The equivariant Ehrhart theory of the permutahedron

Federico Ardila*, Mariel Supina†, and Andrés R. Vindas-Meléndez‡

Abstract. Equivariant Ehrhart theory enumerates the lattice points in a polytope with respect to a group action. Answering a question of Stapledon, we describe the equivariant Ehrhart theory of the permutahedron, and we prove his Effectiveness Conjecture in this special case.

Keywords: Ehrhart theory, permutahedron, quasipolynomial, symmetric group, representation theory, zonotope

1 Introduction

Ehrhart theory measures a polytope $P$ by counting the lattice points in its dilations $tP$ for positive integers $t$. Stapledon [10] introduced equivariant Ehrhart theory as a refinement of Ehrhart theory that takes into account the symmetries of the polytope $P$. He asked for a description of the equivariant Ehrhart theory of the permutahedron under its group of symmetries, the symmetric group. In this extended abstract, we completely answer Stapledon’s question, computing the equivariant Ehrhart polynomials of the standard permutahedra and verifying several conjectures in this special case.

1.1 Ehrhart theory for fixed polytopes of the permutahedron

We consider the action of the symmetric group $S_n$ on the $(n - 1)$-dimensional permutahedron $\Pi_n$. For each permutation $\sigma \in S_n$, we define the fixed polytope $\Pi_n^\sigma \subseteq \Pi_n$ to be the subset of the permutahedron $\Pi_n$ fixed by $\sigma$. Our first main result is a combinatorial formula for the lattice point enumerator $L_{\Pi_n^\sigma}(t) := |t\Pi_n^\sigma \cap \mathbb{Z}^n|:

* federico@sfsu.edu. Supported by NSF grants DMS–1600609, DMS–1855610, and Simons Fellowship grant #613384.
† mariel_supina@berkeley.edu. Supported by the Graduate Fellowships for STEM Diversity.
‡ andres.vindas@uky.edu. Supported by NSF Graduate Research Fellowship DGE–1247392.
Theorem 1.1. Let $\sigma$ be a permutation of $[n] := \{1, 2, \ldots, n\}$ and let $\lambda = (\ell_1, \ldots, \ell_m)$ be the partition of $n$ given by the lengths of the cycles of $\sigma$. Say a set partition $\pi = \{B_1, \ldots, B_k\}$ of $[n]$ is $\lambda$-compatible if for each block $B_i$, either $\ell_j$ is odd for some $j \in B_i$, or the minimum 2-valuation among $\{\ell_j : j \in B_i\}$ is attained at least twice. Also write
\[ v_\pi = \prod_{i=1}^{k} \left( \gcd(\ell_j : j \in B_i) \cdot \left( \sum_{j \in B_i} \ell_j \right)^{|B_i|/2} \right). \] (1.1)

Then the Ehrhart quasipolynomial of the fixed polytope $\Pi^\sigma_n$ is
\[ L_{\Pi^\sigma_n}(t) = \begin{cases} \sum_{\pi \models [n]} v_\pi \cdot t^{m-|\pi|} & \text{if } t \text{ is even} \\ \sum_{\pi \models [n]} v_\pi \cdot t^{m-|\pi|} & \text{if } t \text{ is odd} \\ \lambda-\text{compatible} \end{cases}. \]

1.2 Equivariant Ehrhart theory

Theorem 1.1 fits into the framework of equivariant Ehrhart theory, as we now explain.

Let $G$ be a finite group acting linearly on $\mathbb{Z}^n$ and $P \subseteq \mathbb{R}^n$ be a $d$-dimensional lattice polytope that is invariant under the action of $G$. Let $M$ be the sublattice of $\mathbb{Z}^n$ obtained by translating the affine span of $P$ to the origin, and consider the induced representation $\rho: G \to GL(M)$. We then obtain a family of permutation representations by looking at how $\rho$ permutes the lattice points inside the dilations of $P$. Let $\chi_{tP}: G \to \mathbb{C}$ denote the permutation character associated to the action of $G$ on the lattice points in the $t$th dilate of $P$. For $g \in G$, we have
\[ \chi_{tP}(g) = L_{P^g}(t), \]
where $P^g$ is the polytope of points in $P$ fixed by $g$ and $L_{P^g}(t)$ is its lattice point enumerator.

