1\}$, the above sparse matrices can be written compactly as follows (where a labeled diagram corresponds to the coefficient of the map in those basis elements):*

$R_{CC}(a-b) :$ 

$= \begin{cases} 1 & \text{if } (i,j) = (k,l), \\ b-a & \text{if } (i,j,k,l) \in \{(1,0,0,1), (10,0,0,10), (1,10,10,1)\} \end{cases}$

*where $C \in \{R, G, B\}$ and the two strands are any identical colour*

$R_{RG}(a-b) :$ 

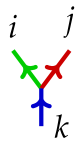
$= \begin{cases} a-b & \text{if } (i,j,k,l) = (0,1,1,0), \\ 1 & \text{if } (i,j,k,l) \in \left\{ \begin{array}{c} 0^4, 1^4, 0^2 1 (10), 0(10)1^2, 0(10)^2 0, \\ 1\,0^2 1, 1^2(10)0, (10)1\,0^2, (10)1^2(10) \end{array} \right\} \end{cases}$

$K_R(a)$ : 

$= 1$ *if* $(i,j) \in \{(1,0),(0,1)\}$

$K_B(a)$ : 

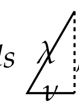
$= \begin{cases} 1 & \text{if } i = j, \\ -2a & \text{if } (i,j) = (1,0) \end{cases}$

$U(a)$ : 

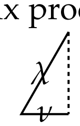
$= 1$ *if* $(i,j,k) \in \{(0,0,0),(0,10,1),(1,0,10),(1,1,1),(10,1,0)\}$

*The subscripts $R, G, B$ on the maps indicate the colours of the incoming edges (listed counter-clockwise). For each map, the matrix entries which are not listed are zero.*

Note that if we take the corresponding bases with lexicographic ordering, with alphabet ordered as $\{0, 10, 1\}$, then the matrices for $R_{CC}$ and $K_B$ are lower-triangular. See [11] for these $R$-matrices and [7, §3] for their representation-theoretic origins.

**Definition 6.** *With the above notation, let* $\mathbf{P}$ *be a half-puzzle with boundary labels* 

*, where $\lambda \in 0^k 1^{2n-k}$ and $\nu \in (10)^{n-k}\{0,1\}^k$. The fugacity $fug(\mathbf{P})$ of $\mathbf{P}$ is the product over all puzzle pieces (dually: vertices) of the entries of the corresponding R-, U-, K-matrices.*

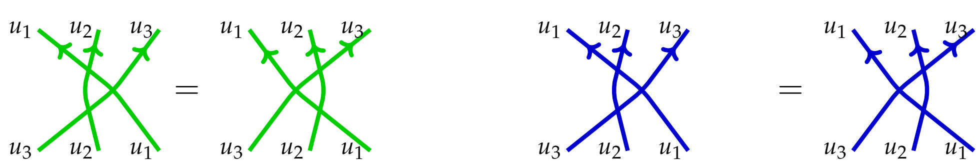
In this way, the summation over half-puzzles reproduces the full matrix product, i.e., the $(\lambda,\nu)$ matrix entry of $\Phi = \sum_{\mathbf{P}} \left\{ fug(\mathbf{P}) \mid \mathbf{P} \text{ is a puzzle with boundary} \right.$ 

$\left. \right\}$.

**Lemma 7.** *The matrices defined in Section 3 satisfy the following identities:*

*i) The Yang–Baxter equation.*

(3.1)

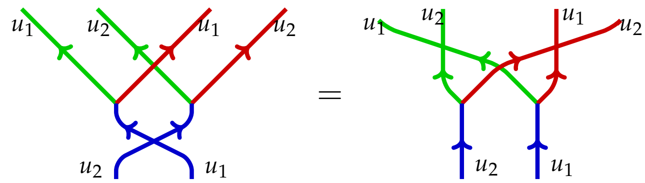


(3.2)

*For example, the linear map form of the Northwest equation is*

$$\begin{aligned} &(R_{RG}(u_2 - u_1) \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes R_{RG}(u_3 - u_1)) \circ (R_{RR}(u_3 - u_2) \otimes \mathrm{Id}) \\ &= (\mathrm{Id} \otimes R_{RR}(u_3 - u_2)) \circ (R_{RG}(u_3 - u_1) \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes R_{RG}(u_2 - u_1)) \end{aligned}$$

