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On enumerating factorizations in reflection groups

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Abstract. We describe an approach, via Malle's permutation Ψ on the set of irreducible characters $\mathrm{Irr}(W)$, that gives a uniform derivation of the Chapuy-Stump formula for the enumeration of reflection factorizations of the Coxeter element. It also recovers its weighted generalization by delMas, Reiner, and Hameister, and further produces structural results for factorization formulas of arbitrary regular elements.

Résumé. Nous décrivons une approche, via la permutation de Malle Ψ sur l'ensemble des caractères irréductibles $\mathrm{Irr}(W)$, qui donne une dérivation uniforme de la formule de Chapuy-Stump pour l'énumération des factorisations de l'élément Coxeter en réflexions. Il récupère également sa généralisation pondérée par delMas, Reiner et Hameister et produit en outre des résultats structurels pour des formules de factorisation d'éléments réguliers arbitraires.

Keywords: Frobenius lemma, Hecke algebras, Malle's permutation

1 Introduction

A famous theorem of Cayley states that there are n^{n-2} vertex-labeled trees on n vertices. The same number, as Hurwitz knew already by the end of the 19th century, enumerates the set of shortest length factorizations $t_1 \cdots t_{n-1} = (12\cdots n) \in S_n$ of the long cycle into transpositions t_i. A natural generalization of this problem, is to enumerate such factorizations of arbitrary length.

It took almost a hundred years for the community to return to this question, but by the end of the 80's Jackson [12, Corol. 4.2] had computed an explicit answer. If $\mathrm{FAC}_{S_n}(t)$ denotes the exponential generating function for the number of arbitrary length factorizations of the long cycle in transpositions (see (3.1)), then Jackson's result can be reinterpreted as follows:

$$\mathrm{FAC}_{S_n}(t) = \frac{e^{t\binom{n}{2}}}{n!}\left(1 - e^{-tn}\right)^{n-1}. \tag{1.1}$$

As it often happens with some of the most fascinating properties of the symmetric group, the previous statements are special cases of more general theorems that hold for

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a bigger class of reflection groups W. A natural analog of the long cycle is the Coxeter element $c \in W$, while transpositions are replaced by reflections. Then, if W is well-generated and of rank n, $\mathcal{R}$ denotes its set of reflections, and h is the order of c, Bessis [2, Prop. 7.6] proved the following enumeration:

$$\#\Big\{(t_1,\cdots,t_n)\in\mathcal{R}^n \mid t_1\cdots t_n = c\Big\} \;=\; \frac{h^n n!}{|W|}. \tag{1.2}$$

The W-analog of Jackson's formula (1.1) regarding arbitrary length factorizations was discovered (and proved) by Chapuy and Stump [8] soon after. If $\mathrm{FAC}_W(t)$ denotes the corresponding exponential generating function, they showed that

$$\mathrm{FAC}_W(t) = \frac{e^{t|\mathcal{R}|}}{|W|}\big(1-e^{-th}\big)^n. \tag{1.3}$$

The reduced case (1.2), which can easily be derived by calculating the leading term of $\mathrm{FAC}_W(t)$, has a long history and appears in connection to many a mathematical endeavour. It originated in singularity theory, in combinatorics it appeared as the number of maximal chains in the noncrossing lattice $NC(W)$, and more importantly it was essential in Bessis' proof of the $K(\pi,1)$-conjecture [2].

Apart from the Weyl group case [16], neither (1.3) nor (1.2) were well understood. Although the statements are uniform for all well-generated groups, the proofs of Bessis and Chapuy-Stump have relied on the Shephard-Todd classification (a common misfortune for theorems regarding reflection groups, but also a driving reason for why the theory is as evolved as it is). Our main contribution is a case-free proof (Corollary 3.8) and generalization of them (Theorems 3.6 and 5.5) which in fact, in terms of Bessis' work [2], completes the first uniform proof of the dual braid presentation of $B(W)$ for real W.

2 Complex reflection groups and regular elements

Given a complex vector space $V \cong \mathbb{C}^n$, we call a *finite* subgroup $W \leq \mathrm{GL}(V)$ a *complex reflection group* if it is generated by unitary reflections. These are $\mathbb{C}$-linear maps t whose fixed spaces $V^t := \ker(t-\mathrm{id})$ are hyperplanes (i.e. $\mathrm{codim}(V^t)=1$). We further say that W is *irreducible* if it has no stable linear subspaces apart from V and $\{0\}$.

Shephard and Todd classified irreducible complex reflection groups into an infinite 3-parameter family $G(r,p,n)$ and 34 exceptional cases indexed G_4 to G_{37}. In what follows we will silently assume that our reflection groups W are in fact irreducible. This simplifies the argument but imposes no serious restrictions on our results, see [10, § 5.1].

We denote by $\mathcal{R}$ the set of reflections of W and we write $\mathcal{A}$ for the associated arrangement of fixed hyperplanes. For such a hyperplane H, let W_H be its pointwise stabilizer. It consists of the identity and the reflections that fix H. Furthermore, because unitary reflections are semisimple, W_H is cyclic.