The permutation characters $\chi_{tP}$ live in the ring $R(G)$ of virtual characters of $G$, which are the integer combinations of the irreducible characters of $G$. The positive integer combinations are called effective; they are the characters of representations of $G$.

Stapledon encoded the characters $\chi_{tP}$ in a power series $H^*[z] \in R(G)[[z]]$ given by
\[ \sum_{t \geq 0} \chi_{tP}(g) z^t = \frac{H^*[z](g)}{(1-z) \det(I-g \cdot z)}. \] (1.2)

We call it the equivariant $H^*$-series because for the identity element $e \in G$, the evaluation $H^*[z](e)$ is the well-studied $h^*$-polynomial of $P$. We say that $H^*[z] =: \sum_{i \geq 0} H^*_i z^i$ is effective if each virtual character $H^*_i$ is a character.

The main open problem in equivariant Ehrhart theory is to characterize when $H^*[z]$ is effective, and Stapledon offered the following conjecture.
Conjecture 1.2 ([10, Effectiveness Conjecture 12.1]). Let $P$ be a lattice polytope fixed by the action of a group $G$. The following conditions are equivalent.

(i) The toric variety of $P$ admits a $G$-invariant non-degenerate hypersurface.

(ii) The equivariant $H^*$-series of $P$ is effective.

(iii) The equivariant $H^*$-series of $P$ is a polynomial.

Our second main result is the following.

Theorem 1.3. Stapledon’s Effectiveness Conjecture holds for the permutahedron under the action of the symmetric group.

Finally, in Proposition 4.9 we verify three other conjectures of Stapledon in this case.

1.3 Organization

In Section 2 we introduce some background on Ehrhart theory and zonotopes. In Section 3 we compute the Ehrhart quasipolynomial of the fixed polytope $\Pi_n^\sigma$, proving Theorem 1.1. In Section 4 we compute the equivariant $H^*$-series $H^*[z]$ for permutahedra and we verify Stapledon’s Effectiveness conjecture in this special case (Theorem 1.3).

2 Preliminaries

2.1 Ehrhart quasipolynomials

Let $P$ be a convex polytope in $\mathbb{R}^n$. The lattice point enumerator of $P$ is the function $L_P : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 0}$ given by $L_P(t) := |tP \cap \mathbb{Z}^n|$. A function $f : \mathbb{Z} \to \mathbb{R}$ is a quasipolynomial if there exists a period $d$ and polynomials $f_0, f_1, \ldots, f_{d-1}$ such that $f(n) = f_i(n)$ whenever $n \equiv i (\text{mod } d)$.

Theorem 2.1 (Ehrhart’s Theorem [4]). If $P$ is a rational polytope, then $L_P(t)$ agrees with a quasipolynomial in $t$ of degree $\dim P$. Its period divides the least common multiple of the denominators of the coordinates of the vertices of $P$.

2.2 Zonotopes

Let $V$ be a finite set of vectors in $\mathbb{R}^n$. The zonotope generated by $V$, denoted $Z(V)$, is defined to be the Minkowski sum of the line segments connecting the origin to $v$ for each $v \in V$. We will also adapt the same notation to refer to any translation of $Z(V)$, that is, the Minkowski sum of any collection of line segments whose direction vectors are the elements of $V$. Zonotopes have a combinatorial decomposition that is useful when calculating volumes and counting lattice points. The following result is due to Shephard.
Proposition 2.2 ([8, Theorem 54]). A zonotope $Z(V)$ can be subdivided into half-open paralleloptopes that are in bijection with the linearly independent subsets of $V$.

A linearly independent subset $S \subseteq V$ corresponds under this bijection to the half-open paralleloptope

$$
\square S := \sum_{v \in S} (0, v).
$$

Theorem 2.3 ([9, Theorem 2.2]). Let $Z(V)$ be a lattice zonotope generated by $V$. Then

$$
L_{Z(V)}(t) = \sum_{S \subseteq V \text{ lin. indep.}} \text{vol}(\square S) \cdot t^{|S|}.
$$

In the statement above and throughout the paper, volumes are normalized so that any primitive lattice paralleloptope has volume 1.

