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123, 2143-avoiding Kazhdan–Lusztig immanants and k-positive matrices

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Abstract. Immanants are functions on square matrices generalizing the determinant and permanent. Stembridge showed that irreducible character immanants are nonnegative on totally nonnegative matrices. Rhoades and Skandera later defined Kazhdan–Lusztig immanants, using specializations of Kazhdan–Lusztig polynomials at 1; results of (Stembridge, 1992) and (Haiman, 1993) show that these are also nonnegative on totally nonnegative immanants. Here, we give conditions on $v \in S_n$ so that the Kazhdan–Lusztig immanant corresponding to v is positive on k-positive matrices.

Keywords: immanants, total positivity

1 Introduction

Given a function $f : S_n \to \mathbb{C}$, the *immanant* associated to f, $\mathrm{Imm}_f : \mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathbb{C}$, is the function

$$M = (m_{ij})_{1\le i,j\le n} \mapsto \mathrm{Imm}_f(M) := \sum_{w\in S_n} f(w)\, m_{1,w(1)} \cdots m_{n,w(n)}. \tag{1.1}$$

Well-studied examples include the determinant, where $f(w) = (-1)^{\ell(w)}$, the permanent, where $f(w) = 1$, and more generally *character immanants*, where f is an irreducible character of S_n.

We will be interested in immanants evaluated on matrices that meet certain positivity conditions.

Definition 1.1. Let $M \in \mathrm{Mat}_{n\times n}(\mathbb{C})$. We call M *k-positive* (respectively, *k-nonnegative*) if all minors of size at most k are positive real numbers (respectively, nonnegative real numbers). If M is n-positive (respectively, n-nonnegative), we also call M *totally positive* (respectively, *totally nonnegative*).

Notice that the entries of M are 1×1 minors, so if M is k-nonnegative for any $k \ge 1$, it has real entries.

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Example 1.2. The matrix

$$\begin{bmatrix} 11 & 9 & 3 \\ 8 & 7 & 3 \\ 2 & 2 & 1 \end{bmatrix}$$

is 2-positive but has negative determinant, so is not 3-nonnegative (totally nonnegative).

The positivity properties of immanants have been of interest since the early 1990s. In [6], Goulden and Jackson conjectured (and Greene [7] later proved) that character immanants of Jacobi–Trudi matrices are polynomials with nonnegative coefficients. This was followed by a number of positivity conjectures by Stembridge [16], including two that were proved shortly thereafter: Haiman showed that character immanants of generalized Jacobi–Trudi matrices are Schur-positive [8] and Stembridge showed that character immanants of totally nonnegative matrices are nonnegative [17].

In [16], Stembridge also asks if certain monomial immanants are nonnegative on k-nonnegative matrices. More generally, it is natural to ask what one can say about the signs of immanants on k-nonnegative matrices. Stembridge's proof in [17] does not extend to k-nonnegative matrices, as it relies on the existence of a certain factorization for totally nonnegative matrices which does not exist for all k-nonnegative matrices.

Here, we will focus on the signs of *Kazhdan–Lusztig immanants*, which were defined by Rhoades and Skandera [13].

Before giving the definition of Kazhdan–Lusztig immanants, we briefly review some basic notions from Coxeter theory, which we will need in what follows. For details, see e.g. [2]. For $1 \leq i \leq n-1$, let $s_i \in S_n$ be the permutation exchanging i and $i+1$ and fixing all other elements of $\{1, \ldots, n\}$. We call $s_1, \ldots, s_{n-1}$ *simple transpositions*. Any permutation $v \in S_n$ can be written as a product of simple transpositions. If an expression for v as a product of simple transpositions uses the smallest possible number of simple transpositions, it is called a *reduced expression*. The *length* of v, denoted $\ell(v)$, is the number of simple transpositions in a reduced expression for v. The length of v is also the number of inversions of v (that is, the number of pairs of integers $1 \leq i < j \leq n$ such that $v(i) > v(j)$). There is a unique element of S_n of maximum length, which is $w_0 := n\ (n-1)\ \ldots\ 2\ 1$. For $w, v \in S_n$, v is smaller than w in the (strong) Bruhat order, which we denote by $v \leq w$, if some (equivalently, every) reduced expression for w contains as a subexpression a reduced expression for v.