Now, if $e_H := |W_H|$ is the size of this cyclic group and t_H is one of its generators, the set of reflections $\mathcal{R}$ can be partitioned as:

$$\mathcal{R} = \bigcup_{H \in \mathcal{A}} \{t_H, \cdots, t_H^{e_H - 1}\}. \tag{2.1}$$

The reflection group W acts on $\mathcal{A}$ determining orbits of hyperplanes which we will denote by $\mathcal{C} \in \mathcal{A}/W$. The size $\omega_{\mathcal{C}}$ of an orbit $\mathcal{C}$ is given by $\omega_{\mathcal{C}} := [W : N_W(H)]$ (for any $H \in \mathcal{C}$). All elements $H \in \mathcal{C}$ have conjugate stabilizers W_H and we write $e_{\mathcal{C}}$ for their common order. With this notation, the cardinalities of the set of reflections $\mathcal{R}$ and of the set of reflecting hyperplanes $\mathcal{A}$ are given by

$$|\mathcal{R}| = \sum_{\mathcal{C} \in \mathcal{A}/W} \omega_{\mathcal{C}}(e_{\mathcal{C}} - 1) \qquad \text{and} \qquad |\mathcal{A}| = \sum_{\mathcal{C} \in \mathcal{A}/W} \omega_{\mathcal{C}}.$$

Notice that if some $e_{\mathcal{C}} \neq 2$, then $|\mathcal{R}|$ and $|\mathcal{A}|$ are not equal.

Braid groups and the full twist

We say that a vector $v \in V$ is *regular* if it is not contained in any reflection hyperplane and we write $V^{\text{reg}} := V \setminus \mathcal{A}$ for the set of regular vectors. We define the *pure braid group* $P(W) := \pi_1(V^{\text{reg}})$ to be the fundamental group of the regular space V^{reg}. It is a theorem of Steinberg that the action of W on V is free precisely on V^{reg}.

Steinberg's theorem implies that the restriction of the quotient map $\rho : V \to V/W$ on V^{reg} is a Galois covering. We define the *braid group* $B(W) := \pi_1(V^{\text{reg}}/W)$ to be the fundamental group of the base of this covering and use the following short exact sequence to obtain a surjection $\pi : B(W) \twoheadrightarrow W$:

$$1 \to \pi_1(V^{\text{reg}}) \xhookrightarrow{\rho_*} \pi_1(V^{\text{reg}}/W) \xrightarrow{\pi} W \to 1. \tag{2.2}$$

Given a choice of a basepoint $x_0 \in V^{\text{reg}}$, a loop $\boldsymbol{b} \in B(W)$ lifts to a path that connects x_0 to $\boldsymbol{b}_*(x_0)$ (we call this the *Galois action* of b). Then, we define $w := \pi(\boldsymbol{b})$ to be the *unique* element $w \in W$ such that $w \cdot x_0 = b_*(x_0)$.

Broué-Malle-Rouquier considered [6, Notation 2.3] a particular element $\boldsymbol{\pi}$ of the pure braid group $P(W)$. It is given as the geometric circle $[0,1] \ni t \to e^{2\pi i t} \cdot x_0$.

Definition 2.1. We call $\boldsymbol{\pi}$ the *full twist*. It is central in $B(W)$ and lies in $P(W)$ [10, §2.2].

Although our initial purpose for this project was to give a uniform proof of the Chapuy-Stump formula (1.3) which regards Coxeter elements, it soon became clear that the techniques developed (see Lemma 3.5) apply to the larger class of Springer-regular elements. The crucial property these elements share is that they lift to roots of $\boldsymbol{\pi}$:

Definition 2.2. [14] We say that an element $g \in W$ is ζ-regular if it has a regular ζ-eigenvector. Its order $d := |g|$ is equal to the order of ζ and is called a *regular number*.

Proposition 2.3. *[5, Prop. 5.24] Let $\zeta = \exp(2\pi i l/d)$ be a primitive d^{th} root of unity, and let g be a ζ-regular element of W. Then, g has a lift $\boldsymbol{g} \in B(W)$ such that $\boldsymbol{g}^d = \boldsymbol{\pi}^l$.*

3 Frobenius lemma via Coxeter numbers

The lemma of Frobenius gives a representation theoretic formula for enumerating factorizations of group elements, when the factors belong to given conjugacy classes:

Theorem 3.1. *[13, App. A.1.3] Let G be a finite group and $A_i \subset G$, $i = 1 \dots l$, subsets that are closed under conjugation. Then the number of factorizations $t_1 \cdots t_l = g$ of an element $g \in W$, where each factor t_i belongs to A_i, is given by*

$$\frac{1}{|G|} \sum_{\chi \in \widehat{G}} \chi(1) \cdot \chi(g^{-1}) \cdot \frac{\chi(A_1)}{\chi(1)} \cdots \frac{\chi(A_l)}{\chi(1)},$$

where $\widehat{G}$ denotes the (complete) set of irreducible characters of G and $\chi(A) := \sum_{g \in A} \chi(g)$.

For a reflection group W, the set of reflections $\mathcal{R}$ is indeed closed under conjugation. If we write $\mathrm{Fact}_{W,g}(l)$ for the cardinality of the following set of factorizations

$$\mathrm{Fact}_{W,g}(l) := \#\{(t_1, \cdots, t_l) \in \mathcal{R}^l \mid t_1 \cdots t_l = g\},$$

then the lemma of Frobenius implies that

$$\mathrm{Fact}_{W,g}(l) = \frac{1}{|W|} \sum_{\chi \in \widehat{W}} \chi(1) \cdot \chi(g^{-1}) \cdot \Big[\frac{\chi(\mathcal{R})}{\chi(1)}\Big]^l.$$