2.3 Fixed polytopes of the permutahedron

The symmetric group $S_n$ acts on $\mathbb{R}^n$ by permuting coordinates of points. The permutahedron $\Pi_n$ is the convex hull of the $n!$ permutations of $[n]$.

Let $\sigma \in S_n$ be a permutation with cycles $\sigma_1, \ldots, \sigma_m$; their lengths form a partition $\lambda = (\ell_1, \ldots, \ell_m)$ of $n$. For each cycle $\sigma_k$ of $\sigma$, let $e_{\sigma_k} = \sum_{i \in \sigma_k} e_i$. The fixed polytope $\Pi_n^\sigma$ is defined to be the polytope consisting of all points in $\Pi_n$ that are fixed under the action of $\sigma$. We will use a few results from [2], which we now summarize.

Theorem 2.4 ([2, Theorem 2.12]). The fixed polytope $\Pi_n^\sigma$ has the following zonotope description:

$$
\Pi_n^\sigma = \sum_{1 \leq i < j \leq m} [\ell_i e_{\sigma_j} - \ell_j e_{\sigma_i}] + \sum_{k=1}^m \frac{\ell_k + 1}{2} e_{\sigma_k}.
$$

Corollary 2.5. The fixed polytope $\Pi_n^\sigma$ is integral or half-integral. It is a lattice polytope if and only if all cycles of $\sigma$ have odd length.

Equation (2.2) also shows that $\Pi_n^\sigma$ is a rational translation of the zonotope $Z(V)$ where $V = \{\ell_i e_{\sigma_j} - \ell_j e_{\sigma_i} : 1 \leq i < j \leq m\}$. The following result characterizes the linearly independent subsets of $V$.

Lemma 2.6 ([2, Lemma 3.2]). The linearly independent subsets of $V$ are in bijection with forests with vertex set $[m]$, where the vector $\ell_i e_{\sigma_j} - \ell_j e_{\sigma_i}$ corresponds to the edge connecting vertices $i$ and $j$. 
In light of this lemma, the fixed polytope $\Pi_n^\sigma$ gets subdivided into half-open paralleloptopes $\square_F$ of the form
\[ \square_F = \sum_{\{i,j\} \in E(F)} \left[ \ell_i e_{\sigma_i}, \ell_j e_{\sigma_j} \right] + \sum_{k=1}^m \frac{\ell_k + 1}{2} e_{\sigma_k} + v_F, \quad v_F \in \mathbb{Z}^n \] (2.3)
for each forest of $F$.

When $F$ is a tree $T$ we have that $\text{vol}(\square_T) = \left( \prod_{i=1}^m \ell_i^{\deg_T(i)-1} \right) \gcd(\ell_1, \ldots, \ell_m)$ by [2, Lemma 3.3]. For a general forest $F$, the paralleloptopes $\square_T$ corresponding to each connected component $T$ of $F$ live in orthogonal subspaces, so

\[ \text{vol}(\square_F) = \left( \prod_{j=1}^m \ell_j^{\deg_f(j)-1} \right) \left( \prod_{\text{conn. comp.}} \gcd(\ell_j : j \in \text{vert}(T)) \right). \] (2.4)

3 Ehrhart quasipolynomial of $\Pi_n^\sigma$

Since $\Pi_n^\sigma$ is a zonotope, we can decompose it into half-open paralleloptopes. However, since $\Pi_n^\sigma$ is half-integral, some of the paralleloptopes in this decomposition may not contain any lattice points.