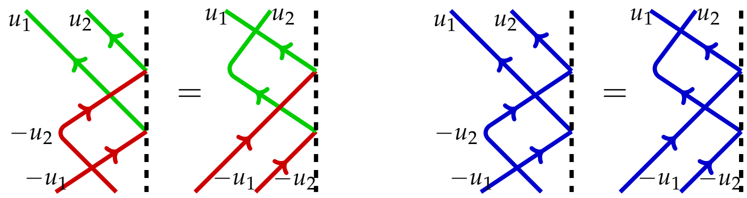
*ii) Swapping of two trivalent vertices.*



(3.3)

$$(\mathrm{Id} \otimes R_{RG}(u_1 - u_2) \otimes \mathrm{Id}) \circ (U(u_1) \otimes U(u_2)) \circ R_{BB}(u_2 - u_1)$$
$$= (R_{GG}(u_2 - u_1) \otimes R_{RR}(u_2 - u_1)) \circ (\mathrm{Id} \otimes R_{RG}(u_2 - u_1) \otimes \mathrm{Id}) \circ (U(u_2) \otimes U(u_1))$$

*iii) The reflection equation.*

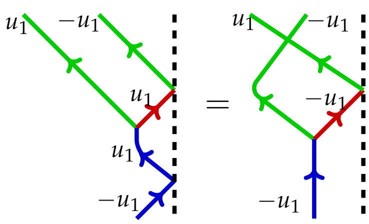


(3.4)

*In linear map terms, the left equation says*

$$(\mathrm{Id} \otimes K_R(-u_2)) \circ R_{RG}(-u_2 - u_1) \circ (\mathrm{Id} \otimes K_R(-u_1)) \circ R_{RR}(-u_1 + u_2)$$
$$= R_{GG}(u_2 - u_1) \circ (\mathrm{Id} \otimes K_R(-u_1)) \circ R_{RG}(-u_1 - u_2) \circ (\mathrm{Id} \otimes K_R(-u_2))$$

*iv) K-fusion.*



$$(\mathrm{Id} \otimes K_R(u_1)) \circ U(u_1) \circ K_B(-u_1) = R_{GG}(-2u_1) \circ (\mathrm{Id} \otimes K_R(-u_1)) \circ U(-u_1) \qquad (3.5)$$

## 4 AJS/Billey formulæ as scattering diagrams

We first discuss the general AJS/Billey formula for restricting an equivariant Schubert class to a torus-fixed point, and then consider the special cases of types $A$ and $C$. Let $G$ be an algebraic group and fix a pinning $G \geq B \geq T$, with $W_G = N_G(T)/T$. Let $B_-$ denote the opposite Borel and $P \geq B$ a parabolic, with Weyl group $W_P$. We recall that Schubert classes are indexed by $W_G/W_P$, which we identify with strings (or signed strings in type $C$), on which $W_G$ acts by permuting/negating positions: $\pi W_P \mapsto \omega \circ \pi^{-1}$ (see Proposition 9 for $\omega$). In particular for $P = B$ our indexing is inverse to the usual convention; this inversion is forced on us by the necessary use for general $P$ of strings-with-repeats, e.g. binary strings rather than Grassmannian permutations.

**Proposition 8.** *1. ([1, 2]) For the Schubert class $S_\pi := \left[\overline{B_-\pi B}/B\right] \in H_T^*(G/B)$, $\pi, \sigma \in W_G$, and $Q = (q_1, \ldots, q_k)$ a reduced word in simple reflections with $\prod Q = \sigma$, the AJS/Billey formula tells us that*

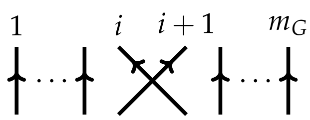
$$S_\pi|_\sigma = \sum_{\substack{R \subseteq Q \\ \prod R = \pi}} \prod_{i=1}^{k} (\widehat{\alpha_{q_i}}^{[q_i \in R]} q_i) \cdot 1 = \sum_{\substack{R \subseteq Q \\ \prod R = \pi}} \prod_{i \in R} \beta_i \;\in\; H_T^*(pt) \tag{4.1}$$

