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Plabic R-Matrices

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Abstract. Postnikov's plabic graphs in a disk are used to parametrize totally positive Grassmannians. In recent years plabic graphs have found numerous applications in math and physics. One of the key features of the theory is that if a plabic graph is reduced, the face weights can be uniquely recovered from boundary measurements. On surfaces more complicated than a disk this property is lost. In this paper we undertake a comprehensive study of a certain semi-local transformation of weights for plabic networks on a cylinder that preserve boundary measurements. We call this a plabic R-matrix. We show that plabic R-matrices have underlying cluster algebra structure, generalizing recent work of Inoue-Lam-Pylyavskyy. Special cases of transformations we consider include geometric R-matrices appearing in Berenstein-Kazhdan theory of geometric crystals, and also certain transformations appearing in a recent work of Goncharov-Shen.

Keywords: plabic graph, R-matrix, cluster algebra

1 Introduction

The relationship between total positivity and networks has been studied extensively [1, 7, 8]. In his groundbreaking paper [18], Postnikov develops a theory of plabic networks for studying the connection between the totally nonnegative Grassmannian and planar directed networks in a disk. Plabic graphs have since been found to have many additional applications. They have been used by Kodama and Williams to study soliton solutions to the KP equation [15, 16], by Arkani-Hamed, et. al., to study scattering amplitudes for $\mathcal{N} = 4$ supersymmetric Yang-Mills [2, 3, 4], and by Gekhtman, Shapiro, and Vainshtein to study Poisson geometry [10, 11].

Postnikov defines a set of local moves and reductions so that the boundary measurement map gives a bijection between move-reduction equivalence classes for plabic networks in a disk and the totally nonnegative Grassmannian. However, there are plabic networks on a cylinder that are not move-reduction equivalent and yet have the same boundary measurements. In particular, we define a semi-local transformation on weights for plabic networks on a cylinder that preserves boundary measurements. We call this a *plabic R-matrix*. Plabic R-matrices are different from Postnikov's moves and reductions

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in that they do not alter the underlying graph. These transformations generalize the geometric R-matrix [5, 14, 17] and transformations used by Goncharov and Shen to study Donaldson-Thomas invariants [13].

This paper is organized as follows. In Section 2 we introduce plabic graphs and directed plabic networks on a cylinder. We explain weights and boundary measurements for these networks. Section 3 begins by defining Postnikov diagrams on a cylinder. Using this construction, we introduce a family of plabic networks on a cylinder called cylindric k-loop plabic networks. We define a transformation called a plabic R-matrix on face weights for such networks. Our first main result, Theorem 1, shows that plabic R-matrices preserve boundary measurements, are involutions, give the only choices of weights that preserve the boundary measurements, and satisfy the braid relation. Section 4 considers the cluster structure of the plabic R-matrix. We first give a brief background on cluster algebras from quivers and y-dynamics. Our second main result, Theorem 2, is that for a cylindric k-loop plabic network the face weighted plabic R-matrix is realized by the y-dynamics of the dual quiver, under a certain mutation sequence.

This paper is an extended abstract to [6].

2 Directed Plabic Networks on a Cylinder

We will always draw a cylinder as a fundamental domain of its universal cover such that it is a rectangle with boundary components on the left and right (see Figure 1).

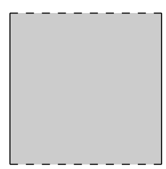


Figure 1: A cylinder, as we will represent them in this paper.

Definition 1. A *plabic graph* on a cylinder is an undirected planar graph, considered up to homotopy. We will assume such a graph has n vertices on the boundary, which we label $b_1, ..., b_n$ starting from the top of the left boundary component to the bottom and then from the bottom of the right boundary component to the top. We will call these *boundary vertices*, and all other vertices *internal vertices*. Each boundary vertex in a plabic graph on a cylinder has degree 1 and each internal vertex is colored black or white.