Definition 1.3. Let $v \in S_n$. The *Kazhdan–Lusztig immanant* $\mathrm{Imm}_v : \mathrm{Mat}_{n \times n}(\mathbb{C}) \to \mathbb{C}$ is given by

$$\mathrm{Imm}_v(M) := \sum_{w \in S_n} (-1)^{\ell(w)-\ell(v)} P_{w_0 w, w_0 v}(1)\ m_{1,w(1)} \cdots m_{n,w(n)} \tag{1.2}$$

where $P_{x,y}(q)$ is the Kazhdan–Lusztig polynomial associated to $x, y \in S_n$ and $w_0 \in S_n$ is the longest permutation.

For the definition of Kazhdan–Lusztig polynomials and their basic properties, see e.g. [2]. It is a fact that $P_{x,y}(q) = 0$ if x is not less than or equal to y in the Bruhat order, and that $P_{x,w_0}(q) = 1$ for all $x \in S_n$. Using these two facts, we can compute that, for example, $\mathrm{Imm}_e(M) = \det M$ and $\mathrm{Imm}_{w_0}(M) = m_{n,1}m_{n-1,2}\cdots m_{1,n}$.

Using results of [8, 17], Rhoades and Skandera [13] show that Kazhdan–Lusztig immanants are nonnegative on totally nonnegative matrices, and are Schur-positive on generalized Jacobi–Trudi matrices. Further, they show that character immanants are nonnegative linear combinations of Kazhdan–Lusztig immanants, so from the perspective of positivity, Kazhdan–Lusztig immanants are the more fundamental object to study.

We will call an immanant *k-positive* if it is positive on all k-positive matrices. We are interested in the following question.

Question 1.4. Let $0 < k < n$ be an integer. For which $v \in S_n$ is $\mathrm{Imm}_v(M)$ k-positive?

Notice that $\mathrm{Imm}_e(M) = \det M$ is k-positive only for $k = n$. On the other hand, Imm_{w_0} is k-positive for all k, since it is positive as long as the entries (i.e. the 1×1 minors) of M are positive. So, the answer to Question 1.4 is a nonempty proper subset of S_n.

Pylyavskyy conjectured that there is a link between $\mathrm{Imm}_v(M)$ being k-positive and v avoiding certain patterns.

Definition 1.5. Let $v \in S_n$, and let $w \in S_m$. Suppose $v = v_1 \cdots v_n$ and $w = w_1 \cdots w_m$ in one-line notation. We say that the pattern $w_1 \cdots w_m$ *occurs* in v if there exist $1 \leq i_1 < \cdots < i_m \leq n$ such that $v_{i_1} \cdots v_{i_m}$ are in the same relative order as $w_1 \cdots w_m$. We say v *avoids* the pattern $w_1 \cdots w_m$ if it does not occur in v.

More precisely, Pylyavskyy conjectured the following.

Conjecture 1.6 ([12])**.** *Let $0 < k < n$ be an integer and let $v \in S_n$ avoid $12 \cdots (k+1)$. Then $\mathrm{Imm}_v(M)$ is k-positive.*

Our main result is a description of some k-positive Kazhdan–Lusztig immanants, in the spirit of Pylyavskyy's conjecture.