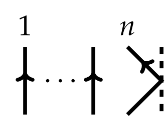
*where $\beta_i := q_1 q_2 \ldots q_{i-1} \cdot \alpha_{q_i}$ and the summation is over reduced subwords $R$ of $Q$.*

*2. To compute a point restriction $S_\lambda|_\mu$ on $G/P$, where $\lambda, \mu \in W_G/W_P$, we use lifts $\widetilde{\lambda}, \widetilde{\mu} \in W_G$ such that $\widetilde{\lambda}$ is the shortest length representative of $\lambda$, and observe that $S_\lambda|_\mu = S_{\widetilde{\lambda}}|_{\widetilde{\mu}}$.*

Below we give a diagrammatic description of the formula from Proposition 8 in the cases when $(G, W_G)$ is $(GL_{2n}, S_{2n})$ or $(Sp_{2n}, S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n)$ using the tensor calculus setting of Section 3. We first introduce some notation.

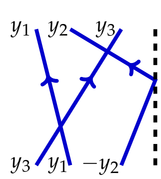
Consider $\sigma$ in $W_G$ (generated by simple reflections $\{s_i^G\}$) and a reduced word for $\sigma$, $Q_\sigma = (q_1, \ldots, q_k)$ (where $q_i = s_{p_i}$ for some $p_i$). We can associate to it a *wiring diagram* $D(Q_\sigma)$ by assigning the diagrams below to the simple reflections $\{s_i^G : 1 \le i \le m_G - 1\}$ for $(m_{GL_{2n}}, m_{Sp_{2n}}) = (2n, n)$, and to $s_n^{Sp_{2n}}$ respectively. Then, a word in simple reflections corresponds to a concatenation of such diagrams.

$s_i^G \mapsto$ 

, for $1 \le i \le m_G - 1$ and $s_n^{Sp_{2n}} \mapsto$ 


Each wire in a wiring diagram is also assigned a spectral parameter. For $G = GL_{2n}$, they are $y_1, \ldots, y_{2n}$ along the top (which we need to later specialize to $y_1, \ldots, y_n, -y_n, \ldots, -y_1$ as in the maps $f_1, h_1$ in Section 2), and for $G = Sp_{2n}$ they are $y_1, \ldots, y_n$.

In the context of the tensor calculus from Section 3, the wiring diagram $D(Q_\sigma)$ can be interpreted as a scattering diagram, i.e., giving a map $(\mathbb{C}_C^3)^{\otimes m_G} \to (\mathbb{C}_C^3)^{\otimes m_G}$; we replace each crossing with $R_{GG}$ in the $GL_{2n}$ (and $C = G$) case or with $R_{BB}$ in the $Sp_{2n}$ (and $C = B$) case, and replace each bounce with $K_B$ which also negates the spectral parameter. For instance, take $G = Sp_6$ and $\sigma = 31\bar{2}$, $Q_\sigma = (s_2, s_3, s_1)$, then

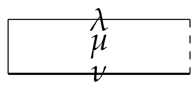
$D(Q_\sigma) =$ 


$$(\mathrm{Id} \otimes R_{BB}(y_3 - y_2)) \circ (Id^{\otimes 2} \otimes K_B(-y_2)) \circ (R_{BB}(y_3 - y_1) \otimes \mathrm{Id}) : (\mathbb{C}_B^3)^{\otimes 3} \to (\mathbb{C}_B^3)^{\otimes 3}$$

**Proposition 9.** *Let $\lambda, \mu$ be strings in $0, 10, 1$ as in Section 2, which we identify with cosets $W_G/W_P$ where $W_G$ is of type $C$ and $P$ is maximal, or of type $A$ and $P$ is maximal or submaximal.*

*Let $\omega_{Gr} = 0\ldots0\ 1\ldots1 \in 0^k1^{2n-k}$ for $G/P = Gr(k, \mathbb{C}^{2n})$, $\omega_{SpGr} = 0\ldots0\ 10\ldots10 \in 0^k(10)^{n-k}$ for $G/P = SpGr(k, \mathbb{C}^{2n})$, or $\omega_{Fl} = 0\ldots0\ 10\ldots10\ 1\ldots1 \in 0^j(10)^{k-j}1^{2n-k}$ for $G/P = Fl(j, k; \mathbb{C}^{2n})$. Make a wiring diagram as just explained, using a reduced word for the shortest lift $\tilde{\mu}$; interpret it as a scattering diagram map, using the $R_{BB}(= R_{GG})$ matrix for crossings and (in type C) $K_B$ for bounces. Then $S_\lambda|_\mu$ is the $(\lambda, \omega_{G/P})$ matrix entry of the resulting product.*