Definition 2 (Section 2.3 of [12])**.** A *trail* in a plabic graph on a cylinder G is a sequence of vertices $v_1, ..., v_{m+1}$ where v_1, v_{m+1} are boundary vertices on different boundary components and for each i, (v_i, v_{i+1}) is an undirected edge in G. Note that a trail carries an orientation; $v_1, ..., v_{m+1}$ is a different trail than $v_{m+1}, ..., v_1$.

If a plabic graph on a cylinder does not have a trail, then this graph could be embedded in a disk. The theory of plabic graphs on a disk was developed by Postnikov [18], so we will only consider plabic graphs on a cylinder that have a trail.

Definition 3. A *directed plabic graph* on a cylinder is a plabic graph with an orientation on each edge such that the black vertices have exactly one outgoing edge and the white vertices have exactly one incoming edge. A *directed plabic network* on a cylinder is a directed plabic graph on a cylinder with a specified trail from the underlying plabic graph, a weight t associated to the trail, and a weight $y_f \in \mathbb{R}_{>0}$ assigned to each face such that $\prod y_f = 1$. The orientation of the edges in the trail do not necessarily need to match the orientation of the trail.

Definition 4. A *path* in a directed plabic graph or network on a cylinder is a sequence of vertices $v_1, ..., v_k$ such that for each i, (v_i, v_{i+1}) is a directed edge in the graph.

Definition 5. We now define the weight, $wt(P, y, t)$, of a path P from b_i to b_j in a directed plabic network with face weights y_f and trail weight t. The path P can be decomposed into a non-self-intersecting path $\widetilde{P}$ from b_i to b_j and a set of non-self-intersecting closed cycles $C_1, ..., C_k$. Then $wt(P, y, t) = wt(\widetilde{P}, y, t) + \sum_{i=1}^{k} wt(C_i, y, t)$

The weight of a closed cycle C with no self-intersections, $wt(C, y, t)$, can be defined in three cases:

(1) If C encloses a region without boundary components and that region lies to the right of C, then $wt(C, y, t)$ is the product of the faces in that region.

(2) If C encloses a region without boundary components and that region lies to the left of C, then $wt(C, y, t)$ is the product of the faces outside that region.

(3) Otherwise, C divides the cylinder into two parts, each containing one boundary component. In this case, $wt(C, y, t)$ is the product of the faces to the right of C.

There are also three cases for defining the weight of a non-self-intersecting path $\widetilde{P}$, $wt(\widetilde{P}, y, t)$:

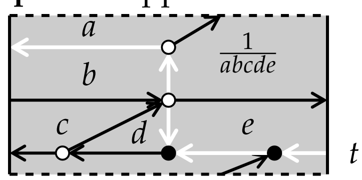
(1) If $\widetilde{P}$ begins and ends on the same boundary component, $wt(\widetilde{P}, y, t)$ is the product of the faces to the right of P.

(2) For a path $\widetilde{P}$ that begins on the same boundary component as the trail and ends on the other boundary component, draw enough copies of the fundamental domain so that $\widetilde{P}$ can be depicted as a connected curve and there is at least one copy of the trail that lies completely to the right of $\widetilde{P}$. Choose one such copy of the trail and call it Q. Then $wt(\widetilde{P}, y, t)$ is t multiplied by the product of the weights of the faces that lie between $\widetilde{P}$ and Q (that is, to the right of $\widetilde{P}$ and to the left Q).

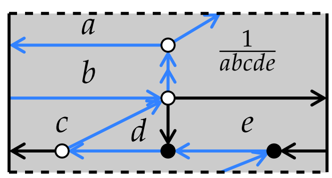
(3) For a path $\widetilde{P}$ that begins on the boundary component where the trail ends and ends on the other boundary component, draw enough copies of the fundamental domain so that $\widetilde{P}$ can be depicted as a connected curve and there is at least one copy of the trail that lies completely to the right of $\widetilde{P}$. Choose one such copy of the trail and call it Q. Then $wt(\widetilde{P}, y, t)$ is t^{-1} multiplied by the product of the weights of the faces that lie between $\widetilde{P}$ and Q (that is, to the right of $\widetilde{P}$ and to the right Q).

