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Divided symmetrization and quasisymmetric functions

Philippe Nadeau*[1] and Vasu Tewari†[2]

[1] *Univ. Lyon, CNRS, Université Claude Bernard Lyon 1, UMR 5208, Institut Camille Jordan, France*
[2] *Department of Mathematics, University of Pennsylvania, PA, USA*

Abstract. We study various aspects of the divided symmetrization operator, which was introduced by Postnikov in the context of volume polynomials of permutahedra. Divided symmetrization is a linear form which acts on the space of polynomials in n indeterminates of degree $n-1$. Our main results are related to quasisymmetric polynomials. We show that divided symmetrization applied to a quasisymmetric polynomial in $m \leq n$ indeterminates has a natural interpretation. We further show that divided symmetrization of any polynomial can be naturally computed with respect to a direct sum decomposition due to Aval–Bergeron–Bergeron, involving the ideal generated by positive degree quasisymmetric polynomials in n indeterminates. Our main motivation for studying divided symmetrization comes from studying the cohomology class of the Peterson variety.

Keywords: divided symmetrization, quasisymmetric functions

1 Introduction

Toward computing volume polynomials of permutahedra, Postnikov [10] introduced an operator called divided symmetrization (abbreviated to DS in the sequel on occasion). It takes a polynomial $f(x_1,\ldots,x_n)$ as input and outputs a symmetric polynomial $\langle f(x_1,\ldots,x_n)\rangle_n$ defined by

$$\langle f(x_1,\ldots,x_n)\rangle_n := \sum_{w\in S_n} w\cdot\left(\frac{f(x_1,\ldots,x_n)}{\prod_{1\leq i\leq n-1}(x_i-x_{i+1})}\right),$$

where S_n denotes the symmetric group on n letters, naturally acting by permuting variables. It can be checked that divided symmetrization acting on homogeneous polynomials of degree strictly less than $n-1$ results in 0. Thus the degree $n-1$ case is the first non-trivial case, and *in this context divided symmetrizations results in scalars*.

*nadeau@math.univ-lyon1.fr.
†vvtewari@math.upenn.edu. Partially supported by the AMS-Simons travel grant

A pertinent and motivating instance is the following. Given $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$, the permutahedron $\mathcal{P}_{\mathbf{a}}$ is the convex hull of all points of the form $(a_{w(1)}, \ldots, a_{w(n)})$ where w ranges over all permutations in S_n. Postnikov [10, Section 3] shows that the volume of $\mathcal{P}_{\mathbf{a}}$ is given by $\frac{1}{(n-1)!}\left\langle (a_1x_1 + \cdots + a_nx_n)^{n-1}\right\rangle_n$.

While a great deal of research has been conducted into various aspects of permutahedra, especially in regard to volumes and lattice point enumeration, divided symmetrization has received limited attention; see [1, 9].

Our own motivation for studying divided symmetrization stems from a problem in Schubert calculus, which we sketch now. The *flag variety* $Fl(n)$ is a complex projective variety structure on the set of complete flags, which are sequences $F_0 = \{0\} \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n = \mathbb{C}^n$ of subspaces such that $\dim F_i = i$. The Schubert varieties $X_w \subset Fl(n)$, for $w \in S_n$, give rise to the basis of Schubert classes σ_w in the integral cohomology $H^*(Fl(n))$. The *Peterson variety* Pet_n is the subvariety of $Fl(n)$ of dimension $n-1$ comprised of flags $(F_i)_{i=0,\ldots,n}$ such that $NF_i \subset F_{i+1}$ for $i < n$, where N is the $n \times n$ matrix with ones above the diagonal and zeros everywhere else. It is a special case of a regular nilpotent Hessenberg variety.

Our initial problem was to compute the number a_w of points in the intersection of Pet_n with a generic translate of a Schubert variety X_w, for w of length $n-1$. Equivalently, a_w is the coefficient of the class $[Pet_n] \in H^*(Fl(n))$ on the class σ_w. We can then show that a_w is given by $\left\langle \mathfrak{S}_w(x_1, \ldots, x_n)\right\rangle_n$ where $\mathfrak{S}_w$ is the celebrated Schubert polynomial attached to w.

The results presented in this extended abstract are thus primarily motivated by understanding the divided symmetrization of Schubert polynomials. The reader is referred to [8] for the version with proofs. We aim to understand more about the structure of the divided symmetrization operator acting on polynomials of degree $n-1$, since Postnikov's work and our own work coming from Schubert calculus both have this condition. Our investigations allows us to uncover a direct (and intriguing) connection between divided symmetrization and quasisymmetric polynomials.