The essentially routine rewriting of Proposition 8 to give Proposition 9 will appear elsewhere. The principal thing one checks is that $R_{BB}$ is the correct $R$-matrix for three labels $\{0, 10, 1\}$. In view of Proposition 9, for $\lambda, \mu, \nu \in W_G/W_P$ as above, we denote



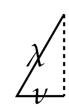
$:=$ the $(\lambda, \nu)$ matrix entry for the scattering diagram map coming from a reduced word for $\tilde{\mu}$.

By the proposition, when $\nu = \omega_{G/P}$ this gives $S_\lambda|_\mu$.

# 5 Proof of Theorem 1B

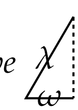
The proof of Theorem 2 is very much as in [7, §3] and will appear elsewhere. Theorem 1A is the $k = n$ special case of Theorem 1B. In fact, we give a more general puzzle rule for equivariant cohomology in Theorem 1C, which in particular implies Theorem 1B.

**Theorem 1C.** *For every $S_\lambda \in H^*_{T^n}(Gr(k, 2n))$, where $\lambda \in 0^k1^{2n-k}$, and $\iota^*$ as in Section 2*

$$\iota^*(S_\lambda) = \sum_{\nu \in (10)^{n-k}\{0,1\}^k} \left( \sum_{\mathbf{P}} \left\{ fug(\mathbf{P}) \mid \mathbf{P} \text{ is a puzzle with boundary} \right. \right.$$


$$\left.\left. \right\} \right) S_\nu$$

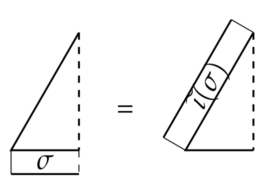
As explained in Section 2, it suffices to check Theorem 1C's equality at each $T^n$-fixed point $\sigma \in (10)^{n-k}\{0,1\}^k$ of $SpGr(k, 2n)$. To do so, we first prove several preliminary results in the language of Section 3.

**Lemma 10.** *For $\omega = \omega_{SpGr}$ as in Proposition 9 and $\lambda \in 0^k1^{2n-k}$, we have*


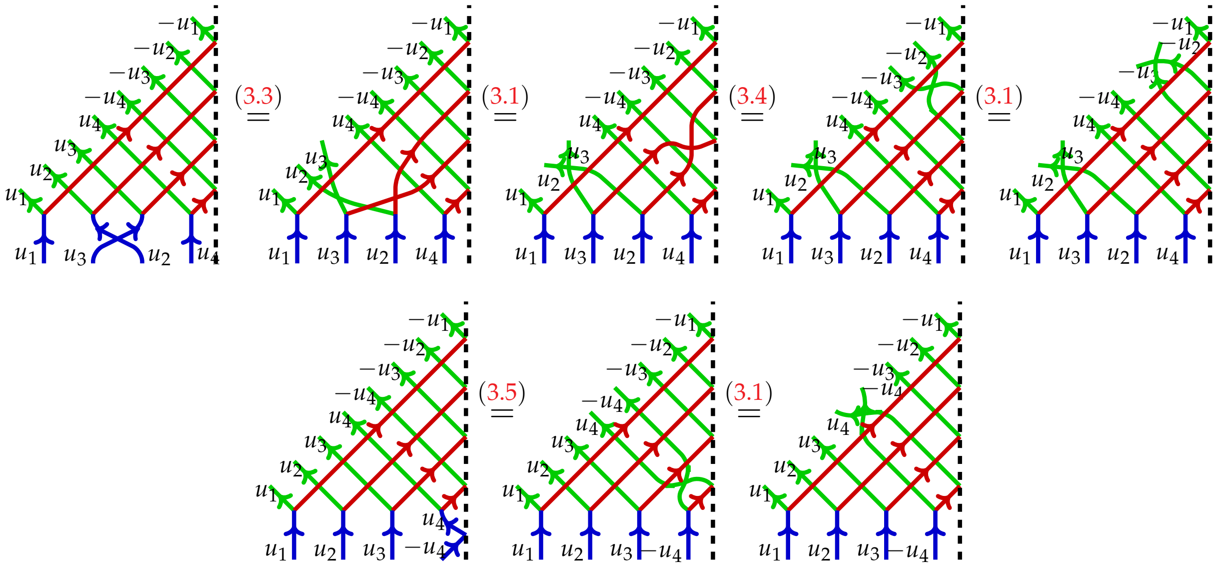
$= \delta_{\lambda, \omega\overline{\omega}}$.