In Cases 2 and 3, $wt(\widetilde{P}, y, t)$ is well-defined because if we choose two trails that lie completely to the right of P, the product of the weights of faces between the trails is 1.

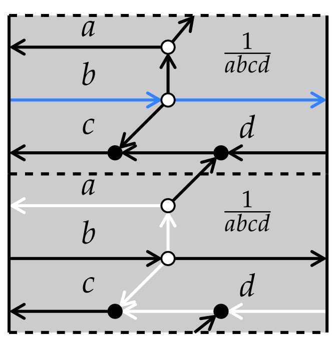
Example 1. Suppose we have the following network with face and trail weights:



The trail and trail weight appear in white, and the trail is oriented from right to left.



For the path P shown in blue, or dark gray in print, (where the upward pointing edge is traversed twice), P can be decomposed into a path $\widetilde{P}$ with 3 edges and cycle C going around the cylinder. Here, $wt(P, y, t) = wt(\widetilde{P}, y, t) + wt(C, y, t) = \frac{acde}{abcde} + \frac{de}{abcde} = \frac{ac+1}{abc}$.



For the path P shown in blue, or dark gray in print, P is going in the opposite direction of the trail. So, $wt(P, y, t) = t^{-1}\left(\frac{acd^2}{abcd}\right) = \frac{d}{tb}$.

Definition 6 (Section 2.1 of [12])**.** A *cut* γ is an oriented non-self-intersecting curve from one boundary component to another, considered up to homotopy. For a path P, the *intersection number*, $int(P)$, is the number of times P crosses γ from the right minus the number where P crosses γ from the left.

For the rest of the paper, we will consider our cylinder to have a fixed cut. We will always assume the cut is disjoint from the set of vertices of a graph and that it corresponds to the top and bottom of our rectangle when we draw a cylinder. The cut is denoted by a directed dashed line.

If P is a path from b to b' where b, b' are on the same boundary component, then C_P is the closed loop created from following the path P and then going down along the boundary from b' to b. If P is a path from b to b' where b, b' are on the different boundary components, then C_P is the closed loop created from following the path P going down on the boundary from b' to the base point of the cut, following the cut (or its reverse), and then down on the boundary from base point of the cut to b. Note that we create C_P on the cylinder, not the universal cover.

Definition 7 (Section 2.1 of [12]). We can glue together the top and bottom of our rectangle, which represents a cylinder, in the plane to form an annulus. Do this such that going up along the boundary of the rectangle corresponds to going clockwise around the boundary of the annulus (see Figure 2). Then for a closed curve C, we define its *winding index*, $wind(C)$. First, we smooth any corners in C. Then $wind(C)$ is the number of counterclockwise $360°$ turns the tangent vector makes when we follow C.

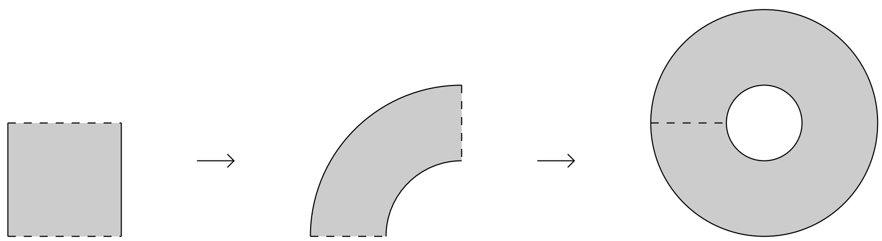
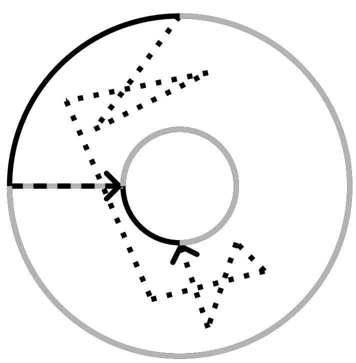


Figure 2: Turning a cylinder into an annulus.

