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Results in Labeled Chip-Firing

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Abstract. In 2016, Hopkins, McConville, and Propp proved that labeled chip-firing on a line always leaves the chips in sorted order if the number of chips is even. Here, we present a new proof of this result, based on a partial ordering of the firing moves near the end of the process, leading to temporary local confluence of the system. We then use the methods from this proof to resolve a series of previously open problems. Finally, we discuss our methods in relation to the general problem of which systems display various confluence properties.

Keywords: chip-firing, labeled chip-firing, confluence

1 Introduction

We work with a labeled variant of the chip-firing process as defined by Hopkins, McConville, and Propp [9]. The problem begins with n labeled chips at the origin in a 1-dimensional lattice. A series of firing moves are performed, in which two chips at the same site are chosen, with the smaller-labeled chip sent left and the larger one sent right. The process continues until all chips are at distinct sites.

The main result from [9] showed that if the number of chips is even, then the chip labels are in sorted order from left to right at the end of the process. Here, we present a new, more direct proof of this result. The basic idea is to consider the firing moves near the end of the process. These moves are highly constrained and locally confluent, which results in the sorting of any reachable configuration.

We then apply these methods to show that sorting occurs for other, similar problems for which sorting was previously conjectured. These include several labeled chip-firing problems from Hopkins et al. [9] and Galashin et al. [6] involving modifications to the original graph or initial configuration for which the process still results in a sorted final configuration.

1.1 Chip-firing

Chip-firing is a discrete dynamical system defined on a graph. The problem is effectively equivalent to the abelian sandpile model first presented Bak et al. [2], although the

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idea of disks being sent between nodes of a graph was introduced by Engel [4], further analyzed by Anderson et al. [1], and later termed "chip-firing" by Bjorner et al. [3]. The problem deals with a collection of indistinguishable chips placed at the nodes of a graph. If a node has at least as many chips as it has neighbors, it can "fire" by sending one chip to each of its neighbors. The process terminates if no site has enough chips to fire.

Chip-firing is known to exhibit strong confluence properties. In particular, there is a notion of local confluence: for any two available local moves, there is a common state that can be reached from either of them in one additional move. For unlabeled chip-firing, this means that any two available local moves may be performed in either order and produce the same resulting configuration. This property, combined with Newman's Lemma on abstract rewriting systems [11], gives a global confluence property, in which any terminating chip-firing process must have a unique final configuration, see [10, Chapter 1].

1.2 Results on the 1-dimensional lattice

Unlabeled chip-firing has been studied extensively on the 1-dimensional lattice. In this problem, n chips are placed at the origin in the infinite 1-dimensional lattice, and the chip-firing process proceeds as above. See [1] and [10]:

Theorem 1.1. *The chip-firing process beginning with* $2m$ *chips at the origin terminates at a final configuration with a chip at every position from* $-m$ *to 1, and from 1 to* m*.*

Reiterating the discussion above, the final configuration is independent of the order of the fires in the process.

We also make repeated use of the following result:

Theorem 1.2. *Over the course of the chip-firing process beginning with* $2m$ *chips at the origin, the number of firing moves at site* k *is* $\binom{m-|k|+1}{2}$ *for* $-m \leq k \leq m$*.*

Thus, the number of fires at each site is also an invariant of the process. This result allows us to unambiguously refer to things like "the third to last firing move at site 0" as a property of the chip-firing process, rather than merely as a property of a particular firing sequence.

One of the crucial pieces of the main result of this paper is a lemma on unlabeled chip-firing that imposes a partial order on the last few firing moves of the process.

1.3 Labeled chip-firing

Labeled chip-firing modifies the process so that the chips are no longer indistinguishable. n chips are placed at the origin in the 1-dimensional lattice, with each chip assigned a distinct integer label. For each firing move, we choose two chips labeled a and b ($a < b$) at a common site i. Chip a is sent to site $i-1$, while chip b is sent to site $i+1$. The process continues until all chips are at distinct sites, and thus no further firing moves may be performed.