*Proof.* This is a straightforward consequence of Definition 5, when considering the $(\lambda, \omega)$ matrix entry of the product of $R$-, $K$-, and $U$-matrices making up the half-puzzle. Alternatively, note that this is half of a classical triangular self-dual puzzle with NW, NE, S boundaries labelled by $\lambda, \overline{\lambda}, \omega\overline{\omega}$, and so the result follows from [7, Proposition 4]. □

**Proposition 11.** *Given $\sigma \in (10)^{n-k}\{0,1\}^k$, fixing the Northwest and South boundaries to be strings of length $2n$ and $n$ respectively, one has*

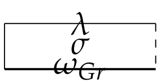


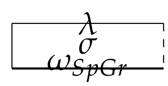
*Proof.* It suffices to consider $\widetilde{\sigma}$ (from Proposition 8) a simple reflection. For the purposes of illustration, we set $n = 4$ and demonstrate the equality in the case of an $s_i$ where $i < n$, as well as for $s_n$.

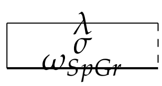



□

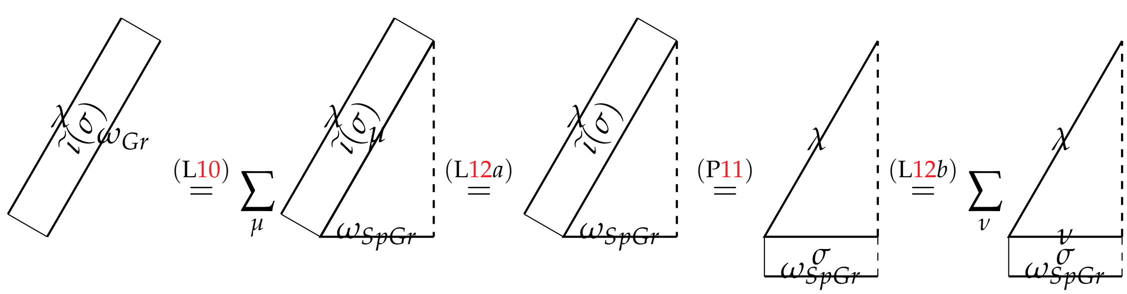
**Lemma 12.** *a) [7, Proposition 4] Type A. Let $\sigma \in 0^k 1^{2n-k}$ and $\lambda$ be a string of length $2n$:*

*If*  *$\neq 0$ for $\omega_{Gr}$ as in Proposition 9, then $\lambda$ consists only of 0s and 1s (no 10s).*

*b) Type C. Let $\sigma \in (10)^{n-k}\{0,1\}^k$ and $\lambda$ be a string of length $n$: If*  *$\neq 0$ for $\omega_{SpGr}$ as in Proposition 9, then $\lambda$ has the same number of 10s as $\omega_{SpGr}$.*

*Proof.* To prove part b), recall that  is the $(\lambda, \omega_{SpGr})$ matrix entry for the composition of $R_{BB}$ and $K_B$ maps. From Definition 5, we see that both of these maps preserve the number of 10s in a string, hence so will compositions of these maps. □

*Proof of Theorem 1C.* In $H_T^*(\mathrm{pt})$, we have the following equality



The left side corresponds to $\iota^*(S_\lambda)|_\sigma$ by Proposition 9. In the second and fourth equality, the strings $\mu$ and $\nu$ have content $0^k 1^{2n-k}$ and $(10)^{n-k}\{0,1\}^k$ respectively, and all other terms of the sum vanish. □

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