Theorem 1.7. *Let $0 < k < n$ be an integer and let $v \in S_n$ be 123-, 2134-, $1(2k)(2k-1)\ldots2$-, and $(2k-1)(2k-2)\ldots1(2k)$-avoiding. Then $\mathrm{Imm}_v(M)$ is k-positive.*

Example 1.8. Consider $v = 2413$ in S_4. It avoids the patterns 123, 2134, 1432, and 3214, so Theorem 1.7 guarantees that

$$\begin{aligned}\mathrm{Imm}_v(M) = {} & m_{12}m_{24}m_{31}m_{43} - m_{14}m_{22}m_{31}m_{43} - m_{13}m_{24}m_{31}m_{42} + m_{14}m_{23}m_{31}m_{42} \\ & - m_{12}m_{24}m_{33}m_{41} + m_{14}m_{22}m_{33}m_{41} + m_{13}m_{24}m_{32}m_{41} - m_{14}m_{23}m_{32}m_{41}\end{aligned}$$

is positive on all 2-positive 4×4 matrices.

Remark 1.9. For $k = 2$, Theorem 1.7 concerns permutations which are 123-, 2134-, 1432-, and 3214-avoiding. According to [19], the number of such permutations in S_n is given by the coefficient of x^n in the power series

$$\sum_{i=0}^{\infty} c_n x^n = \frac{1-x}{1-2x-x^3}.$$

For other k, the number of permutations avoiding 123, 2134, $1(2k)(2k-1)\ldots 2$, and $(2k-1)(2k-2)\ldots 1(2k)$ is not known.

Note that Theorem 1.7 supports Conjeture 1.6. Indeed, if $k = 1$, the $(2k-1)(2k-2)\ldots 1(2k)$-avoiding condition reduces to avoiding 12. For $k \geq 2$, the 123-avoiding condition ensures that v also avoids $12\cdots(k+1)$.

Before outlining the the proof of Theorem 1.7, we would like to provide some additional motivation and context for Question 1.4. For an arbitrary reductive group G, Lusztig [11] defined the totally positive part $G_{>0}$ and showed that elements of the dual canonical basis of $\mathcal{O}(G)$ are positive on $G_{>0}$. Fomin and Zelevinsky [5] later showed that for semisimple groups, $G_{>0}$ is characterized by the positivity of generalized minors, which are dual canonical basis elements corresponding to the fundamental weights of G and their images under Weyl group action. Note that the generalized minors are a finite subset of the (infinite) dual canonical basis, but their positivity guarantees the positivity of all other elements of the basis.

In the case we are considering, $G = GL_n(\mathbb{C})$, $G_{>0}$ consists of the totally positive matrices and generalized minors are just ordinary minors. Skandera [15] showed that Kazhdan–Lusztig immanants are part of the dual canonical basis of $\mathcal{O}(GL_n(\mathbb{C}))$, which gives another perspective on their positivity properties. (In fact, Skandera proved that every dual canonical basis element can be obtained from a Kazhdan–Lusztig immanant evaluated on matrices with repeated rows and columns.) In light of these facts, Question 1.4 becomes a question of the following kind.

Question 1.10. Suppose some finite subset S of the dual canonical basis is positive on $M \in G$. Which other elements of the dual canonical basis are positive on M? In particular, what if S consists of the generalized minors corresponding to the first k fundamental weights and their images under the Weyl group action?

These questions have a similar flavor to positivity tests arising from cluster algebras, which is different than the approach we take here. The coordinate ring of GL_n is a cluster algebra, with some clusters given by double wiring diagrams [1]. The minors are cluster variables. If we restrict our attention to the minors of size at most k in the clusters for GL_n, we obtain a number of sub-cluster algebras, investigated by the first author in [3]. The cluster monomials in those sub-algebras will be positive on k-positive matrices. Thus, one strategy to show $\mathrm{Imm}_v(M)$ is k-positive is to show it is a cluster monomial in a sub-cluster algebra. Interestingly, the Kazhdan–Lusztig immanants of 123-, 2143-, 1423-, and 3214-avoiding permutations do appear in sub-cluster algebras of this kind for $k = 2$. In general, however, it is not known if Imm_v is a cluster variable in the cluster structure on GL_n, or in the sub-cluster algebras using only minors of size at most k. It is conjectured that cluster monomials form a (proper) subset of the dual canonical basis, so the cluster algebra approach would at best provide a partial answer to Question 1.10.