After this, the exponential generating function for reflection factorizations of g equals:

$$\mathrm{FAC}_{W,g}(t) := \sum_{l \geq 0} \mathrm{Fact}_{W,g}(l) \cdot \frac{t^l}{l!} = \frac{1}{|W|} \sum_{\chi \in \widehat{W}} \chi(1) \cdot \chi(g^{-1}) \cdot \exp\Big[t \cdot \frac{\chi(\mathcal{R})}{\chi(1)}\Big]. \tag{3.1}$$

Now, a priori the evaluations $\chi(\mathcal{R})$ are complex numbers, but the decomposition (2.1) of the set $\mathcal{R}$ forces them to in fact be integers:

Proposition 3.2. *The numbers $\chi(\mathcal{R})$ are integers, and they further respect the tight bounds:*

$$-|\mathcal{A}| \cdot \chi(1) \;\leq\; \chi(\mathcal{R}) \;\leq\; |\mathcal{R}| \cdot \chi(1).$$

Proof. We keep the notation from (2.1) and choose a generator t_H for each of the cyclic groups W_H and write $e_H := |W_H|$ for its order.

For each eigenvalue λ of t_H in the representation U_χ associated to χ, the contribution of the set of reflections $\{t_H, \cdots, t_H^{e_H - 1}\}$ in the evaluation of $\chi(\mathcal{R})$ equals $\sum_{k=1}^{e_H - 1} \lambda^k$. Since $\lambda^{e_H} = 1$, this quantity is either $e_H - 1$ or -1 depending on whether λ itself is 1 or not.

This implies the integer property as well as the inequalities, after noticing that the multiset of eigenvalues of t_H acting on U_χ has $\chi(1)$-many elements. In particular, in order to recover the second inequality we use, after (2.1), that $\sum_{H \in \mathcal{A}} (e_H - 1) = |\mathcal{R}|$.

For the tightness statement, the higher bound is achieved when each eigenvalue of each t_H equals 1; of course this happens only in the trivial representation. For the lower bound, we need all $\lambda \neq 1$, which happens for instance in the det representation. □

The character values $\chi(\mathcal{R})$ on the sum of reflections are related to a statistic of the associated representation called the *Coxeter number* and denoted by c_χ. We postpone to Section 4.1 the discussion about its origin and for now we only give the definition:

Definition 3.3. [11, §1.3] We define the Coxeter number c_χ associated to the character χ, as the normalized trace of the central element $\sum_{t\in\mathcal{R}}(\mathbf{1}-t)$. That is,

$$c_\chi := \frac{1}{\chi(1)}\cdot\big(|\mathcal{R}|\chi(1)-\chi(\mathcal{R})\big) = |\mathcal{R}| - \frac{\chi(\mathcal{R})}{\chi(1)}.$$

We record the following as an immediate corollary of Theorem 3.1:

Corollary 3.4. *The exponential generating function* $\mathrm{FAC}_{W,g}(t)$ *for arbitrary length reflection factorizations of an element* $g\in W$ *is given by:*

$$\mathrm{FAC}_{W,g}(t) = \frac{e^{t|\mathcal{R}|}}{|W|}\sum_{\chi\in\widehat{W}}\chi(1)\cdot\chi(g^{-1})\cdot e^{-t\cdot c_\chi}. \tag{3.2}$$

The following lemma is the main technical ingredient for the proof of Theorem 3.6; we postpone an argument until Section 5, where we prove the more general Lemma 5.2. It relies on a cyclic action on the set $\mathrm{Irr}(W)$ of irreducible representations of W which is induced by a Galois action (see Definition 4.4) on the modules of the Hecke algebra $\mathcal{H}(W)$.

Lemma 3.5. *For a complex reflection group* W*, and a* ***regular*** *element* $g\in W$*, the total contribution in* (3.2) *of those characters* $\chi\in\widehat{W}$ *for which* c_χ *is not a multiple of* $|g|$ *is* 0.

Theorem 3.6. *For a complex reflection group* W*, and a regular element* $g\in W$*, the exponential generating function* $\mathrm{FAC}_{W,g}(t)$ *of reflection factorizations of* g *takes the following form:*

$$\mathrm{FAC}_{W,g}(t) = \frac{e^{t|\mathcal{R}|}}{|W|}\cdot\Big[(1-X)^{l_R(g)}\cdot\Phi(X)\Big]\Big|_{X=e^{-t|g|}}.$$

Here $l_R(g)$ *is the reflection length of* g *and* $\Phi(X)$ *is a polynomial of degree* $\frac{|\mathcal{R}|+|\mathcal{A}|}{|g|}-l_R(g)$ *in* X*, which is not further divisible by* $(1-X)$ *and has constant term equal to* 1.

Proof. After Lemma 3.5 we only consider terms of the form $\chi(1)\cdot\chi(g^{-1})\cdot e^{-t\cdot k|g|}$, $k\in\mathbb{Z}$ in the evaluation of (3.2). Furthermore, rephrasing Proposition 3.2 in terms of the Coxeter numbers (via Definition 3.3) forces $k\in\{0,\ldots,\frac{|\mathcal{R}|+|\mathcal{A}|}{|g|}\}$. This means that if we set $X=e^{-t|g|}$, we can rewrite (3.2) as

$$\mathrm{FAC}_{W,g}(t) = \frac{e^{t|\mathcal{R}|}}{|W|}\cdot\tilde{\Phi}(X),$$

where $\tilde{\Phi}(X)$ is a priori a polynomial in $\mathbb{C}[X]$ of degree $(|\mathcal{R}|+|\mathcal{A}|)/|g|$. From the proof of Lemma 3.5 we have that the constant term of $\tilde{\Phi}(X)$ is equal to $\chi_{\text{triv}}(1)\cdot\chi_{\text{triv}}(g^{-1})=1$.