The ring of quasisymmetric functions in infinitely many variables $\mathbf{x} = \{x_1, x_2, \ldots\}$ was introduced by Gessel [5] and has since acquired great importance in algebraic combinatorics. A distinguished linear basis for this ring is given by the fundamental quasisymmetric functions F_α where α is a composition. Given a positive integer n, consider a quasisymmetric function $f(\mathbf{x})$ of degree $n-1$. We denote the quasisymmetric polynomial obtained by setting $x_i = 0$ for all $i > m$ by $f(x_1, \ldots, x_m)$ and refer to the evaluation of $f(x_1, \ldots, x_m)$ at $x_1 = \cdots = x_m = 1$ by $f(1^m)$. Our first main result states the following:

Theorem 1.1. *For a quasisymmetric function f of degree $n-1$, we have*

$$\sum_{j\geq 0} f(1^j)t^j = \frac{\sum_{m=0}^{n}\left\langle f(x_1, \ldots, x_m)\right\rangle_n t^m}{(1-t)^n}.$$

A natural candidate for f comes from Stanley's theory of P-partitions [11, 12]: To any naturally labeled poset P on $n-1$ elements, one can associate a quasisymmetric function $K_P(\mathbf{x})$ with degree $n-1$. Let $\mathcal{L}(P)$ denote the set of linear extensions of P. Note that elements in $\mathcal{L}(P)$ are permutations in S_{n-1}. Under this setup, we obtain the following corollary of Theorem 1.1.

Corollary 1.2. *For $m \leq n$, we have*

$$\left\langle K_P(x_1,\ldots,x_m)\right\rangle_n = |\{\pi \in \mathcal{L}(P) \mid \ \pi \text{ has } m-1 \text{ descents}\}|.$$

We further establish connections between a quotient ring of polynomials investigated by [4, 3]. Let $\mathcal{J}_n$ denote the ideal in $\mathbb{Q}[\mathbf{x}_n] := \mathbb{Q}[x_1,\ldots,x_n]$ generated by homogeneous quasisymmetric polynomials in $x_1,\ldots,x_n$ of positive degree. Let R_n be the degree $n-1$ homogeneous component of $\mathbb{Q}[\mathbf{x}_n]$, and let $K_n := R_n \cap \mathcal{J}_n$. Aval–Bergeron–Bergeron [3] provide a distinguished basis for a certain complementary space $K_n^\dagger$ of K_n in R_n, defined in Section 5. This leads to our second main result.

Theorem 1.3. *If $f \in K_n$, then $\langle f\rangle_n = 0$. More generally, if $f \in R_n$ is written $f = g + h$ with $g \in K_n^\dagger$ and $h \in K_n$ according to (5.1), then*

$$\langle f\rangle_n = g(1,\ldots,1).$$

Outline of the article: Section 2 sets up the necessary notations and definitions. In Section 3 we discuss the case of monomials of degree $n-1$ and define our distinguished class of Catalan compositions that plays a crucial role. In Section 4 we focus on quasisymmetric polynomials, beginning with the basis of quasisymmetric monomials. Our central result stated above as Theorem 1.1 is proved there. Section 5 deepens the connection with quasisymmetric polynomials by way of Theorem 5.2.

2 Background

Throughout, for a nonnegative integer n, we set $[n] := \{i \mid 1 \leq i \leq n\}$. In particular, $[0] = \emptyset$. We denote the set of variables $\{x_1,\ldots,x_n\}$ by $\mathbf{x}_n$. Furthermore, set $\mathbb{Q}[\mathbf{x}_n] := \mathbb{Q}[x_1,\ldots,x_n]$. We refer the reader to [11, 12] for any undefined terminology.

2.1 Compositions

Given a nonnegative integer k and a positive integer n, a *weak composition* of k with n *parts* is a sequence $(c_1,\ldots,c_n)$ of nonnegative integers whose sum is k. We denote the set of compositions $n-1$, which play a special role in what follows, by $\mathcal{W}_n'$. The *size* of a weak composition $\mathbf{c} = (c_1,\ldots,c_n)$ is the sum of its parts and is denoted by $|\mathbf{c}|$. A *strong composition* is a weak composition all of whose parts are positive. Given a weak composition $\mathbf{c}$, we denote the underlying strong composition obtained by omitting zero parts by $\mathbf{c}^+$. Henceforth, by the term composition, we always mean strong composition. If the size of a composition α is k, we denote this by $\alpha \vDash k$. We denote the number of parts of α by $\ell(\alpha)$.