2 Results

To prove Theorem 1.7, we first note that Equation (1.2) has a much simpler form when v is 1324- and 2143-avoiding. By [9], $P_{x,y}(q)$ is the Poincaré polynomial of the local intersection cohomology of the Schubert variety indexed by y at any point in the Schubert variety indexed by x; by [10], the Schubert variety indexed by y is smooth precisely when y is 4231- and 3412-avoiding. These results imply that $P_{x,y}(q) = 1$ for y avoiding 4231 and 3412. Together with the fact that $P_{x,y}(q) = 0$ for $x \nleq y$ in the Bruhat order, this gives the following lemma.

Lemma 2.1. *Let $v \in S_n$ be* 1324- *and* 2143*-avoiding. Then*

$$\mathrm{Imm}_v(M) = (-1)^{\ell(v)} \sum_{w \geq v} (-1)^{\ell(w)} \, m_{1,w(1)} \cdots m_{n,w(n)}. \tag{2.1}$$

The coefficients in the formula in Lemma 2.1 suggest a strategy for analyzing $\mathrm{Imm}_v(M)$ for $v \in S_n$ 1324- and 2143-avoiding: find some matrix $\tilde{M}$ such that $\det(\tilde{M}) = \pm \mathrm{Imm}_v(M)$. If such a matrix $\tilde{M}$ exists, the sign of $\mathrm{Imm}_v(M)$ is the sign of some determinant, which we have tools (e.g. the Desnanot–Jacobi identity) to analyze. The most straightforward candidate for $\tilde{M}$ is a matrix obtained from M by replacing some entries with 0.

Definition 2.2. Let $P \subseteq \{1, \dots, n\}^2$ and let $M = (m_{ij})$ be in $\mathrm{Mat}_{n \times n}(\mathbb{C})$. The *restriction of M to P*, denoted $M|_P$, is the matrix with entries

$$\tilde{m}_{ij} = \begin{cases} m_{ij} & \text{if } (i,j) \in P \\ 0 & \text{else.} \end{cases}$$

For a fixed $v \in S_n$ that avoids 1324 and 2143, suppose there exists $P \subseteq \{1, \dots, n\}^2$ such that $\mathrm{Imm}_v(M) = \pm \det M|_P$. Given the terms appearing in Equation (2.1), P must include $\{(i, w(i)) : w \in [v, w_0]\}$, the union of graphs of permutations in the Bruhat interval $[v, w_0]$. In fact, the minimal choice of P suffices.

Proposition 2.3. *Let $v \in S_n$ be* 1324- *and* 2143*-avoiding, and let $M \in Mat_{n \times n}(\mathbb{C})$. Denote by* $\mathrm{gr}[v, w_0] := \{(i, w(i)) : w \in [v, w_0]\}$. *Then*

$$\det(M|_{\mathrm{gr}[v,w_0]}) = \sum_{w \geq v} (-1)^{\ell(w)} \, m_{1,w(1)} \cdots m_{n,w(n)}.$$

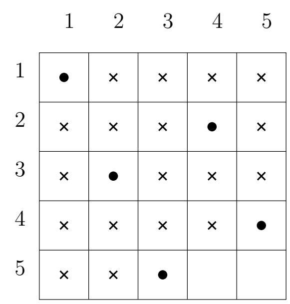


Figure 1: An example of $\mathrm{gr}[v, w_0]$ for $v = 14253$. Dots mark the positions $(i, v(i))$, and crosses mark all other elements of $\mathrm{gr}[v, w_0]$.

Before discussing the proof of Proposition 2.3, we mention that $\mathrm{gr}[v, w_0]$ is easy to characterize as a subset of $\mathbb{Z}^2$.

Lemma 2.4. *Let $v \in S_n$. Then $\mathrm{gr}[v, w_0]$ consists exactly of the points (i, j) such that there exist $k, \ell \in \{1, \ldots, n\}$ where $k \leq i \leq \ell$ and $v_k \leq j \leq v_\ell$.*

Remark 2.5. Index the rows and columns of an $n \times n$ grid so that row indices increase going down and column indices increase going right. Lemma 2.4 gives a method for drawing $\mathrm{gr}[v, w_0]$ in the grid. First, put a dot in positions $(i, v(i))$ for $i = 1, \ldots, n$. Then, put a cross in all positions that are weakly southeast of one dot and weakly northwest of another (see Figure 1).