Example 2.



Here we have the cylinder depicted as an annulus. The dashed line is the cut. A path P is shown as a dotted line. P crosses the cut once from left to right, so $int(P) = -1$. The extension of P to C_P is shown in gray. We can see $wind(C_P) = -3$.

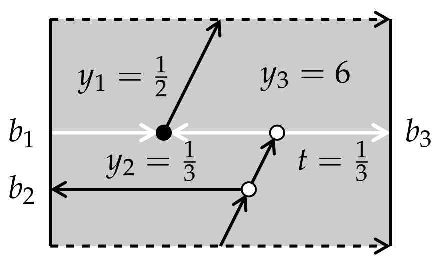
Definition 8. Let b_i be a source and b_j be a sink in a directed plabic network on a cylinder. Let the face weights be the formal variables y_f and the trail weight be the formal variable t. Then the *formal boundary measurement* M_{ij}^{form} is the formal power series

$$M_{ij}^{\text{form}} := \sum_{\substack{\text{paths } P \text{ from} \\ b_i \text{ to } b_j}} (-1)^{wind(C_P)-1} \zeta^{int(P)} wt(P, y, t).$$

Lemma 1 (Section 2.2 of [12]). *If N is a directed plabic network on a cylinder, then the formal power series M_{ij}^{form} sum to rational expressions in the variables y_f, t and ζ.*

Definition 9. The *boundary measurements* M_{ij} for a directed plabic network on a cylinder are rational functions in ζ obtained by writing the formal boundary measurements M_{ij}^{form} as rational expressions, and then specializing them by assigning y_f the weight of the face f and t the weight of the trail.

Example 3. Suppose we have the following network where the trail appears in white and is oriented from left to right:



$$
\begin{aligned}
M_{12}^{\text{form}} &= \frac{\zeta}{y_1} - \frac{\zeta^2}{y_1^2 y_2} + \frac{\zeta^3}{y_1^3 y_2^2} - \dots \\
&= \frac{\zeta}{y_1} \sum_{i=0}^{\infty} \left(\frac{-\zeta}{y_1 y_2} \right)^i \\
&= \frac{y_2 \zeta}{y_1 y_2 + \zeta} \\
M_{13}^{\text{form}} &= -t y_3 \zeta + t y_3^2 \zeta^2 - t y_3^3 \zeta^3 + \dots \\
&= -t y_3 \zeta \sum_{i=0}^{\infty} (-y_3 \zeta)^i \\
&= \frac{-t y_3 \zeta}{1 + y_3 \zeta}
\end{aligned}
$$

Substituting our values for t and the y_f's, we find $M_{12} = \frac{2\zeta}{1+6\zeta}$ and $M_{13} = \frac{-6\zeta}{1+6\zeta}$.

3 The Plabic R-Matrix

A *Postnikov diagram* on a cylinder is a set of directed curves, called *strands*, such that when we draw the strands on the universal cover of the cylinder they meet the following conditions:

(1) Each strand begins and ends on the boundary or is infinite.

(2) No three strands intersect at the same point.

(3) All intersections are transverse (the tangent vectors are independent).

(4) There are a finite number of intersections in each fundamental domain.

(5) Along any strand, the strands that cross it alternate crossing from the left and crossing from the right. Similarly, along each boundary component, the strands that touch the boundary alternate beginning and ending there.

(6) Strands do not have self-intersections, except in the case where a strand has a loop around a vertex of degree 1 attached to a boundary vertex.

(7) If two strands intersect at u and v, then one strand is oriented from u to v and one strand is oriented from v to u.