Now, since $\tilde{\Phi}(X)$ essentially encodes the generating function $\text{FAC}_{W,g}(t)$, the combinatorial properties of the latter impose restrictions on its structure. In particular, consider the root factorization of the polynomial:

$$\tilde{\Phi}(X)=a(\alpha_1-X)(\alpha_2-X)\cdots(\alpha_r-X).$$

If we revert to $X=e^{-t|g|}$, each of the linear terms above has a Taylor expansion that starts with $(\alpha_i-1)+t|g|+\cdots$. This means that it contributes to the leading term of $\text{FAC}_{W,g}(t)$ either by a factor of (α_i-1) or by a factor of $t|g|$, depending on whether α_i equals 1 or not.

On the other hand, the combinatorial definition of $\text{FAC}_{W,g}(t)$ in (3.1) implies that its leading term is a multiple of $t^{l_R(g)}$. Therefore, exactly $l_R(g)$-many of the roots of $\tilde{\Phi}$ must be equal to 1 and this completes the proof. The statements about the degree and the constant term follow from the analogous results for $\tilde{\Phi}$ described previously. □

Remark 3.7. In the previous argument, the existence of a reflection length and therefore the knowledge that the first few terms of the generating function $\text{FAC}_{W,g}(t)$ are zero, came for free but was very useful nonetheless. It is hoped that similar ideas might apply to other groups with natural length functions, such as $\text{GL}_n(\mathbb{F}_q)$. Moreover, one might construct special length functions to support different enumerative questions (as we pursue in Proposition 3.10 and in Definition 5.3).

Corollary 3.8. *For a complex reflection group W and a regular element $g\in W$ of order $|g|=d_n$, the exponential generating function for reflection factorizations of g is given by:*

$$\text{FAC}_{W,g}(t)=\frac{e^{t|\mathcal{R}|}}{|W|}\cdot\big(1-e^{-t|g|}\big)^{l_R(g)}.$$

Sketch: This is essentially due to a theorem of Bessis [1, Prop. 4.2] which states that when d_n is regular, the minimal number of reflections needed to generate W is $(|\mathcal{R}|+|\mathcal{A}|)/d_n$. The remaining details appear in [10, Corol. 3.9]. □

Remark 3.9. When W is a well-generated group and c a Coxeter element of W, we always have $|c|=d_n$. The previous corollary therefore completes a proof of the Chapuy-Stump formula (1.3) and extends it to the groups for which d_n is regular.

We finish this section with an example where we can push the previous ideas slightly further by considering a different length function, the transitive factorization length:

Proposition 3.10. *[10, Prop. 3.11] The exponential generating function for transitive reflection factorizations of the regular element $g=(12\cdots n-1)(n)\in S_n$ is given by*

$$\text{TR-FAC}_{S_n,g}(t)=\frac{e^{t\binom{n}{2}}}{n!}\cdot\big(1-e^{-t(n-1)}\big)^n.$$

4 Hecke algebras and the technical lemma

The following definition of Hecke algebras, which recovers the usual Iwahori-Hecke algebras when W is a Coxeter group, is due to Broué, Malle, and Rouquier, and was introduced in their seminal paper [6].

Let $\mathcal{C} \in \mathcal{A}/W$ denote an orbit of hyperplanes, and $e_{\mathcal{C}}$ the common order of the pointwise stabilizers W_H (for $H \in \mathcal{C}$). Consider now a set of $\sum_{\mathcal{C}\in\mathcal{A}/W} e_{\mathcal{C}}$ many variables $\boldsymbol{u} := (u_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W, 0\leq j\leq e_{\mathcal{C}}-1)}$ and let $\mathbb{Z}[\boldsymbol{u},\boldsymbol{u}^{-1}]$ be the Laurent polynomial ring on them.

Definition 4.1. [6, Defn. 4.21] The *generic Hecke algebra* $\mathcal{H}(W)$ associated to W is the quotient of the group ring $\mathbb{Z}[\boldsymbol{u},\boldsymbol{u}^{-1}]B(W)$ of the braid group, over the ideal generated by the elements of the form

$$(\boldsymbol{s}-u_{\mathcal{C},0})(\boldsymbol{s}-u_{\mathcal{C},1})\cdots(\boldsymbol{s}-u_{\mathcal{C},e_{\mathcal{C}}-1}), \tag{4.1}$$

which we call *deformed order relations*. Here $\boldsymbol{s}$ runs over all possible braid reflections (these are a set of topological generators of $B(W)$ [10, § 2.1]) around the stratum $\mathcal{C}$ of $\mathcal{H}$.

Broué, Malle, Rouquier also made various conjectures about these Hecke algebras, the most important of which was until recently known as ***"The BMR freeness conjecture"***:

Theorem. *[4, after Thm. 3.5] The algebra $\mathcal{H}(W)$ is a free $\mathbb{Z}[\boldsymbol{u},\boldsymbol{u}^{-1}]$-module of rank $|W|$.*

The Hecke algebra is by construction a deformation of the group algebra of W. Indeed, for $\zeta_n := \exp(2\pi i/n)$, the specialization $\sigma:\ u_{\mathcal{C},j} \to \zeta_{e_{\mathcal{C}}}^j$ turns the defining relations (4.1) to order relations of the form $\boldsymbol{s}^{e_{\mathcal{C}}} = 1$, where we have $W \cong B(W)/\langle \boldsymbol{s}^{e_{\mathcal{C}}}\rangle$, [10, § 2.1].