Returning to Proposition 2.3, notice that by definition,

$$\det(M|_{\mathrm{gr}[v,w_0]}) = \sum_{\substack{w \in S_n \\ \mathrm{gr}(w) \subseteq \mathrm{gr}[v,w_0]}} (-1)^{\ell(w)}\, m_{1,w(1)} \cdots m_{n,w(n)} \tag{2.2}$$

where $\mathrm{gr}(w) := \{(i, w(i)) : i = 1, \ldots, n\}$ is the graph of w. So the content of Proposition 2.3 is that for permutations v avoiding the appropriate patterns, $[v, w_0] = \{w \in S_n : \mathrm{gr}(w) \subseteq \mathrm{gr}[v, w_0]\}$. This does not hold for arbitrary permutations v, as the following example shows.

Example 2.6. Consider $v = 14253$. Then $x = 12453$ is not in the Bruhat interval $[v, w_0]$, but $\mathrm{gr}(x) \subseteq \mathrm{gr}[v, w_0]$. This can easily be seen from Figure 1.

Proposition 2.3 follows from Lemma 2.4 and a result of Sjöstrand [14]. Lemma 2.4 shows that for arbitrary $v \in S_n$, $\mathrm{gr}[v, w_0]$ is equal to what Sjöstrand calls the "right convex hull" of v. Theorem 4 of [14] establishes that the right convex hull of v contains only the graphs of $w \geq v$ if and only if v avoids the patterns 1324, 24153, 31524, and 426153. The latter 3 patterns contain an occurrence of 2143, so if v avoids 2143, it also surely avoids these patterns.

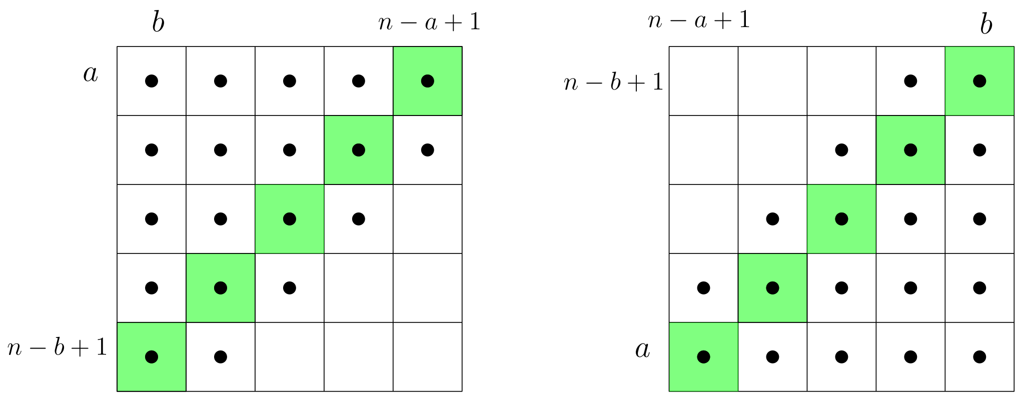


Figure 2: $P^{\ell}_{a,b}$ (on the left) and $P^{r}_{a,b}$ (on the right). The boxes on the antidiagonal are shown in green.

In light of Proposition 2.3, it remains to analyze the sign of $\det(M|_{\mathrm{gr}[v,w_0]})$. In particular, we would like to find some k for which $\det(M|_{\mathrm{gr}[v,w_0]})$ has the same sign for all k-positive matrices M. We will do this in the special case when v avoids 123 and 2143 by describing $\mathrm{gr}[v,w_0]$ more explicitly as a region in an $n \times n$ grid. We restrict to this case so we can use a result of Tenner:

Theorem 2.7 ([18])**.** *A permutation $y \in S_n$ avoids 321 and 3412 if and only if in every (equivalently, one) reduced expression for y, each simple transposition appears at most once.*

If v avoids 123 and 2143, then w_0v will avoid 321 and 3412. So, we can use Tenner's result to describe $\mathrm{gr}[v,w_0]$ for v 123- and 2143-avoiding. The building blocks for these graphs are pentagonal collections of dots, which we define next.