**Example 3.1.** The fixed polytope $\Pi_4^{(12)}$ of Figure 1, which corresponds to the cycle type $\lambda = (2, 1, 1)$, is
\[ \Pi_4^{(12)} = [2e_3, e_{12}] + [2e_4, e_{12}] + [e_4, e_3] + \frac{3}{2} e_{12} + e_3 + e_4. \]

Figure 2 shows its decomposition into parallelograms indexed by the forests on vertex set $\{12, 3, 4\}$. The three trees give parallelograms with volumes 2, 1, 1 that contain 2, 1, 1 lattice points, respectively. The three forests with one edge give segments of volumes...
1, 1, 1 and 1, 1, 0 lattice points, respectively. The empty forest gives a point of volume 1 and 0 lattice points. Hence the Ehrhart quasipolynomial of $\Pi_{4}^{(12)}$ is

$$L_{\Pi_{4}^{(12)}}(t) = \begin{cases} (2 + 1 + 1)t^2 + (1 + 1 + 1)t + 1 & \text{if } t \text{ is even} \\ (2 + 1 + 1)t^2 + (1 + 1 + 0)t + 0 & \text{if } t \text{ is odd} \end{cases}.$$ 

Following the reasoning of Example 3.1, we will find the Ehrhart quasipolynomial of $\Pi_{n}^{\sigma}$ by examining its decomposition into half-open parallelotopes. In order to find the number of lattice points in each parallelotope $\Box_F$, the following observation is crucial.

**Lemma 3.2.** [1, 6] If $\Box$ is a half-open lattice parallelotope in $\mathbb{Z}^n$ and $\mathbf{v} \in \mathbb{Q}^n$, the number of lattice points in $\Box + \mathbf{v}$ is

$$|\Box + \mathbf{v} \cap \mathbb{Z}^n| = \begin{cases} \text{vol}(\Box) & \text{if the affine span of } \Box + \mathbf{v} \text{ intersects the lattice } \mathbb{Z}^n \\ 0 & \text{otherwise} \end{cases}.$$ 

We now apply Lemma 3.2 to the parallelotopes $\Box_F$. Surprisingly, whether $\text{aff}(\Box_F)$ contains lattice points does not depend on the forest $F$, but only on the set partition $\pi$ of the vertex set $[m]$ induced by the connected components of $F$. To make this precise we need a definition. Recall that the 2-valuation of a positive integer is the largest power of 2 dividing that integer; for example, $\text{val}_2(24) = 3$.

**Definition 3.3.** Let $\lambda = (\ell_1, \ldots, \ell_m)$ be a partition of the integer $n$. A set partition $\pi = \{B_1, \ldots, B_k\}$ of $[m]$ is called $\lambda$-compatible if for each block $B_i \in \pi$, at least one of the following conditions holds:

(i) $\ell_j$ is odd for some $j \in B_i$, or

(ii) the minimum 2-valuation among $\{\ell_j : j \in B_i\}$ occurs an even number of times.

**Example 3.4.** Let $\lambda = (\ell_1, \ell_2, \ell_3)$ and $\text{val}_2(\lambda) = (v_1, v_2, v_3)$ and assume that $v_1 \geq v_2 \geq v_3$. Table 1 shows which partitions of $[3]$ are $\lambda$-compatible depending on $\text{val}_2(\lambda)$. 

![Figure 2: Decomposition of the fixed polytope $\Pi_{4}^{(12)}$ into half-open parallelepipeds.](image-url)
Table 1: λ-compatibility for \( m = 3 \).

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**Lemma 3.5.** Let \( \sigma \in S_n \) have cycle type \( \lambda = (\ell_1, \ldots, \ell_m) \). Let \( F \) be a forest on \([m]\) whose connected components induce the partition \( \pi = \{B_1, \ldots, B_k\} \) of \([m]\). Then \( \text{aff}(\square_F) \) intersects the lattice \( \mathbb{Z}^n \) if and only if \( \pi \) is \( \lambda \)-compatible.

**Proof.** First we claim that

\[
\text{aff}(\square_F) = \left\{ \sum_{j=1}^m x_j \sigma_j : \sum_{j \in B_i} \ell_j x_j = \sum_{j \in B_i} \frac{\ell_j(\ell_j + 1)}{2} \right\} \quad (3.1)
\]

This affine subspace intersects the lattice \( \mathbb{Z}^n \) if and only if (3.1) has integer solutions. Elementary number theory tells us that this is the case if and only if each block \( B_i \) satisfies

\[
\gcd(\ell_j : j \in B_i) \left| \sum_{j \in B_i} \frac{\ell_j(\ell_j + 1)}{2} \right. \quad (3.2)
\]

It is always true that \( \gcd(\ell_j : j \in B_i) \) divides \( \sum_{j \in B_i} \ell_j(\ell_j + 1) \), so (3.2) holds if and only if

\[
\text{val}_2(\gcd(\ell_j : j \in B_i)) < \text{val}_2\left( \sum_{j \in B_i} \ell_j(\ell_j + 1) \right) \quad (3.3)
\]

We consider two cases.