Any ring map $\theta : \mathbb{Z}[\boldsymbol{u},\boldsymbol{u}^{-1}] \to R$ defines an R-module structure on the Hecke algebra and will be called a *specialization* of it. We will further say that θ is an *admissible* specialization if it factors through σ (i.e. if there is a map f such that $f\circ\theta(u_{\mathcal{C},j}) = \zeta_{e_{\mathcal{C}}}^j$).

Two particular specializations are fundamental in what follows. We first pick a set of parameters $\boldsymbol{x} := (x_{\mathcal{C}})_{\mathcal{C}\in\mathcal{A}/W}$ and the single parameter x and define the ring maps:

$$\begin{aligned} &\theta_{\boldsymbol{x}} : \mathbb{Z}[\boldsymbol{u},\boldsymbol{u}^{-1}] \to \mathbb{Z}[\boldsymbol{x},\boldsymbol{x}^{-1}] \quad\text{and}\quad \theta_x : \mathbb{Z}[\boldsymbol{u},\boldsymbol{u}^{-1}] \to \mathbb{Z}[x,x^{-1}] \\ &\theta_{\boldsymbol{x}}(u_{\mathcal{C},j}) = \begin{cases} x_{\mathcal{C}} & \text{if } j=0 \\ \zeta_{e_{\mathcal{C}}}^j & \text{if } j\neq 0 \end{cases} \qquad \theta_x(u_{\mathcal{C},j}) = \begin{cases} x & \text{if } j=0 \\ \zeta_{e_{\mathcal{C}}}^j & \text{if } j\neq 0 \end{cases} \end{aligned} \tag{4.2}$$

Both $\theta_{\boldsymbol{x}}$ and θ_x are admissible specializations (as seen by further sending $x_{\mathcal{C}}$ or x to 1). We write $\mathcal{H}_{\boldsymbol{x}}(W)$ and $\mathcal{H}_x(W)$ for the corresponding Hecke algebras; by Tits' deformation theorem [10, § 4.1] (and because of the BMR-freeness theorem) there is a spectra-preserving bijection between their representations and those of W. Therefore, given a character χ of W, we will write $\chi_{\boldsymbol{x}}$ and χ_x for the corresponding characters of $\mathcal{H}_{\boldsymbol{x}}(W)$ and $\mathcal{H}_x(W)$ respectively.

4.1 Local Coxeter numbers and Malle's character permutation

We are now going to define a local version of Coxeter numbers (see Definition 3.3) and study how they naturally appear in our key technical lemma (Proposition 4.5).

Definition 4.2. We define the *local* Coxeter number $c_{\chi,\mathcal{C}}$ associated to the character χ and the hyperplane orbit $\mathcal{C} \in \mathcal{A}/W$, as the normalized trace

$$c_{\chi,\mathcal{C}} := \frac{1}{\chi(1)} \cdot \chi\Big(\sum_{V^t \in \mathcal{C}} (\mathbf{1} - t) \Big).$$

Here, the sum is taken over all reflections t whose fixed hyperplane belongs to the orbit $\mathcal{C}$. Notice that these numbers refine the Coxeter numbers in the sense that $c_\chi = \sum c_{\chi,\mathcal{C}}$.

The following statement is proved similarly to Proposition 3.2 (see [10, Corol. 4.16]). The integer property is [7, Corol. 4.17].

Proposition 4.3. *The numbers $c_{\chi,\mathcal{C}}$ are integers and they satisfy* $\quad 0 \leq c_{\chi,\mathcal{C}} \leq e_{\mathcal{C}} \cdot \omega_{\mathcal{C}}$.

The local Coxeter numbers $c_{\chi,\mathcal{C}}$ appear in the following character calculation from [7] (and we have reinterpreted their formulas, but one can find more details in [10, § 4.2-4.3]). Let $\boldsymbol{\pi}$ be the full twist as in Definition 2.1 and w a regular element. By Proposition 2.3 w has a lift $\boldsymbol{w}$ in $B(W)$, that satisfies $\boldsymbol{w}^d = \boldsymbol{\pi}^l$ for suitable l and d. Then, for the characters $\chi_{\boldsymbol{x}}$ of $\mathcal{H}_{\boldsymbol{x}}(W)$ (see after (4.2)), and the elements $T_{\boldsymbol{\pi}}$ and $T_{\boldsymbol{w}}$ of the Hecke algebra we have:

$$\chi_{\boldsymbol{x}}(T_{\boldsymbol{\pi}}) = \chi(1) \prod_{\mathcal{C}\in\mathcal{A}/W} x_{\mathcal{C}}^{e_{\mathcal{C}}\omega_{\mathcal{C}} - c_{\chi,\mathcal{C}}} \quad \text{and} \quad \chi_{\boldsymbol{x}}(T_{\boldsymbol{w}}) = \chi(w) \prod_{\mathcal{C}\in\mathcal{A}/W} x_{\mathcal{C}}^{(e_{\mathcal{C}}\omega_{\mathcal{C}} - c_{\chi,\mathcal{C}})l/d}. \tag{4.3}$$

Moreover, after the further specialization $x_{\mathcal{C}} \to x$ of θ_x from (4.2) and for the characters χ_x of $\mathcal{H}_x(W)$, we have (recalling that $\sum e_{\mathcal{C}}\omega_{\mathcal{C}} = |\mathcal{R}| + |\mathcal{A}|$):

$$\chi_x(T_{\boldsymbol{\pi}}) = \chi(1) \cdot x^{|\mathcal{R}|+|\mathcal{A}|-c_\chi} \quad \text{and} \quad \chi_x(T_{\boldsymbol{w}}) = \chi(w) \cdot x^{(|\mathcal{R}|+|\mathcal{A}|-c_\chi)l/d}. \tag{4.4}$$