Given $\alpha = (\alpha_1, \dots, \alpha_{\ell(\alpha)}) \vDash k$ for k a positive integer, we associate a subset $\mathrm{Set}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \cdots + \alpha_{\ell(\alpha)-1}\} \subseteq [k-1]$. This establishes a bijection between compositions of k and subsets of $[k-1]$. Given $S \subseteq [k-1]$, we define $\mathrm{comp}(S)$ to be the composition of k associated to S under the preceding correspondence. The inclusion order on subsets allows us to define the *refinement order* on compositions. More specifically, given α and β both compositions of k, we say that β *refines* α, denoted by $\alpha \preccurlyeq \beta$, if $\mathrm{Set}(\alpha) \subseteq \mathrm{Set}(\beta)$. For instance, we have $\alpha = (1,3,2,2) \preccurlyeq (1,2,1,1,1,2) = \beta$ as $\mathrm{Set}(\alpha) = \{1,4,6\}$ is a subset of $\mathrm{Set}(\beta) = \{1,3,4,5,6\}$.

2.2 Polynomials

Recall from the introduction that we refer to $\mathbb{Q}^{(n-1)}[\mathbf{x}_n]$ as R_n. We say that $f \in \mathbb{Q}[\mathbf{x}_n]$ is *symmetric* if $w(f) = f$ for all $w \in S_n$. The space of symmetric polynomials in $\mathbb{Q}[\mathbf{x}_n]$ is denoted by Λ_n, and we denote its degree d homogeneous component by $\Lambda_n^{(d)}$. To keep our exposition brief, we refer the reader to [12, Chapter 7] and [6] for in-depth exposition on symmetric polynomials. Instead, we discuss the space of quasisymmetric polynomials, which includes Λ_n and has come to occupy a central role in algebraic combinatorics since its introduction by Gessel [5].

Given a weak composition $\mathbf{c} = (c_1, \dots, c_n)$, let

$$\mathbf{x}^{\mathbf{c}} := \prod_{1 \le i \le n} x_i^{c_i}.$$

A polynomial $f \in \mathbb{Q}[\mathbf{x}_n]$ is called *quasisymmetric* if the coefficients of $\mathbf{x}^{\mathbf{a}}$ and $\mathbf{x}^{\mathbf{b}}$ in f are equal whenever $\mathbf{a}^+ = \mathbf{b}^+$. We denote the space of quasisymmetric polynomials in $x_1, \dots, x_n$ by QSym_n and its degree d homogeneous component by $\mathrm{QSym}_n^{(d)}$. A basis for $\mathrm{QSym}_n^{(d)}$ is given by the *monomial quasisymmetric polynomials* $M_\alpha(x_1, \dots, x_n)$ indexed by compositions $\alpha \vDash d$. More precisely, we set

$$M_\alpha(x_1, \dots, x_n) = \sum_{\mathbf{a}^+ = \alpha} \mathbf{x}^{\mathbf{a}}.$$

The reader may verify that $f = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2 x_3$ is a quasisymmetric polynomial in $\mathbb{Q}[\mathbf{x}_3]$, and it can be expressed as $M_{(2,1)}(x_1, x_2, x_3) + M_{1,1,1}(x_1, x_2, x_3)$. We note here that $M_\alpha(x_1, \dots, x_n) = 0$ if $\ell(\alpha) > n$.

The more important basis for QSym_n consists of the *fundamental quasisymmetric polynomials* $F_\alpha(x_1,\ldots,x_n)$ indexed by compositions α. These are defined by the relation

$$F_\alpha(x_1,\ldots,x_n) = \sum_{\alpha \preccurlyeq \beta} M_\beta(x_1,\ldots,x_n). \tag{2.1}$$

3 Divided symmetrization of monomials of degree $n-1$

If $f = \mathbf{x}^{\mathbf{c}}$ where $\mathbf{c} \in \mathcal{W}'_n$, then [10, Proposition 3.5] gives us a precise combinatorial description for $\langle f \rangle_n$. Given $\mathbf{c} := (c_1,\ldots,c_n) \in \mathcal{W}'_n$, define the subset $S_{\mathbf{c}} \subseteq [n-1]$ by

$$S_{\mathbf{c}} := \{k \in \{1,\ldots,n-1\} \mid \sum_{i=1}^{k} c_i < k\}. \tag{3.1}$$