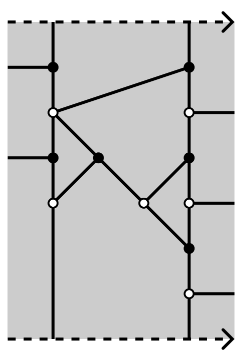
Postnikov diagrams are considered up to homotopy. We can obtain a plabic graph from a Postnikov diagram as follows:

(1) Place a black vertex in every face oriented counterclockwise and a white vertex in every face oriented clockwise.

(2) If two oriented faces share a corner, connect the vertices in these two faces.

Definition 10. A *cylindric k-loop plabic graph* is a plabic graph on a cylinder that arises from a Postnikov diagram where exactly k of the strands are loops around the cylinder with the same orientation.

Cylindric k-loop plabic graphs have k strings of vertices around the cylinder. Those strings alternate black and white vertices, and the black vertices only have additional edges on the left of the strand while the white vertices only have additional edges to the right of the strand (see Figure 3).


plabic graph

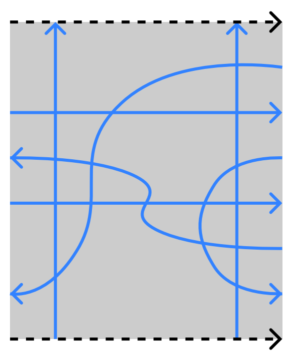

Postnikov diagram

Figure 3: A cylindric 2-loop plabic graph and its Postnikov diagram.

For a cylindric k-loop plabic graph, any vertices that are not on one of the strings of vertices defined by the k loops and lie between two of these strings are called *interior vertices*. We will assume throughout the rest of this paper that cylindric k-loop plabic graphs have no interior vertices. This assumption is justified in Section 4 of [6]. We label the k strings of vertices 1 through k, from left to right.

Definition 11. The *canonical orientation* of a cylindric k-loop plabic graph is the orientation where the edges on the strings are oriented up and the other edges are oriented from left to right.

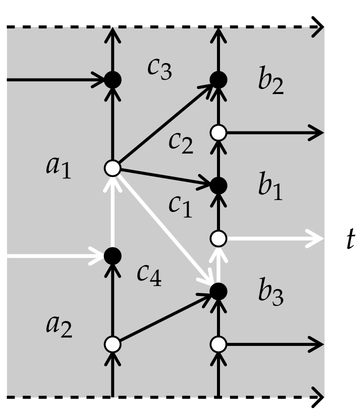
Suppose we have a cylindric k-loop plabic graph with the canonical orientation. We will be defining a transformation $T_{\varkappa,f}$, which will alter the weights of the faces bordering strings $\varkappa$ and $\varkappa+1$. We begin by fixing $1 \leq \varkappa < k$.

We now choose a trail in our cylindric k-loop plabic graph. Choose an edge (u,v) from string $\varkappa$ to string $\varkappa+1$ such that the edge is directly counter-clockwise from an edge on string $\varkappa$ around u and directly counter-clockwise from an edge on string $\varkappa+1$ around v. Such an edge must exist. Let (w,u) be an edge on string $\varkappa$. Choose the edge from string $\varkappa-1$ (or the left boundary if $\varkappa=1$) to string $\varkappa$ that is directly counter-clockwise from (w,u) around w. Let P be this edge, followed by (w,u), then (u,v). From v, go up string $\varkappa+1$ and make the sharpest right to string $\varkappa+2$ (or the right boundary if $\varkappa=k-1$). Add these two edges to P, then extend P to the left and right so it is a directed path from the left boundary to the right boundary. Let this path be our trail with trail weight t. Let the weights of the faces bordering string $\varkappa$ on the left be $a_1, a_2, ..., a_\ell$ starting above the trail and going up. In the same way, let the weights of the faces bordering string $\varkappa+1$ on the right be $b_1, b_2, ..., b_m$ and the weights of the faces between strings $\varkappa$ and $\varkappa+1$ be $c_1, c_2, ..., c_{n-1}, c_n = \frac{1}{a_1a_2...a_\ell b_1b_2...b_mc_1c_2...c_{n-1}}$. We will consider all of these indices to be modular. For any set S we'll define $\mathbf{a}_S := \prod_{i\in S} a_i$, and similarly for $\mathbf{b}_S$ and $\mathbf{c}_S$. Let $d := \mathbf{a}_{[1,\ell]}\,\mathbf{b}_{[1,m]}$ for ease of notation.