Malle's permutation is a Galois action on the characters

The fake degree $P_\chi(q) := \sum q^{e_i(\chi)}$ of a character $\chi \in \widehat{W}$ is a polynomial that records its exponents $e_i(\chi)$ (see [14, §4.4]). Beynon and Lusztig [3, Prop. A] had observed a remarkable reciprocity property for these polynomials. They satisfy

$$P_\chi(q) = q^{c_\chi} P_{\iota(\chi)}(q^{-1}),$$

where c_χ is the Coxeter number as given in Definition 3.3 and ι is a permutation on $\mathrm{Irr}(W)$ that for Weyl groups is the identity apart from two characters of E_7 and four of E_8.

Malle later on [15, Thm. 6.5] extended this reciprocity result for all complex reflection groups, defining a permutation Ψ on $\mathrm{Irr}(W)$ that is induced by a Galois action on the irreducible characters of the Hecke algebra (the two permutations satisfy $\iota(\chi) = \Psi(\chi^*)$). This permutation of Malle is the missing ingredient for Lemma 3.5; the characters χ for which $|g|$ does not divide c_χ are grouped together by Ψ and their contributions cancel.

Recall from (4.2) the specializations of the Hecke algebra $\mathcal{H}_{\boldsymbol{x}}(W)$ and $\mathcal{H}_x(W)$. It is a theorem of Malle [15, Thm. 5.2] that one can find split extensions $\mathbb{C}(\boldsymbol{y})/\mathbb{C}(\boldsymbol{x})$ and $\mathbb{C}(y)/\mathbb{C}(x)$ of the coefficient fields, by introducing parameters $\boldsymbol{y} := (y_{\mathcal{C}})_{\mathcal{C}\in\mathcal{A}/W}$ and y, which satisfy $y_{\mathcal{C}}^{N_W} = x_{\mathcal{C}}$ and $y^{N_W} = x$ for some number N_W. This means that all representations of $\mathcal{H}_{\boldsymbol{x}}(W)$ and $\mathcal{H}_x(W)$ are realizable over $\mathbb{C}(\boldsymbol{y})$ and $\mathbb{C}(y)$ respectively.

Definition 4.4. We consider the permutations $\Psi_{\mathcal{C}}$ and Ψ on the sets $\mathrm{Irr}(\mathcal{H}_{\boldsymbol{x}}(W))$ and $\mathrm{Irr}(\mathcal{H}_x(W))$ that are induced by the Galois automorphisms $\Phi_{\mathcal{C}}$ (for $\mathcal{C} \in \mathcal{A}/W$) and Φ:

$$\begin{aligned} &\Phi_{\mathcal{C}} \in \mathrm{Gal}\big(K(\boldsymbol{y})/K(\boldsymbol{x})\big) \qquad\qquad && \Phi \in \mathrm{Gal}\big(K(y)/K(x)\big) \\ &y_{\mathcal{C}} \to e^{2\pi i/N_W}\cdot y_{\mathcal{C}} && y \to e^{2\pi i/N_W}\cdot y \end{aligned}$$

In particular, they are defined via $\Psi_{\mathcal{C}}(\chi_{\boldsymbol{y}})(T_{\boldsymbol{g}}) := \Phi_{\mathcal{C}}\big(\chi_{\boldsymbol{y}}(T_{\boldsymbol{g}})\big)$ and similarly for Ψ. By Tits' deformation theorem, they induce permutations on the set $\widehat{W}$ of irreducible characters of W, which we also denote by $\Psi_{\mathcal{C}}$ and Ψ.

The following lemma is an almost immediate application of the character formulas (4.3) and (4.4), in conjunction with Tits' deformation theorem, and Definition 4.4:

Proposition 4.5 (The key technical lemma)**.** *[10, Prop. 4.19]*
Let g be a ζ-regular element of W, with $\zeta = e^{2\pi i l/d}$ of order d, $\chi \in \widehat{W}$ an irreducible character, and $\mathcal{C} \in \mathcal{A}/W$ an orbit of hyperplanes. Then, we have

$$\Psi_{\mathcal{C}}(\chi)(g) = \exp\big(-2\pi i\cdot\frac{lc_{\chi,\mathcal{C}}}{d}\big)\cdot\chi(g) \quad \textit{and} \quad \Psi(\chi)(g) = \exp\big(-2\pi i\cdot\frac{lc_{\chi}}{d}\big)\cdot\chi(g).$$

5 The weighted enumeration

The following section studies the weighted enumeration of reflection factorizations as considered in [9]. It provides a uniform proof of their result and extends it in a similar direction as with the Chapuy-Stump formula (1.3).