For instance, if $\mathbf{c} = (0,3,0,0,0,1,3,0,) \in \mathcal{W}'_8$, then $S_{\mathbf{c}} = \{1,4,5,6\} \subseteq [7]$. For a subset $S \subseteq [n-1]$, let

$$\beta_n(S) := |\{w \in S_n \mid \mathrm{Des}(w) = S\}|, \tag{3.2}$$

where $\mathrm{Des}(w) := \{1 \le i \le n-1 \mid w_i > w_{i+1}\}$ is the set of descents of w. Postnikov [10] shows that $\langle \mathbf{x}^{\mathbf{c}} \rangle_n$ for $\mathbf{c} \in \mathcal{W}'_n$ equals $\beta_n(S_{\mathbf{c}})$ up to sign.

Lemma 3.1 (Postnikov)**.** *If $\mathbf{c} = (c_1,\ldots,c_n) \in \mathcal{W}'_n$, then*

$$\langle \mathbf{x}^{\mathbf{c}} \rangle_n = (-1)^{|S_{\mathbf{c}}|} \beta_n(S_{\mathbf{c}}). \tag{3.3}$$

Example 3.2. *Consider computing $\langle x_1x_3 \rangle_3$. Then $\mathbf{c} = (1,0,1)$ and $S_{\mathbf{c}} = \{2\}$. By Lemma 3.1, $\langle x_1x_3 \rangle_3$ up to sign is the number of permutations in S_3 with the only descent in position 2. Thus, $\langle x_1x_3 \rangle_3 = -2$.*

In theory, one can use Lemma 3.1 to compute $\langle f \rangle_n \in R_n$, but this typically results in signed expressions, which may not be useful especially if we know that $\langle f \rangle_n$ is in fact positive. Nevertheless, there is a distinguished class of weak compositions for which Lemma 3.1 simplifies immensely, and equally importantly, this class also motivates the hitherto unknown connection between divided symmetrization and the work of Aval–Bergeron–Bergeron [3].

3.1 Catalan compositions and monomials

We now focus on $\langle \mathbf{x}^{\mathbf{c}} \rangle_n$ where $\mathbf{c}$ belongs to a special subset of $\mathcal{W}'_n$. Consider $\mathcal{CW}_n$ defined as

$$\mathcal{CW}_n = \{\mathbf{c} \in \mathcal{W}'_n \mid \sum_{i=1}^{k} c_i \ge k \text{ for } 1 \le k \le n-1\}. \tag{3.4}$$

This description immediately implies that $|\mathcal{CW}_n| = \mathrm{Cat}_{n-1}$, the $(n-1)$-th Catalan number equal to $\frac{1}{n}\binom{2n-2}{n-1}$. In view of this, we refer to elements of $\mathcal{CW}_n$ as *Catalan compositions*. Observe that $S_{\mathbf{c}} = \emptyset$ if and only if $\mathbf{c} \in \mathcal{CW}_n$. By Lemma 3.1, we have that $\langle \mathbf{x}^{\mathbf{c}} \rangle_n = 1$ when $\mathbf{c} \in \mathcal{CW}_n$, since the only permutation whose descent set is empty is the identity permutation. We refer to monomials $\mathbf{x}^{\mathbf{c}}$ where $\mathbf{c} \in \mathcal{CW}_n$ as *Catalan monomials*. For example, the Catalan monomials of degree 3 are given by $\{x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2^2, x_1x_2x_3\}$.

Remark 3.3. We refer to the image of a Catalan monomial under the involution $x_i \mapsto x_{n+1-i}$ for all $1 \le i \le n$ as an *anti-Catalan monomial*. These monomials are characterized by $S_{\mathbf{c}} = [n-1]$. By Lemma 3.1, the divided symmetrization of an anti-Catalan monomial yields $(-1)^{n-1}$.

The preceding discussion implies the following fact: If $f(x_1,\dots,x_n) \in R_n$ is such that each monomial appearing in f is a Catalan monomial, then $\langle f \rangle_n = f(1^n)$. Here, $f(1^n)$ refers to the usual evaluation of $f(x_1,\dots,x_n)$ at $x_1 = \cdots = x_n = 1$. This statement is a shadow of a more general result that we establish in the context of super-covariant polynomials in Section 5. For the moment though, we demonstrate its efficacy by discussing a specific instance in a family of polynomials introduced by Postnikov in regards to mixed volumes of hypersimplices [10, Section 9].