(i) Suppose \( \ell_j \) is odd for some \( j \in B_i \). Then \( \gcd(\ell_j : j \in B_i) \) is odd, whereas \( \sum_{j \in B_i} \ell_j(\ell_j + 1) \) is always even. Hence (3.3) always holds in this case.

(ii) Suppose that \( \ell_j \) is even for all \( j \in B_i \). For each \( \ell_j \), write \( \ell_j = 2^{p_j} q_j \) for some integer \( p_j \geq 1 \) and odd integer \( q_j \). Then \( \text{val}_2(\gcd(\ell_j : j \in B_i)) = \min_{j \in B_i} p_j \), we will call this integer \( p \). We have

\[
\text{val}_2\left( \sum_{j \in B_i} \ell_j(\ell_j + 1) \right) = \text{val}_2\left( \sum_{j \in B_i} 2^{p_j} q_j(\ell_j + 1) \right) = p + \text{val}_2\left( \sum_{j \in B_i} 2^{p_j-p} q_j(\ell_j + 1) \right).
\]
Note that $q_j(\ell_j + 1)$ is odd for each $j$. If the minimum 2-valuation $p$ of $\{\ell_j : j \in B_i\}$ occurs an odd number of times, then $\sum_{j \in B_i} 2^{p-q_j(\ell_j + 1)}$ will be odd and we will have $\text{val}_2(\sum_{j \in B_i} \ell_j(\ell_j + 1)) = p$. Otherwise, this sum will be even and we will have $\text{val}_2(\sum_{j \in B_i} \ell_j(\ell_j + 1)) > p$. Therefore (3.3) holds if and only if the minimum 2-valuation among the $\ell_j$ for $j \in B_i$ occurs an even number of times. This is precisely the condition of $\lambda$-compatibility.

We now have all of the tools to compute the Ehrhart quasipolynomial of $\Pi_n^\sigma$. Recall the definition of $\lambda$-compatibility in Definition 3.3 and the definition of $v_\pi$ in (1.1).

**Theorem 1.1.** Let $\sigma$ be a permutation of $[n]$ with cycle type $\lambda = (\ell_1, \ldots, \ell_m)$. Then the Ehrhart quasipolynomial of the fixed polytope $\Pi_n^\sigma$ is

$$L_{\Pi_n^\sigma}(t) = \begin{cases} \sum_{\pi \models [m]} v_\pi \cdot t^{m-|\pi|} & \text{if } t \text{ is even} \\ \sum_{\pi \models [m]} v_\pi \cdot t^{m-|\pi|} & \text{if } t \text{ is odd} \end{cases}$$

\[\lambda\text{-compatible}\]

**Proof.** We calculate the number of lattice points in each integer dilate $t\Pi_n^\sigma$ by decomposing it into half-open parallelotopes and adding up the number of lattice points inside of each parallelotope.

First, suppose that $t$ is even. Then $t\Pi_n^\sigma$ is a lattice polytope, all parallelotopes in the decomposition of $t\Pi_n^\sigma$ have vertices on the integer lattice, and each $i$-dimensional parallelotope $\square$ contains $\text{vol}(\square)t^i$ lattice points [3, Lemma 9.2]. The parallelotopes correspond to linearly independent subsets of the vector configuration $\{\ell_i e_{\sigma j} - \ell_j e_{\sigma i} : 1 \leq i < j \leq m\}$, which are in bijection with forests on $[m]$. It follows from Theorem 2.3 and (2.4) that when $t$ is even,