The example below shows a complete firing sequence for $n = 4$. We begin with chips labeled 1 through 4 at the origin (the adjacent blank spaces represent the neighboring sites, which initially have no chips).

$$\begin{array}{ccccc} & & 4 & & \\ & & 3 & & \\ & & 2 & & \\ \_ & \_ & \underline{1} & \_ & \_ \end{array}$$

We can then choose to fire any pair of chips at the origin. We choose chips 3 and 4, sending 3 to the left and 4 to the right.

$$\begin{array}{ccccc} & & 2 & & \\ \_ & \underline{3} & \underline{1} & \underline{4} & \_ \end{array}$$

We now fire chips 1 and 2:

$$\begin{array}{ccccc} & 1 & & 2 & \\ \_ & \underline{3} & \_ & \underline{4} & \_ \end{array}$$

There are now two sites that can fire. We choose to fire 2 and 4.

$$\begin{array}{ccccc} & 1 & & & \\ \_ & \underline{3} & \underline{2} & \_ & \underline{4} \end{array}$$

This leaves two more firing moves:

$$\begin{array}{ccccc} & & 3 & & \\ \underline{1} & \_ & \underline{2} & \_ & \underline{4} \end{array}$$

$$\begin{array}{ccccc} \underline{1} & \underline{2} & \_ & \underline{3} & \underline{4} \end{array}$$

There are no more available firing moves, so we have reached a final configuration. Note that in this instance, the chips are in sorted order from left to right. In fact, this system was shown in [9] to always terminate in a unique final configuration, in which all chips end in sorted order, as long as the number of chips n is even.

This is a notable result, as it shows global confluence for a system in which local confluence, and thus Newman's Lemma, do not apply. It is also what makes the main theorem much more difficult to prove than corresponding results in unlabeled chip-firing.

Some results are known involving global confluence without local confluence, including confluence of flow-firing [5] and forms of root system firing [7] and [6]. There have been attempts to generalize the results of [9] to produce similar results for other systems related to labeled chip-firing. In particular, this includes several conjectures from [9] and [8] in which labeled chip-firing is extended to several modified versions of the 1-dimensional grid graph. Additionally, the root system firing of Galashin et al. treat the labeled chip-firing problem as chip-firing in Type A, and then generalize the problem to apply to more general classes of firing moves [7] and to other types of root systems [6]. However, while global confluence results of this form appear to be true in many special cases, few results in labeled chip-firing are known beyond the even case of labeled chip-firing in [9].

Here, we present a novel proof of the even labeled case that is able to encompass a number of these problems. We define a partial order on all firing moves based on which moves are guaranteed to happen before others. The Hasse diagram for this ordering for $n = 10$ and $n = 11$ is shown in Figure 1, with $n = 20$ and $n = 21$ shown in Figure 2. We use this ordering to show that the end of the process is highly constrained and locally confluent. If chips satisfy certain location bounds (which we prove are guaranteed by the chip-firing process), then this final collection of moves must sort them.

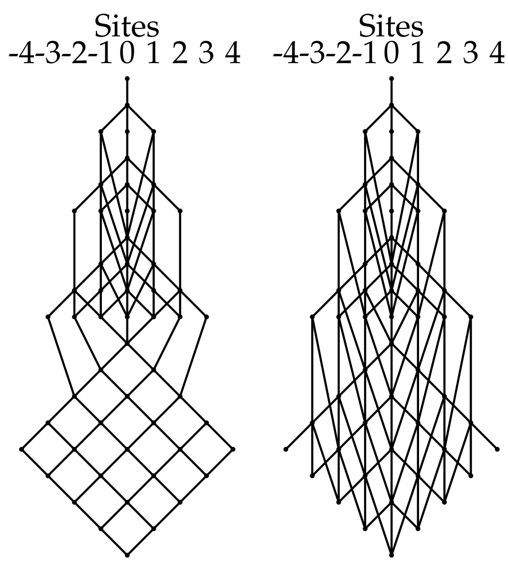

Figure 1: The full firing order poset for $n = 10$ (left) and $n = 11$ (right). These are identical to the Hasse diagrams for several other similar problems from Section 3, including the r edge case (for $10r \leq n < 11r$ and $11r \leq n < 12r$, respectively) and for the case with s self-loops at the origin (for $n - s = 10$ and $n - s = 11$, respectively).