Example 3.4. *For $\mathbf{c} = (c_1,\dots,c_n) \in \mathcal{W}_n'$, consider the polynomial*

$$y_{\mathbf{c}} \coloneqq y_{\mathbf{c}}(x_1,\dots,x_n) = x_1^{c_1}(x_1+x_2)^{c_2}\cdots(x_1+\cdots+x_n)^{c_n}. \tag{3.5}$$

Following Postnikov, define the mixed Eulerian number *$A_{\mathbf{c}}$ to be $\langle y_{\mathbf{c}} \rangle_n$. In the case $\mathbf{c} = (0^k, n-1, 0^{n-k-1})$, it turns out that $A_{\mathbf{c}}$ equals the Eulerian number $A_{n-1,k}$, the number of permutations in S_{n-1} with k descents, whence the name. We record here a simple proof of a fact proved by Postnikov [10, Theorem 16.3 part (9)] via a different approach. Suppose $\mathbf{c} = (c_1,\dots,c_n) \in \mathcal{CW}_n$. Then*

$$A_{\mathbf{c}} = 1^{c_1}2^{c_2}\cdots n^{c_n}. \tag{3.6}$$

Indeed, consider the monomial expansion $y_{\mathbf{c}} = x_1^{c_1}(x_1+x_2)^{c_2}\cdots(x_1+\cdots+x_n)^{c_n}$. Equation (3.6) is an immediate consequence of the fact that all monomials in the support of $y_{\mathbf{c}}$ are Catalan monomials if $\mathbf{c}$ is itself Catalan.

4 DS of quasisymmetric polynomials

We proceed to discuss the divided symmetrization of quasisymmetric polynomials of degree $n-1$. We begin by noting the elementary fact that if $f \in \mathbb{Q}[\mathbf{x}_n]$ is homogeneous of degree $k < n-1$, then $\langle f \rangle_n = 0$. This fact taken in conjunction with our Lemma 4.1 is extremely useful toward computing divided symmetrizations.

Lemma 4.1. *Let $f \in R_n$ be such that $f = (x_i - x_{i+1})g(x_1, \ldots, x_i)h(x_{i+1}, \ldots, x_n)$. Then*

$$\langle f \rangle_n = \binom{n}{i} \langle g(x_1, \ldots, x_i) \rangle_i \langle h(x_1, \ldots, x_{n-i}) \rangle_{n-i}.$$

In particular, $\langle f \rangle_n = 0$ if $\deg(g) \neq i - 1$ (or equivalently $\deg(h) \neq n - i - 1$).

Using this lemma, we can compute the divided symmetrization of monomial quasisymmetric polynomials. This is precisely the content of our next result which shows in particular that $\langle M_\alpha(x_1, \ldots, x_m) \rangle_n$ depends solely on n, m and $\ell(\alpha)$.

Proposition 4.2. *Fix a positive integer n, and let $\alpha \vDash n - 1$. Then*

$$\langle M_\alpha(x_1, \ldots, x_m) \rangle_n = (-1)^{m-\ell(\alpha)} \binom{n - 1 - \ell(\alpha)}{m - \ell(\alpha)} \tag{4.1}$$

for any $m \in \{\ell(\alpha), \ldots, n - 1\}$, and

$$\langle M_\alpha(x_1, \ldots, x_n) \rangle_n = 0. \tag{4.2}$$

Let us give an idea of the proof, which is given in [8]. Note that the right hand side only depends on α through its number of parts $\ell(\alpha)$. The idea is then to perform elementary transformations on the compositions α of a given length, and check that the value of $\langle M_\alpha(x_1, \ldots, x_m) \rangle_n$ is preserved; this uses Lemma 4.1 in a crucial way. These transformations allow us to reach a "hook composition", for which we can compute the quantity of interest directly. Only the case $m = n$ has to be treated differently at this step, which explains in part why it is stated separately in the Proposition.

In the next subsection, we will see how Proposition 4.2 implies a pleasant result (Theorem 4.3) for all quasisymmetric polynomials.

4.1 Truncated quasisymmetric functions and DS

In this subsection, we give a natural interpretation of $\langle f(x_1, \ldots, x_m) \rangle_n$ for $m \leq n$ when f is a quasisymmetric polynomial in $x_1, \ldots, x_m$ with degree $n - 1$. To this end, we briefly discuss a generalization of Eulerian numbers that is pertinent for us.