We will say a_j is associated to i if the highest edge on the left string bordering the face labeled a_j also borders the face c_i. Similarly, b_j is associated to i if the lowest edge on the right string bordering the face labeled b_j also borders the face c_i. Let $\mathcal{A}_{[i,j]} := \{k \mid a_k \text{ is associated to } \ell \in [i,j]\}$ and $\mathcal{B}_{[i,j]} := \{k \mid b_k \text{ is associated to } \ell \in [i,j]\}$. Define

$$\widehat{\lambda}_i(a,b,c) := \begin{cases} \mathbf{b}_{[1,m]}\,\mathbf{a}_{\mathcal{A}_{[1,i]}}\left(\sum_{j=i}^{n}\mathbf{c}_{[i,j]} + \sum_{j=1}^{i-1}\mathbf{c}_{[i,j]}\right) & i=1, \\ \mathbf{c}_{[1,i-1]}\,\mathbf{b}_{[1,m]}\,\mathbf{b}_{\mathcal{B}_{[1,i-1]}}\,\mathbf{a}_{\mathcal{A}_{[1,i]}}\left(\sum_{j=i}^{n}\mathbf{c}_{[i,j]} + \sum_{j=1}^{i-1}\mathbf{c}_{[i,j]}\right) & i>1. \end{cases}$$

Example 4. Consider the network below.



In this network, a_1 is associated to 3, a_2 to 4, b_1 to 1, b_2 to 2, and b_3 to 3.

$$\begin{aligned} \widehat{\lambda_1}(a,b,c) &= \mathbf{b}_{[1,3]}\,\mathbf{a}_{\mathcal{A}_{[1,1]}}\left(\sum_{j=1}^{4}\mathbf{c}_{[1,j]} + \sum_{j=1}^{0}\mathbf{c}_{[1,j]}\right) \\ &= b_1b_2b_3c_1 + b_1b_2b_3c_1c_2 + b_1b_2b_3c_1c_2c_3 + \tfrac{1}{a_1a_2} \\ \widehat{\lambda_2}(a,b,c) &= \mathbf{c}_{[1,1]}\,\mathbf{b}_{[1,3]}\,\mathbf{b}_{\mathcal{B}_{[1,1]}}\,\mathbf{a}_{\mathcal{A}_{[1,2]}}\left(\sum_{j=2}^{4}\mathbf{c}_{[2,j]} + \sum_{j=1}^{1}\mathbf{c}_{[2,j]}\right) \\ &= b_1^2b_2b_3c_1c_2 + b_1^2b_2b_3c_1c_2c_3 + \tfrac{b_1}{a_1a_2} + \tfrac{b_1c_1}{a_1a_2} \end{aligned}$$

Definition 12. Suppose we have a cylindric k-loop plabic network with the canonical orientation, a choice of $1 \leq \varkappa < k$, a trail chosen as above, and face and trail weights as above. Define $T_{\varkappa,f}$ to be the transformation on face and trail weights from (a,b,c,t) to (a',b',c',t) where

$$a'_i = \frac{\widehat{\lambda}_j(a,b,c)}{\widehat{\lambda}_p(a,b,c)\,\mathbf{b}_{\mathcal{B}_{[p,j-1]}}} \quad \text{where } a_i \text{ is associated to } j, a_{i-1} \text{ is associated to } p$$

$$b'_i = \frac{\widehat{\lambda}_q(a,b,c)}{\widehat{\lambda}_j(a,b,c)\,\mathbf{a}_{\mathcal{A}_{[j+1,q]}}} \quad \text{where } b_i \text{ is associated to } j, b_{i+1} \text{ is associated to } q$$

$$c'_i = \frac{\mathbf{a}_{\mathcal{A}_{[i,i+1]}}\,\mathbf{b}_{\mathcal{B}_{[i-1,i]}}\,c_i\widehat{\lambda}_{i-1}(a,b,c)}{\widehat{\lambda}_{i+1}(a,b,c)}$$

We call $T_{\varkappa,f}$ the face weighted *plabic R-matrix*.