In Section 2, we present the proof that labeled chip-firing beginning with an even number of chips at the origin terminates in a unique configuration with the chips in a sorted order. In Section 3, we apply these methods to prove a series of related conjectures on sorting via chip-firing on modified versions of the one-dimensional grid graph. In Section 4, we discuss how these methods can be applied to the case where the number of chips is odd.

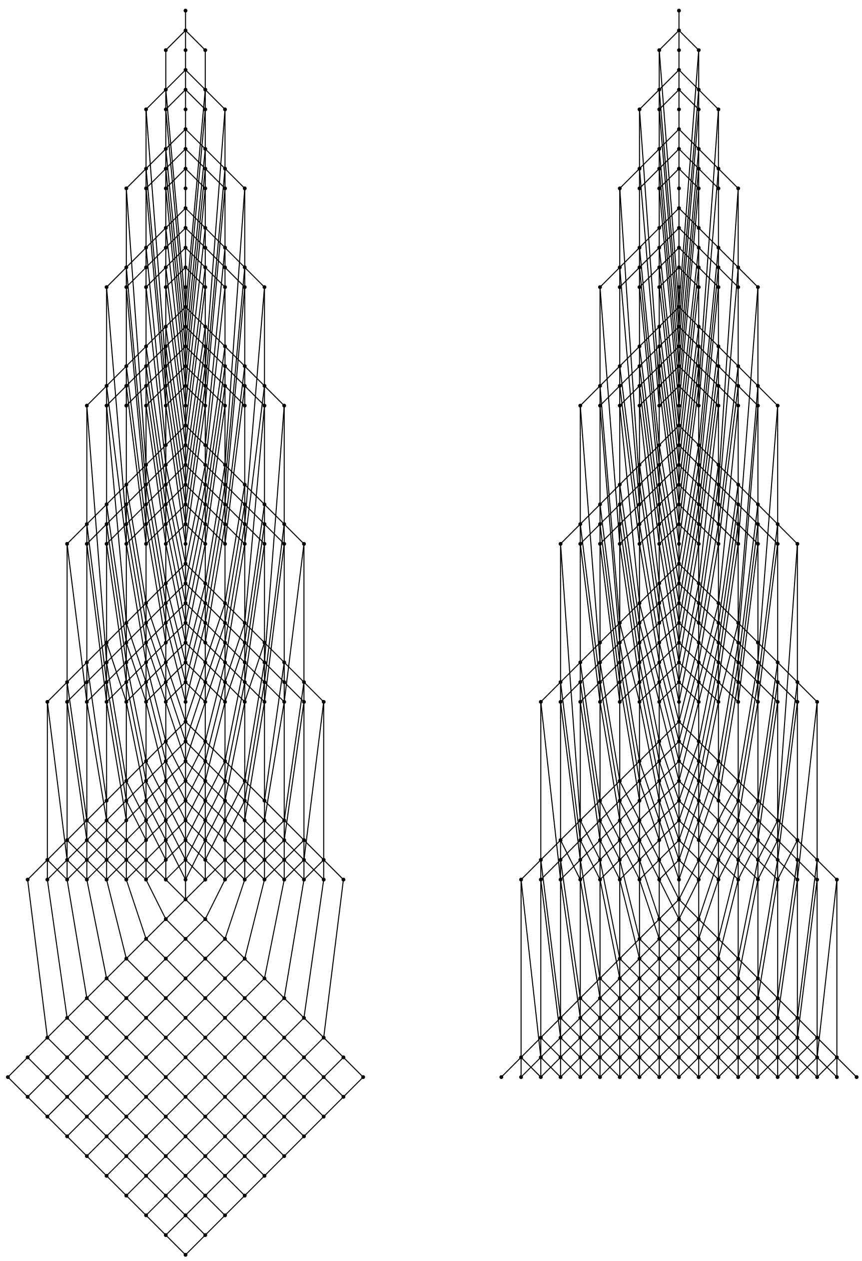

Figure 2: Firing order poset for $n = 20$ (left) and $n = 21$ (right).

2 Main Result

We now proceed to our main result: labeled chip-firing on a line with an even number of chips at the origin results in a sorted final configuration. Let $n = 2m$, and label the chips $-m, -m+1, \cdots, -1, 1, 2, \cdots, m$.