If $\phi(x)$ is a univariate polynomial satisfying $\deg(\phi) < n$, then (cf. [11, Chapter 4]) there exist scalars $h_m^{(n)}(\phi)$ such that

$$\sum_{j \geq 0} \phi(j) t^j = \frac{\sum_{m=0}^{n-1} h_m^{(n)}(\phi) t^m}{(1 - t)^n}. \tag{4.3}$$

By extracting coefficients, the $h_m^{(n)}(\phi)$ are uniquely determined by the following formulas for $j = 0, \dots, n-1$:

$$\phi(j) = \sum_{i=0}^{j} \binom{n-1+i}{i} h_{j-i}^{(n)}(\phi). \tag{4.4}$$

Stanley calls the $h_i^{(n)}(\phi)$ the *ϕ-eulerian numbers* (cf. [11, Chapter 4.3]), and the numerator the *ϕ-eulerian polynomial*, since if $\phi(j) = j^{n-1}$ we get the classical Eulerian numbers $A_{n,i}$ and polynomial $A_n(t)$.

Let QSym denote the ring of quasisymmetric functions. Let $\mathbf{x}$ denote the infinite set of variables $\{x_1, x_2, \dots\}$. Elements of QSym may be regarded as bounded-degree formal power series $f \in \mathbb{Q}[\![\mathbf{x}]\!]$ such that for any composition $(\alpha_1, \dots, \alpha_k)$ the coefficient of $x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$ equals that of $x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ whenever $i_1 < \cdots < i_k$. Denote the degree $n-1$ homogeneous summand of QSym by $\mathrm{QSym}^{(n-1)}$, and pick $f \in \mathrm{QSym}^{(n-1)}$. Let

$$r_m(f)(x_1, \dots, x_m) := f(x_1, \dots, x_m, 0, 0, \dots) \tag{4.5}$$

be the quasisymmetric polynomial obtained by *truncating* f. Additionally, set

$$\phi_f(m) := r_m(f)(1, \dots, 1). \tag{4.6}$$

This is $ps_m^1(f)$ in the notation of [12, Section 7.8]. On occasion we will abuse notation and write $f(x_1, \dots, x_m)$ for $r_m(f)$ and similarly $f(1^m)$ for $\phi_f(m)$.

Observe that *$\phi_f(m)$ is a polynomial in m of degree at most $n-1$.* By linearity it is enough to check this on a basis. If we pick $M_\alpha(\mathbf{x})$, we have that $\phi_{M_\alpha}(m)$ is the number of monomials in $M_\alpha(x_1, \dots, x_m)$, that is

$$\phi_{M_\alpha}(m) = \binom{m}{\ell(\alpha)}, \tag{4.7}$$

a polynomial of degree $\ell(\alpha) \le n-1$.

Therefore the ϕ_f-Eulerian numbers $h_m^{(n)}(\phi_f)$ are well defined for $m \le n-1$. Our first main result, presented as Theorem 1.1 in the introduction, is that these can be obtained by divided symmetrization:

Theorem 4.3. *For any $f \in \mathrm{QSym}^{(n-1)}$, we have that $\langle r_n(f) \rangle_n = 0$ and $\langle r_m(f) \rangle_n = h_m^{(n)}(\phi_f)$ for $m < n$. In other words, we have the identity*

$$\sum_{j \ge 0} f(1^j) t^j = \frac{\sum_{m=0}^{n} \langle f(x_1, \dots, x_m) \rangle_n t^m}{(1-t)^n}. \tag{4.8}$$

The following remarkable fact, which further emphasizes the role essayed by fundamental quasisymmetric polynomials in the context of divided symmetrization, is an immediate consequence of Theorem 4.3.

Corollary 4.4. *For any $\alpha \vDash n-1$, and for $m \leq n$, we have*

$$\left\langle F_\alpha(x_1, \ldots, x_m) \right\rangle_n = \delta_{m,\ell(\alpha)}. \tag{4.9}$$

Thus if $f \in \mathrm{QSym}^{(n-1)}$ expands as $f = \sum_{\gamma \vDash n-1} c_\gamma F_\gamma$, then for any positive integer $m < n$

$$\left\langle f(x_1, \ldots, x_m) \right\rangle_n = \sum_{\substack{\gamma \vDash n-1 \\ \ell(\gamma)=m}} c_\gamma.$$

If a quasisymmetric function expands nonnegatively in terms of fundamental quasisymmetric functions, we call it *F-positive*. The upshot of Corollary 4.4 is that the divided symmetrization of F-positive quasisymmetric polynomial is itself nonnegative. F-positive quasisymmetric functions abound in combinatorics, with the ubiquitous Schur functions serving as a prototypical instance. Indeed, given a partition $\lambda \vdash n-1$ and $m \leq n$, Corollary 4.4 implies the following relation for the divided symmetrization of the Schur polynomial $s_\lambda(x_1, \ldots, x_m)$:

$$\left\langle s_\lambda(x_1, \ldots, x_m) \right\rangle_n = |\{T \in \mathrm{SYT}(\lambda) \mid |\mathrm{Des(T)}| = m-1\}|. \tag{4.10}$$

Here $\mathrm{SYT}(\lambda)$ denotes the set of *standard Young tableaux* of shape λ, and $\mathrm{Des}(T)$ refers to the descent set of the standard Young tableau T.