$$L_{\Pi_n^\sigma}(t) = \sum_{\pi \models [m]} v_\pi \cdot t^{m-|\pi|}.$$  

Next, suppose $t$ is odd. Then $t\Pi_n^\sigma$ is half-integral, but it may not be a lattice polytope. As before, we may decompose $t\Pi_n^\sigma$ into half-open parallelotopes that are in bijection with forests on $[m]$. Lemma 3.2, Lemma 3.5, and [3, Lemma 9.2] tell us that $\square_F$ contains $\text{vol}(\square_F)t^{m-|\pi|}$ lattice points if the set partition $\pi$ induced by $F$ is $\lambda$-compatible, and 0 otherwise. Therefore if $t$ is odd,

$$L_{\Pi_n^\sigma}(t) = \sum_{\pi \models [m]} v_\pi \cdot t^{m-|\pi|}$$

\[\lambda\text{-compatible}\]

as desired. \qed
4 The equivariant $H^*$-series of the permutahedron

We now compute the equivariant $H^*$-series of the permutahedron and characterize when it is polynomial and when it is effective, proving Stapledon’s Effectiveness Conjecture 1.2 in this special case.

The Ehrhart series of a rational polytope $P$ is

$$Ehr_p(z) = 1 + \sum_{t=1}^{\infty} L_p(t) \cdot z^t.$$ 

In computing the Ehrhart series of $\Pi_n^\sigma$, Eulerian polynomials naturally arise. The Eulerian polynomial $A_k(z)$ is defined by the identity

$$\sum_{t \geq 0} t^k z^t = \frac{A_k(z)}{(1-z)^{k+1}}.$$ 

**Proposition 4.1.** Let $\sigma \in S_n$ have cycle type $\lambda = (\ell_1, \ldots, \ell_m)$. The Ehrhart series of $\Pi_n^\sigma$ is

$$Ehr_{\Pi_n^\sigma}(z) = \sum_{\pi \in [m]^{\lambda\text{-compatible}}} v_{\pi} \cdot A_{m-|\pi|}(z) \frac{(1-z)^{m-|\pi|+1}}{(1-z)^{m-|\pi|+1}} + \sum_{\pi \in [m]^{\lambda\text{-incompatible}}} v_{\pi} \cdot 2^{m-|\pi|} \cdot A_{m-|\pi|}(z^2) \frac{(1-z^2)^{m-|\pi|+1}}{(1-z^2)^{m-|\pi|+1}}$$

and the $H^*$-series of the permutahedron equals $H^*[z](\sigma) = (\prod_{i=1}^{m} (1-z^{\ell_i})) \cdot Ehr_{\Pi_n^\sigma}(z)$.

**Proof.** Omitted.

Table 2 shows the equivariant $H^*$-series of $\Pi_4$. Stapledon writes that “The main open problem is to characterize when $H^*[z]$ is effective”, and he conjectures the following characterization:
Conjecture 1.2 ([10, Effectiveness Conjecture 12.1]). Let $P$ be a lattice polytope fixed by the action of a group $G$. The following conditions are equivalent.

(i) The toric variety of $P$ admits a $G$-invariant non-degenerate hypersurface.

(ii) The equivariant $H^*$-series of $P$ is effective.

(iii) The equivariant $H^*$-series of $P$ is a polynomial.

He shows that $(i) \implies (ii) \implies (iii)$, so only the reverse implications are conjectured. Our next goal is to verify Stapledon’s conjecture for the action of $S_n$ on the permutahedron $\Pi_n$.

4.1 Polynominality of $H^*[z]$ 

Lemma 4.2. Let $\sigma \in S_n$ have cycle type $\lambda = (\ell_1, \ldots, \ell_m)$. The equivariant $H^*$-series evaluated at $\sigma$, $H^*[z](\sigma)$, is a polynomial if and only if the number of even parts in $\lambda$ is $0$, $m - 1$, or $m$.

Proof. Omitted. \hfill $\square$

Proposition 4.3. The equivariant $H^*$-series of the permutahedron $\Pi_n$ is a polynomial if and only if $n \leq 3$.

Proof. When $n \leq 3$, all partitions of $n$ have 0, 1, or all odd parts. Hence $H^*[z](\sigma)$ is a polynomial for all $\sigma \in S_n$, so $H^*[z]$ is a polynomial.