Theorem 2.1. *The chip-firing process with* $2m$ *chips at the origin terminates with all chips in sorted order from left to right.*

The proof consists of three main steps:

1. Prove that the last few firing moves in the process follow a locally confluent grid structure.
2. Bound the positions that chips can reach throughout the firing process.
3. Combine (1) and (2) to uniquely constrain a chip's final location.

The first step is a result on unlabeled chip-firing. Define the set P of firing moves in the process. Moves take the following form:

$$k_j = j^{th} \text{ to last firing move at site } k$$

The elements of the poset are ordered according to the relation $\geq_P$ (or just $\geq$), defined as follows:

$$k_j \geq_P k'_{j'} \text{ if move } k_j \text{ must occur before move } k'_{j'}$$
$$\text{in any complete sequence of firing moves}$$

Consider the diagram shown in Figure 3. The diagram consists of an m by m diamond with move 0_1 at the bottom, 0_m at the top, and $\pm(m-1)_1$ at the corners. For all vertices at the top of the diamond, except for the two at the left and the two at the right, there are red edges directed away from the diamond. The top vertex of the diamond has two such edges.

The first step in the proof is to show that this grid structure exists in the Hasse diagram for P (See Figure 3).

Lemma 2.2 (Grid Structure)**.** *Consider move* k_j *with* $-m+1 \leq k \leq m-1$, *and* $1 \leq j \leq m - |k|$. *Then*

1. *If* $k = 0$, *then* $0_j \leq 1_j$ *and* $0_j \leq -1_j$.
2. *If* $k \neq 0$, *then* $k_j \leq (k + sgn(k))_j$ *and* $k_j \leq (k - sgn(k))_{j+1}$, *if such moves exist.*
3. *Move* k_j *must take place with exactly 2 chips present at site* k.

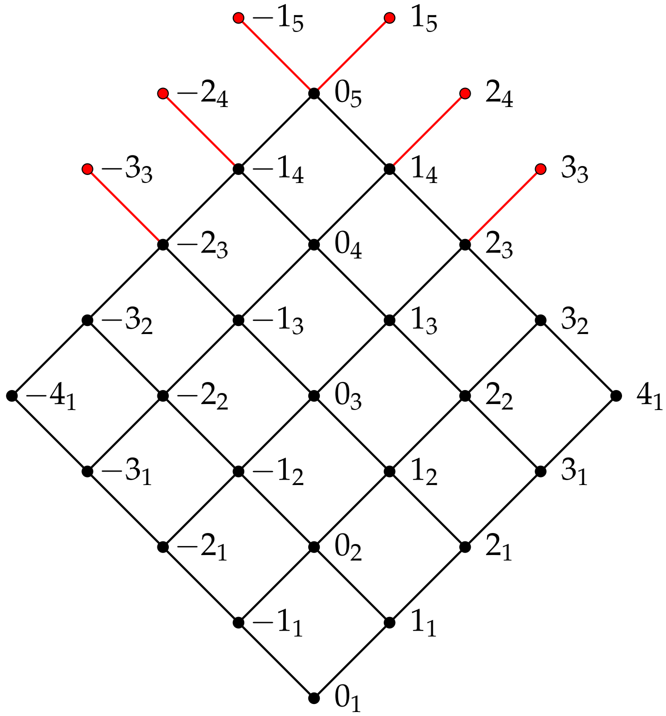

Figure 3: Firing move poset, $n = 10$.

The first two conditions in the lemma impose the relevant structure on the order of the firing moves, while the last condition ensures that each relevant firing move can only have one possible outcome. The proof is by induction on the row of vertices in the diamond, starting at the bottom. We do not go through the details of the induction, but the key idea is to keep track of the number of chips that have left or returned to a given site after each firing move. A counting argument using the results of Theorem 1.2 establishes the relative ordering of certain firings.

Next, we consider bounds on the locations of specific labeled chips. The following is a much weaker bound than the one provided in [9], but this weak bound is all that is needed for our result:

Lemma 2.3 (Position Bounds)**.** *The position of chip k ($k < 0$) must never exceed $k + m$ at any point in the chip-firing process. Similarly, the position of chip k ($k > 0$) must never be less than $k - m$ at any point in the chip-firing process.*

The proof follows quickly by induction on the chip k, based on the fact that the smallest chip at a site cannot fire to the right.