Recall from the introduction that we were initially interested in the values a_w given by the DS of Schubert polynomials $\mathfrak{S}_w$, where $w \in S_n$ has length $n-1$. Now it is well-known that if w is a *Grassmannian* permutation of shape λ and descent m, one has $\mathfrak{S}_w = s_\lambda(x_1, \ldots, x_m)$. Thus (4.10) tells us precisely that in this case the intersection number a_w is the number of standard tableaux of shape λ with $m-1$ descents.

Example 4.5. *The two standard Young tableaux of shape $(2,1)$ have exactly 1 descent each. It follows that $\left\langle s_{(2,1)}(x_1, x_2) \right\rangle_4 = 2$.*

$$\begin{array}{|c|c} \hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \qquad \begin{array}{|c|c} \hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

It further implies that $\left\langle s_{(2,1)}(x_1, x_2, x_3) \right\rangle_4 = 0$. An alternative way to infer the latter fact is to realize that $s_{(2,1)}(x_1, x_2, x_3)$ belongs to the ideal in $\mathbb{Q}[\mathbf{x}_4]$ generated by positive degree homogeneous symmetric polynomials in $\{x_1, \ldots, x_4\}$. This motivates the more general picture that follows next.

5 Connection to the super-covariant ring

We proceed to another perspective on divided symmetrization, one which relates it to the study of super-covariant polynomials initiated by Aval–Bergeron [4] and Aval–Bergeron–Bergeron [3].

Let $\mathcal{J}_n$ denote the ideal generated by homogeneous quasisymmetric polynomials in $x_1, \dots, x_n$ with positive degree. The super-covariant ring $\mathbf{SC}_n$ is defined as

$$\mathbf{SC}_n = \mathbb{Q}[\mathbf{x}_n]/\mathcal{J}_n.$$

The central result of Aval–Bergeron–Bergeron [3, Theorem 4.1] establishes that $\mathbf{SC}_n$ is finite-dimensional with a natural basis given by monomials indexed by Dyck paths. Consider the set of weak compositions defined by

$$\mathcal{B}_n = \{(c_1, \dots, c_n) \mid \sum_{1 \le j \le i} c_j < i \text{ for all } 1 \le i \le n\}.$$

Theorem 5.1 ([3]). *The set of monomials* $\{\mathbf{x}^{\mathbf{c}} \mod \mathcal{J}_n \mid \mathbf{c} \in \mathcal{B}_n\}$ *forms a basis for* $\mathbf{SC}_n$.

In particular, $\mathbf{SC}_n$ is finite-dimensional with dimension given by the nth Catalan number Cat_n. We are specifically interested in the degree $n-1$ graded piece of $\mathbf{SC}_n$, that is, $R_n/(R_n \cap \mathcal{J}_n)$. The Aval–Bergeron–Bergeron basis for this piece is given by familiar objects: it comprises what we have referred to as anti-Catalan monomials. In particular, the dimension of $R_n/(R_n \cap \mathcal{J}_n)$ equals Cat_{n-1}.

Since the involution on $\mathbb{Q}[\mathbf{x}_n]$ that send $x_i \mapsto x_{n+1-i}$ for $1 \le i \le n$ preserves the ideal $\mathcal{J}_n$, it sends any basis modulo $\mathcal{J}_n$ to another such basis. So if we set

$$K_n := R_n \cap \mathcal{J}_n \quad \text{and} \quad K_n^{\dagger} := \mathrm{Vect}\,(\mathbf{x}^{\mathbf{c}} \mid \mathbf{c} \in \mathcal{CW}_n)$$

then Theorem 5.1 implies that we have a vector space decomposition

$$R_n = K_n^{\dagger} \oplus K_n. \tag{5.1}$$

We can now state our structural result, which is Theorem 1.3 in the introduction. It characterizes divided symmetrization with respect to the direct sum in (5.1).