Definition 2.8. Let $0 < a,b \leq n$. If (a,b) is above the antidiagonal, we define $P^{\ell}_{a,b}$ as the left-justified collection of dots in rows a through $n-b+1$ where row i has $\min\{n-b-i+3, n-b-a+2\}$ dots and the first dot in each row is in column b. Similarly, if (a,b) is below the antidiagonal, $P^{r}_{a,b}$ is the right-justified collection of dots in rows $n-b+1$ through a where row i has $\min\{b-n+i+1, b-n+a\}$ dots and the last dot in each row is in column b. We call $P^{\ell}_{a,b}$ and $P^{r}_{a,b}$ *pentagonal shapes*. See Figure 2 for an example.

Notice that $P^{\ell}_{a,b}$ is related to $P^{r}_{a,b}$ by reflection across the antidiagonal.

Proposition 2.9. *Let $n > 2$. Suppose $v \in S_n$ avoids 123 and 2143 and $\ell(v) = \binom{n-1}{2}$. Define $L := \{i : v(i) < n-i\}$ and $R := \{i : v(i) > n-i+2\}$. Then*

$$\mathrm{gr}[v,w_0] = \left(\bigcup_{i \in L} P^{\ell}_{i,v(i)}\right) \cup \left(\bigcup_{i \in R} P^{r}_{i,v(i)}\right).$$

Moreover, the pentagonal shapes alternate between $P^{\ell}_{a,b}$ and $P^{r}_{c,d}$ down the antidiagonal, and $P^{\ell}_{a,b}$ intersects $P^{r}_{c,d}$ either nowhere or in a 2×2 square of dots.

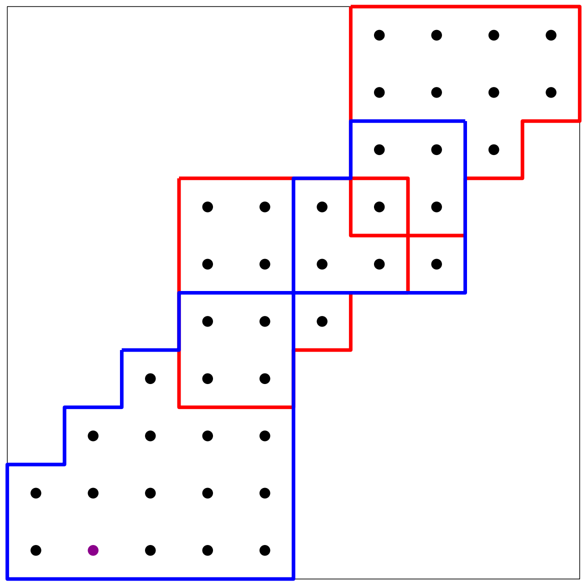


Figure 3: Let $v = 4\ 10\ 9\ 2\ 8\ 6\ 3\ 2\ 1\ 5$. It is 123- and 2143-avoiding and every simple transposition occurs once in each reduced expression for $w_0 v$. The decomposition of $\mathrm{gr}[v, w_0]$ into a union of pentagon shapes, guaranteed by Proposition 2.9, is shown here. All $P^{\ell}_{a,b}$'s are shown in red; all $P^{r}_{a,b}$'s are shown in blue.

We have the following immediate corollary.

Corollary 2.10. *Let $v \in S_n$ avoid 123 and 2143. Then $\mathrm{gr}[v, w_0]$ is block antidiagonal. Consider each minimal block of $\mathrm{gr}[v, w_0]$ as a $m \times m$ grid. Then either the block consists of a single dot, consists of a 2×2 square of dots, or has shape described by Proposition 2.9.*

With this description of $\mathrm{gr}[v, w_0]$ we can prove the following.