Theorem 1. *T_f has the following properties:*

1. *It preserves the boundary measurements.*
2. *It is an involution.*
3. *(a,b,c,t) and (a',b',c',t) are the only choices of face and trail weights on a fixed cylindric 2-loop plabic network with the canonical orientation that preserve the boundary measurements.*
4. *It satisfies the braid relation. That is, $T_{\varkappa,f}T_{\varkappa+1,f}T_{\varkappa,f} = T_{\varkappa+1,f}T_{\varkappa,f}T_{\varkappa+1,f}$ for $1 \leq \varkappa < k-1$.*

Example 5. Continuing Example 4, when we apply $T_{1,f}$, we get the following face weights:

$$a'_1 = \frac{a_1a_2b_1b_2b_3c_1c_2c_3+1+c_1+c_1c_2}{a_2b_1b_2b_3(1+c_1+c_1c_2+c_1c_2c_3)}$$
$$a'_2 = \frac{a_2(1+c_1+c_1c_2+c_1c_2c_3)}{a_1a_2b_1b_2b_3c_1c_2c_3+1+c_1+c_1c_2}$$
$$b'_1 = \frac{b_1(a_1a_2b_1b_2b_3c_1c_2+a_1a_2b_1b_2b_3c_1c_2c_3+1+c_1)}{a_1a_2b_1b_2b_3c_1+a_1a_2b_1b_2b_3c_1c_2+a_1a_2b_1b_2b_3c_1c_2c_3+1}$$
$$b'_2 = \frac{b_2(a_1a_2b_1b_2b_3c_1c_2c_3+1+c_1+c_1c_2)}{a_1a_2b_1b_2b_3c_1c_2+a_1a_2b_1b_2b_3c_1c_2c_3+1+c_1}$$
$$b'_3 = \frac{a_1a_2b_1b_2b_3c_1+a_1a_2b_1b_2b_3c_1c_2+a_1a_2b_1b_2b_3c_1c_2c_3+1}{a_1a_2b_1b_2(a_1a_2b_1b_2b_3c_1c_2c_3+1+c_1+c_1c_2)}$$
$$c'_1 = \frac{a_1a_2b_1b_2b_3c_1(1+c_1+c_1c_2+c_1c_2c_3)}{a_1a_2b_1b_2b_3c_1c_2+a_1a_2b_1b_2b_3c_1c_2c_3+1+c_1}$$
$$c'_2 = \frac{c_2(a_1a_2b_1b_2b_3c_1+a_1a_2b_1b_2b_3c_1c_2+a_1a_2b_1b_2b_3c_1c_2c_3+1)}{a_1a_2b_1b_2b_3c_1c_2c_3+1+c_1+c_1c_2}$$
$$c'_3 = \frac{c_3(a_1a_2b_1b_2b_3c_1c_2+a_1a_2b_1b_2b_3c_1c_2c_3+1+c_1)}{1+c_1+c_1c_2+c_1c_2c_3}$$
$$c'_4 = \frac{a_1a_2b_1b_2b_3c_1c_2c_3+1+c_1+c_1c_2}{c_1c_2c_3(a_1a_2b_1b_2b_3c_1+a_1a_2b_1b_2b_3c_1c_2+a_1a_2b_1b_2b_3c_1c_2c_3+1)}$$

4 Cluster Structure of $T_{\kappa,f}$

We begin by reviewing some important concepts from the theory of cluster algebras.