Definition 5.1. Consider a set of variables $\boldsymbol{w} := (w_{\mathcal{C}})_{(\mathcal{C}\in\mathcal{A}/W)}$ and a weight function

$$\mathrm{wt} : \mathcal{R} \to \{w_{\mathcal{C}} \mid \mathcal{C} \in \mathcal{A}/W\},$$

such that $\mathrm{wt}(t) = w_{\mathcal{C}}$ if $\mathcal{C}$ is the orbit that contains the fixed hyperplane V^t. Then, for a given element $g \in W$, we count its weighted reflection factorizations via:

$$\mathrm{FAC}_{W,g}(\boldsymbol{w}, z) := \sum_{\substack{(t_1,\cdots,t_N)\in\mathcal{R}^N \\ t_1\cdots t_N = g}} \mathrm{wt}(t_1)\cdots\mathrm{wt}(t_N)\cdot\frac{z^N}{N!}.$$

Because the sets $\mathcal{C}^{\mathrm{ref}} := \{t \in \mathcal{R} \mid V^t \in \mathcal{C}\}$ are closed under conjugation, the Lemma of Frobenius can again be used to express $\mathrm{FAC}_{W,g}(\boldsymbol{w}, z)$ as a finite sum of exponentials. Assuming that there are $r = |\mathcal{A}/W|$ different orbits of hyperplanes, denoted $\mathcal{C}_1, \cdots, \mathcal{C}_r$, Theorem 3.1 implies that

$$\mathrm{FAC}_{W,g}(\boldsymbol{w}, z) = \frac{1}{|W|}\sum_{\chi\in\widehat{W}}\chi(1)\cdot\chi(g^{-1})\cdot\exp\left[zw_{\mathcal{C}_1}\cdot\frac{\chi(\mathcal{C}_1^{\mathrm{ref}})}{\chi(1)}\right]\cdots\exp\left[zw_{\mathcal{C}_r}\cdot\frac{\chi(\mathcal{C}_1^{\mathrm{ref}})}{\chi(1)}\right].$$

By Definition 4.2 we can rewrite the quantities in the exponentials in terms of local Coxeter numbers. Indeed, we have $c_{\chi,\mathcal{C}} = |\mathcal{C}^{\mathrm{ref}}| - \chi(\mathcal{C}^{\mathrm{ref}})/\chi(1)$ and if we define $\mathrm{wt}(\mathcal{R}) := \sum_{t\in\mathcal{R}}\mathrm{wt}(t)$, the previous expression becomes a direct analog of (3.2) :

$$\mathrm{FAC}_{W,g}(\boldsymbol{w}, z) = \frac{e^{z\cdot\mathrm{wt}(\mathcal{R})}}{|W|}\sum_{\chi\in\widehat{W}}\chi(1)\cdot\chi(g^{-1})\cdot\left(e^{-zw_{\mathcal{C}_1}}\right)^{c_{\chi,\mathcal{C}_1}}\cdots\left(e^{-zw_{\mathcal{C}_r}}\right)^{c_{\chi,\mathcal{C}_r}}. \tag{5.1}$$

Lemma 5.2. *For a complex reflection group W, and a **regular** element $g \in W$, the total contribution in (5.1) of those characters $\chi \in \widehat{W}$ for which **any** $c_{\chi,\mathcal{C}}$ is not a multiple of $|g|$ is 0.*

Proof. We have to start by ordering the orbits $\mathcal{C} \in \mathcal{A}/W$ (arbitrarily); we will then apply the following idea sequentially: We partition the set of irreducible characters $\chi \in \widehat{W}$ into orbits under the action of $\Psi_{\mathcal{C}_1}$. Pick a character χ whose orbit is *not* a singleton and let k be the smallest number such that $\Psi^k_{\mathcal{C}_1}(\chi) = \chi$ (by Proposition 4.5, we will have $k = \frac{|g|}{\gcd(c_{\chi,\mathcal{C}_1}, |g|)}$). Now, the degrees of characters and the (local) Coxeter numbers are respected by $\Psi_{\mathcal{C}_1}$ (see [10, Prop. 4.18]), and so it is enough to show that

$$\sum_{j=1}^{k}\Psi^j_{\mathcal{C}_1}(\chi)(g^{-1}) = 0.$$

Indeed, this follows immediately from Proposition 4.5 as $\Psi^j_{\mathcal{C}_1}(\chi)(g^{-1}) = \xi^j\chi(g^{-1})$ for some k^{th} root of unity ξ. Notice now that we can continue with the *remaining* characters and the orbit $\mathcal{C}_2$ without worrying that we might eventually cancel the same character twice. $\square$

Before we proceed with our structural result for weighted enumeration formulas, we introduce the following combinatorial generalizations of the length function $l_R(g)$:

Definition 5.3. For an arbitrary element $g \in W$ and an orbit $\mathcal{C} \in \mathcal{A}/W$, we define $n_{\mathcal{C}}(g)$ to be the smallest number of reflections in $\mathcal{C}^{\mathrm{ref}}$ that may appear in *any* reflection factorization of g (i.e. not necessarily reduced).

Remark 5.4. Notice that it is not always true that $\sum n_{\mathcal{C}}(g) = l_R(g)$. Indeed, the element $g := (12\bar{1}\bar{2}) = -\mathbf{1}$ in B_2 (which is the square of the Coxeter element) can be written both as $g = (12)(1\bar{2})$ and as $g = (1\bar{1})(2\bar{2})$, so that $n_1(g) = n_2(g) = 0$.