Suppose $n \geq 4$. Then there always exists some partition of $n$ with more than 1 but fewer than all odd parts: if $n$ is even we can take the partition $(n - 2, 1, 1)$, and if $n$ is odd we can take the partition $(n - 3, 1, 1, 1)$. Therefore $H^*[z]$ is not polynomial. \hfill $\square$

4.2 Effectiveness of $H^*[z]$ 

Proposition 4.4. The equivariant $H^*$-series of the permutahedron $\Pi_n$ is effective if and only if $n \leq 3$.

Proof. We prove this by computing the decomposition of the $H^*$ characters into irreducibles. \hfill $\square$

4.3 $S_n$-invariant non-degenerate hypersurfaces in the permutahedral variety

We begin by explaining condition (i) of Conjecture 1.2, which arises from Khovanskii’s notion of non-degeneracy [5]. We refer the reader to [10, Section 7] for more details.
Let \( P \subset \mathbb{R}^n \) be a lattice polytope that is invariant under the action of a finite group \( G \). For \( v \in \mathbb{Z}^n \) we write \( x^v := x_1^{v_1} \cdots x_n^{v_n} \). The coordinate ring of the projective toric variety \( X_P \) of \( P \) has the form \( \mathbb{C}[x^v : v \in P \cap \mathbb{Z}^n] \), so a hypersurface in \( X_P \) is given by a linear equation \( \sum_{v \in P \cap \mathbb{Z}^n} a_v x^v = 0 \) for some complex coefficients \( a_v \). The group \( G \) acts on the monomials \( x^v \) by its action on the lattice points \( v \in P \cap \mathbb{Z}^n \), so the equation of a \( G \)-invariant hypersurface should have \( a_v = a_u \) whenever \( u \) and \( v \) are in the same \( G \)-orbit. A projective hypersurface in \( X_P \) with equation \( f(x_1, \ldots, x_n) = 0 \) is smooth if the gradient \((\partial f / \partial x_1, \ldots, \partial f / \partial x_n)\) is never zero when \((x_1, \ldots, x_n) \in (\mathbb{C}^*)^n\). There is a unique polynomial in the \( a_v \)s, called the discriminant, such that the hypersurface is smooth when the discriminant does not vanish at the coefficients \( a_v \). A hypersurface in the toric variety of \( P \) is non-degenerate if it is smooth and for each face \( F \) of \( P \), the hypersurface \( \sum_{v \in F \cap \mathbb{Z}^n} a_v x^v = 0 \) is also smooth.

The permutahedral variety \( X_{\Pi_n} \) is the projective toric variety associated to the permutahedron \( \Pi_n \).

**Proposition 4.5.** The permutahedral variety \( X_{\Pi_n} \) admits an \( S_n \)-invariant non-degenerate hypersurface if and only if \( n \leq 3 \).

**Proof.** We prove this by checking gradients when \( n = 1, 2 \). For \( n = 3 \), we compute a discriminant using a formula from [7].

### 4.4 Stapledon’s Effectiveness Conjecture

Theorem 1.3 now follows as a corollary.

**Theorem 1.3.** Stapledon’s Effectiveness Conjecture holds for the permutahedron under the action of the symmetric group.

**Proof.** This follows immediately from Propositions 4.3 to 4.5.

### 4.5 Other conjectures

**Conjecture 4.6** ([10, Conjecture 12.2]). If \( H^*[z] \) is effective, then \( H^*[1] \) is a permutation representation.

**Conjecture 4.7** ([10, Conjecture 12.3]). For any \( g \in G \), the quantity \( H^*[1](g) \) is a non-negative integer.

**Conjecture 4.8** ([10, Conjecture 12.4]). If \( H^*[z] \) is a polynomial and the \( i \)th coefficient of the \( h^* \)-polynomial of \( P \) is positive, then the trivial representation occurs with non-zero multiplicity in the virtual character \( H^* \).

**Proposition 4.9.** Conjectures 4.6 to 4.8 hold for permutahedra under the action of the symmetric group.

**Proof.** Omitted.
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References