Theorem 5.2. *If* $f \in K_n$, *then* $\langle f \rangle_n = 0$. *More generally, if* $f \in R_n$ *is written* $f = g + h$ *with* $g \in K_n^{\dagger}$ *and* $h \in K_n$ *according to* (5.1), *then*

$$\langle f \rangle_n = g(1, \dots, 1).$$

Notice that the first statement in Theorem 5.2 states that divided symmetrization vanishes on the degree $n-1$ piece of the ideal of positive degree homogeneous quasisymmetric polynomials.

Example 5.3. *We revisit the computation of* $\langle x_1x_3\rangle_3$*, invoking Theorem 5.2 this time. Note that*

$$x_1x_3 = x_1F_1(x_1,x_2,x_3) - (x_1^2 + x_1x_2),$$

and that x_1x_2 *and* x_1^2 *are both Catalan monomials. Using* $f = x_1F_1(x_1,x_2,x_3)$ *and* $g = -(x_1^2 + x_1x_2)$ *in Theorem 5.2 we conclude that* $\langle x_1x_3\rangle_3 = -2$.

As a further demonstration of Theorem 5.2, we revisit the divided symmetrization of fundamental quasisymmetric polynomials in this new light.

5.1 Fundamental quasisymmetric polynomials revisited

Before stating the main result in this subsection, we need two operations for compositions. Given compositions $\gamma = (\gamma_1,\dots,\gamma_{\ell(\gamma)})$ and $\delta = (\delta_1,\dots,\delta_{\ell(\delta)})$, we define the *concatenation* $\gamma\cdot\delta$ and *near-concatenation* $\gamma\odot\delta$ as the compositions $(\gamma_1,\dots,\gamma_{\ell(\gamma)},\delta_1,\dots,\delta_{\ell(\delta)})$ and $(\gamma_1,\dots,\gamma_{\ell(\gamma)}+\delta_1,\delta_2,\dots,\delta_{\ell(\delta)})$ respectively. For instance, we have $(3,2)\cdot(1,2) = (3,2,1,2)$ and $(3,1)\odot(1,1,2) = (3,2,1,2)$.

Given finite alphabets $\mathbf{x}_n = \{x_1,\dots,x_n\}$ and $\mathbf{y}_m = \{y_1,\dots,y_m\}$, define the formal sum $\mathbf{x}_n + \mathbf{y}_m$ to be the alphabet $\{x_1,\dots,x_n,y_1,\dots,y_m\}$ where the total order is given by $x_1 < \cdots < x_n < y_1 < \cdots < y_m$. Following Malvenuto-Reutenauer [7], we have

$$F_\alpha(\mathbf{x}_n + \mathbf{y}_m) = \sum_{\gamma\cdot\delta=\alpha \text{ or } \gamma\odot\delta=\alpha} F_\gamma(\mathbf{x}_n)F_\delta(\mathbf{y}_m). \tag{5.2}$$

As explained in [7, Section 2], the equality in (5.2) relies on the coproduct in the Hopf algebra of quasisymmetric functions. By utilizing the antipode on this Hopf algebra [7, Corollary 2.3], one can evaluate quasisymmetric functions at formal differences of alphabets. See [2, Section 2.3] for a succinct exposition on the same. The analogue of (5.2) is

$$F_\alpha(\mathbf{x}_n - \mathbf{y}_m) = \sum_{\gamma\cdot\delta=\alpha \text{ or } \gamma\odot\delta=\alpha} (-1)^{|\delta|}F_\gamma(\mathbf{x}_n)F_{\delta^t}(\mathbf{y}_m), \tag{5.3}$$

where $\delta^t := \mathrm{comp}([|\delta|-1]\setminus\mathrm{Set}(\delta))$. For instance, if $\delta = (3,2,1,2) \vDash 8$, then $\mathrm{Set}(\delta) \subseteq [7]$ is given by $\{3,5,6\}$. Thus we obtain $\delta^t = \mathrm{comp}(\{1,2,4,7\}) = (1,1,2,3,1)$.

To conclude this article, we have the following result which renders Corollary 4.4 transparent.

Proposition 5.4. *Let* $\alpha \vDash n-1$ *and let* m *be a positive integer satisfying* $\ell(\alpha) \le m \le n$*. If* $m > \ell(\alpha)$*, then* $F_\alpha(x_1,\dots,x_m) \in \mathcal{J}_n$*. In particular, we have*

$$\langle F_\alpha(x_1,\dots,x_m)\rangle_n = \delta_{m,\ell(\alpha)}.$$

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