A *quiver* Q is a directed graph with vertices labeled $1, ..., m$ and no loops or 2-cycles. If k is a vertex in a quiver Q, a *quiver mutation* at k, $\mu_k(Q)$ is defined from Q as follows:

(1) for each pair of edges $i \to k$ and $k \to j$, add a new edge $i \to j$,

(2) reverse any edges incident to k,

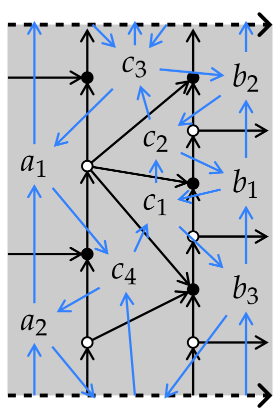
(3) remove any 2-cycles.

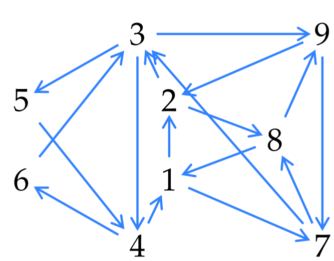
In this abstract, we will let a *y-seed* be a pair $(Q, \mathbf{y})$ where Q is a quiver and $\mathbf{y} = (y_1, ..., y_m) \in \mathbb{R}^m$ with m the number of vertices of Q. For a more general definition of a y-seed, see [9]. If k is a vertex in Q, a *y-seed mutation* of $(Q, \mathbf{y})$ at k is $\mu_k(Q, \mathbf{y}) = (Q', \mathbf{y}')$ where $Q' = \mu_k(Q)$, $\mathbf{y}' = (y'_1, ..., y'_m)$, and

$$y'_i = \begin{cases} y_k^{-1} & i = k, \\ y_i(1 + y_k^{-1})^{-\#\{\text{edges } k \to i \text{ in } Q\}} & i \neq k, \#\{\text{edges } k \to i \text{ in } Q\} \geq 0, \\ y_i(1 + y_k)^{\#\{\text{edges } i \to k \text{ in } Q\}} & i \neq k, \#\{\text{edges } i \to k \text{ in } Q\} \geq 0. \end{cases}$$

Definition 13. The *dual quiver* of a plabic graph is the quiver with vertices corresponding to the faces of the plabic graph and edges crossing every bicolored edge of the plabic graph. We orient the edges of the quiver so that the black vertex is always on the right and the white vertex is on the left.

Example 6. We have the dual quiver to the network from Examples 4 and 5 below. On the left, we can see how to draw the quiver from the network. On the right we have the quiver redrawn more clearly.





If Q is the dual quiver to a cylindric k-loop plabic network with no interior vertices and $1 \leq \varkappa < k$, the vertices corresponding to the faces between strings $\varkappa$ and $\varkappa + 1$

and the arrows between these vertices form a cycle. Label these vertices $1, ..., n$ going around the cycle. Let $\tau_\varkappa := \mu_1\mu_2...\mu_{n-2}s_{n-1,n}\mu_n\mu_{n-1}...\mu_1$ where $s_{n-1,n}$ is the operation that transposes vertices $n-1$ and n.

Theorem 2. *Let Q be the dual quiver to a cylindric k-loop plabic network with no interior vertices. If we set the y-variable for each vertex equal to the weight of the corresponding face and apply $\tau_\varkappa$, the y-variables we obtain are the same as the face variables with the transformation $T_{\varkappa,f}$ applied to them.*

Example 7. Let Q be the quiver in Example 6 and let our vector of y-variables be $(c_1, c_2, c_3, c_4, a_1, a_2, b_1, b_2, b_3)$. If apply τ_1, we are mutating at vertex 1, mutating at vertex 2, mutating at vertex 3, mutating at vertex 4, swapping vertices 3 and 4, mutating at vertex 2, and lastly mutating at vertex 1. When we do this, we end with the quiver Q and the vector of y-variables $(c'_1, c'_2, c'_3, c'_4, a'_1, a'_2, b'_1, b'_2, b'_3)$ from Example 5.

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