Theorem 2.11. *Let $v \in S_n$ be 123- and 2143-avoiding, and suppose k is the size of the largest square appearing in $\mathrm{gr}[v, w_0]$. Then $\mathrm{Imm}_v(M)$ is k-positive.*

Proof sketch. By Proposition 2.3, it suffices to show that $\det(M|_{\mathrm{gr}[v,w_0]})$ is nonzero and has sign $\ell(v)$. We prove this first for v such that $\mathrm{gr}[v, w_0]$ has a single antidiagonal block, in which case $\ell(v) = n - 1$. The proof is by induction, and relies on the Desnanot–Jacobi identity (also known as Dodgson condensation), which is as follows.

For $N \in \mathrm{Mat}_{n \times n}(\mathbb{C})$ and $I, J \subseteq \{1, \ldots, n\}$, let N_I^J denote the submatrix obtained by removing rows indexed by I and columns indexed by J. Then for $1 \le a < a' \le n$ and $1 \le b < b' \le n$, we have

$$\det N \det N_{a,a'}^{b,b'} = \det N_a^b \det N_{a'}^{b'} - \det N_a^{b'} \det N_{a'}^{b}. \tag{2.3}$$

Proposition 2.9 plays a crucial role in understanding the determinants of the matrices appearing in the Desnanot–Jacobi identity for $N = M|_{\mathrm{gr}[v,w_0]}$.

Once we know the case when $\mathrm{gr}[v, w_0]$ is a single block, we use Corollary 2.10 and the following lemma to establish the result for all v avoiding 123 and 2143.

Lemma 2.12. *Let $M \in Mat_{n\times n}(\mathbb{C})$ be block-antidiagonal, with blocks $M_1, \dots, M_r$ of size $n_1, \dots n_r$. Then*

$$\det M = (-1)^{\binom{n}{2}} \prod_{i=1}^{r} (-1)^{\binom{n_i}{2}} \det M_i.$$

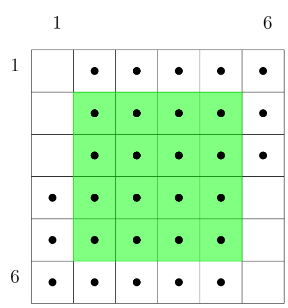
□

To obtain Theorem 1.7 from Theorem 2.11, we simply translate the condition "k is the size of the largest square appearing in $\mathrm{gr}[v, w_0]$" into pattern avoidance language.

The assumptions of Theorem 2.11 are stronger than those of Proposition 2.3, so it's natural to conjecture that they can be weakened. In work following the completion of this abstract [4], we prove the following statement.

Theorem 2.13 ([4])**.** *Suppose $v \in S_n$ avoids 1324 and 2143, and let k be the size of the largest square in $\mathrm{gr}[v, w_0]$. Then $\mathrm{Imm}_v(M)$ is k-positive.*

Example 2.14. Let $v = 236145$. Then $\mathrm{gr}[v, w_0]$ is the following grid:



The largest square in $\mathrm{gr}[v, w_0]$ is of size 4 (one choice of such a square is highlighted in green). Since v avoids 1324 and 2143, we know that $\mathrm{Imm}_v(M) = (-1)^{\ell(v)} \det(M|_{\mathrm{gr}[v,w_0]})$. Theorem 2.13 states that if M is 4-positive, then $\mathrm{Imm}_v(M)$ is positive.

To prove Theorem 2.13, we again use the Desnanot–Jacobi identity to determine the sign of $\det(M|_{\mathrm{gr}[v,w_0]})$ for v and M satisfying the appropriate conditions. The key observation is that, for certain choices of deleted rows and columns, every term of the Desdanot–Jacobi identity is of the form $\det(M|_{\mathrm{gr}[v',w_0]})$ for some v' which avoids 1324 and 2143. After this observation, the argument proceeds by induction.

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