Theorem 5.5. *For a complex reflection group W and a regular element $g \in W$, the exponential generating function $\mathrm{FAC}_{W,g}(\boldsymbol{w}, z)$ of weighted reflection factorizations of g takes the form:*

$$\mathrm{FAC}_{W,g}(\boldsymbol{w}, z) = \frac{e^{z\cdot \mathrm{wt}(\mathcal{R})}}{|W|} \cdot \Big[\Phi(\boldsymbol{X}) \cdot \prod_{\mathcal{C}\in\mathcal{A}/W} (1 - X_{\mathcal{C}})^{n_{\mathcal{C}}(g)}\Big]\Big|_{X_{\mathcal{C}} = e^{-zw_{\mathcal{C}}|g|}}.$$

Here, $\Phi(\boldsymbol{X})$ is a polynomial of degree $(e_{\mathcal{C}} \cdot \omega_{\mathcal{C}})/|g| - n_{\mathcal{C}}(g)$ on each of its variables $X_{\mathcal{C}}$, it has constant term $\Phi(\mathbf{0}) = 1$, and it is not further divisible by any $(1 - X_{\mathcal{C}})$. The exponents satisfy

$$\frac{e_{\mathcal{C}}\omega_{\mathcal{C}}}{|g|} \;\geq\; n_{\mathcal{C}}(g) \;\geq\; l_R(g) - \frac{|\mathcal{R}| + |\mathcal{A}| - e_{\mathcal{C}}\omega_{\mathcal{C}}}{|g|}.$$

Remark: The proof [10, Thm. 5.5] is very similar to that of Theorem 3.6. One significant difference is that at some point it becomes necessary to write a generating function for factorizations (of arbitrary length) that have precisely $n_{\mathcal{C}}$ reflections from the orbit $\mathcal{C}$. This is done again through the Lemma of Frobenius. □

Corollary 5.6. *For a complex reflection group W and a regular element $g \in W$ of order $|g| = d_n$, the weighted reflection factorizations of g are counted by the formula:*

$$\mathrm{FAC}_{W,g}(\boldsymbol{w}, z) = \frac{e^{z\cdot \mathrm{wt}(\mathcal{R})}}{|W|} \cdot \prod_{\mathcal{C}\in\mathcal{A}/W} \big(1 - e^{-zw_{\mathcal{C}}|g|}\big)^{n_{\mathcal{C}}(g)},$$

where the exponents are explicitly given by $n_{\mathcal{C}}(g) = (e_{\mathcal{C}}\omega_{\mathcal{C}})/|g|$.

Remark: This is essentially the same argument as in Corollary 3.8. □

Remark 5.7. For well-generated groups W, we always have $|c| = d_n$ so that the previous statement recovers the main theorem of [9] and extends it to the groups with d_n regular. Notice that while in well-generated groups we have at most two orbits of hyperplanes, the groups $G_7, G_{11}, G_{15}, G_{19}$ have three orbits. For all of them but G_{15}, d_n is regular.

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References

[1] D. Bessis. "Zariski theorems and diagrams for braid groups". *Invent. Math.* **145**.3 (2001), pp. 487–507. Link.

[2] D. Bessis. "Finite complex reflection arrangements are $K(\pi, 1)$". *Ann. of Math. (2)* **181**.3 (2015), pp. 809–904. Link.

[3] W. M. Beynon and G. Lusztig. "Some numerical results on the characters of exceptional Weyl groups". *Math. Proc. Cambridge Philos. Soc.* **84**.3 (1978), pp. 417–426. Link.

[4] C. Boura, E. Chavli, M. Chlouveraki, and K. Karvounis. "The BMM symmetrising trace conjecture for groups G_4, G_5, G_6, G_7, G_8". 2018. arXiv:1802.07482.

[5] M. Broué. *Introduction to Complex Reflection Groups and their Braid Groups*. Lecture Notes in Mathematics 1988. Springer-Verlag, Berlin, 2010.

[6] M. Broué, G. Malle, and R. Rouquier. "Complex reflection groups, braid groups, Hecke algebras". *J. Reine Angew. Math.* **500** (1998), pp. 127–190.

[7] M. Broué and J. Michel. "Sur certains éléments réguliers des groupes de Weyl et les variétés de Deligne-Lusztig associées". *Finite Reductive Groups (Luminy, 1994)*. Progr. Math. 141. Birkhäuser, Boston, MA, 1997, pp. 73–139.

[8] G. Chapuy and C. Stump. "Counting factorizations of Coxeter elements into products of reflections". *J. Lond. Math. Soc. (2)* **90**.3 (2014), pp. 919–939. Link.

[9] E. delMas, T. Hameister, and V. Reiner. "A refined count of Coxeter element reflection factorizations". *Electron. J. Combin.* **25**.1 (2018), Paper 1.28, 11 pp. Link.

[10] T. Douvropoulos. "On enumerating factorizations in reflection groups". 2018. arXiv:1811.06566.

[11] I. G. Gordon and S. Griffeth. "Catalan numbers for complex reflection groups". *Amer. J. Math.* **134**.6 (2012), pp. 1491–1502. Link.

[12] D. M. Jackson. "Some combinatorial problems associated with products of conjugacy classes of the symmetric group". *J. Combin. Theory Ser. A* **49**.2 (1988), pp. 363–369. Link.

[13] S. K. Lando and A. K. Zvonkin. *Graphs on Surfaces and their Applications*. Encyclopaedia of Mathematical Sciences 141. Springer-Verlag, Berlin, 2004. Link.

[14] G. I. Lehrer and D. E. Taylor. *Unitary Reflection Groups*. AustMS Lecture Series 20. Cambridge University Press, Cambridge, 2009.

[15] G. Malle. "On the rationality and fake degrees of characters of cyclotomic algebras". *J. Math. Sci. Univ. Tokyo* **6**.4 (1999), pp. 647–677.

[16] J. Michel. "Deligne-Lusztig theoretic derivation for Weyl groups of the number of reflection factorizations of a Coxeter element". *Proc. Amer. Math. Soc.* **144**.3 (2016), pp. 937–941. Link.