Finally, we show that each chip must end up at the correct final location after going through some of the firing moves in the diamond. Assign coordinates to the vertices in the diamond. The coordinates $(0,0)$ are assigned to the bottom vertex in the diamond. A move of one unit up and to the right corresponds to an increase of 1 in the first coordinate, while a move of one unit up and to the left corresponds to an increase of 1 in the second coordinate.

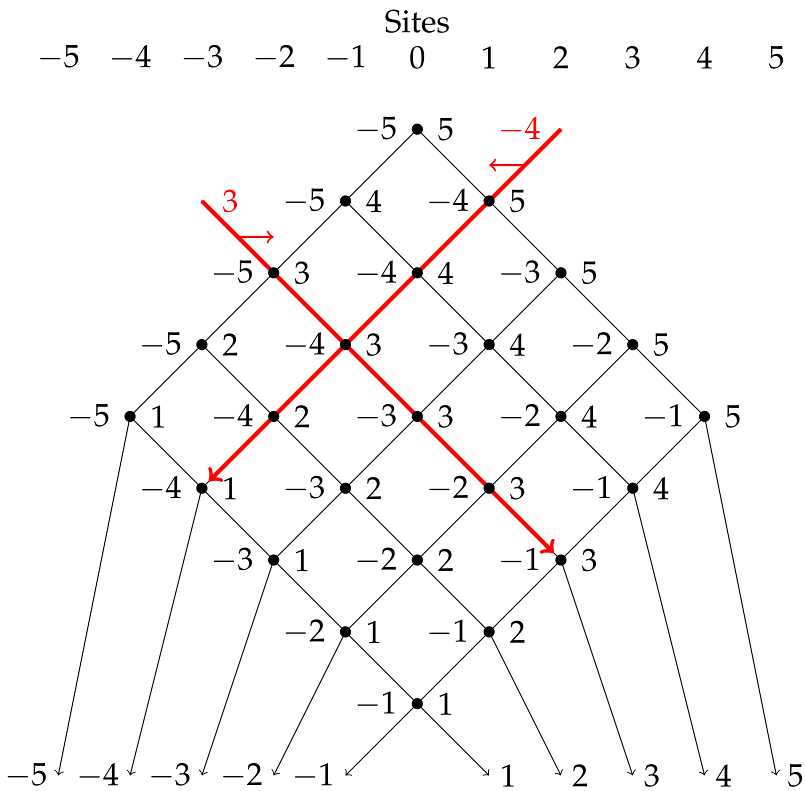

Figure 4: Diagram for the main theorem. For each firing move shown, the chip written to the left of the vertex must be at or to the left of that site when the firing move occurs, while the chip written to the right must be at or to the right of that site. The arrows at the bottom show how the last firing moves at each site send each of the chips to their final, sorted positions. As examples, the red arrows shown are the respective left and right bounds for the positions of chips 3 and -4 when given firing moves occur.

This gives us the following main lemma. Visually, Lemma 2.4 proves that for $k < 0$, chip k must always stay at or to the left of the line $y = -k - 1$ in the grid, while for $k > 0$, chip k must stay at or to the right of the line $x = k - 1$.

Lemma 2.4. *For each chip k with $-m \le k < 0$, and each x with $0 \le x \le m - 1$, the position of chip k may not exceed $x + k + 1$ immediately preceding firing move $(x, -k - 1)$. Similarly, for each $0 < k \le m$ and $0 \le y \le m - 1$, the position of chip k must be at least $k - 1 - y$ immediately preceding firing move $(k - 1, y)$.*

This lemma is illustrated in Figure 4. Each chip has a corresponding line in the diamond that it must never cross. The only final configuration satisfying the conditions of Lemma 2.4 is the one in which all chips are sorted. Again, this follows directly by induction on the chip number, and then on the firing move in a given line, based on the fact that if a site has only 2 chips, then any firing move at that site must send the smallest chip to the left.

3 Related Results

Using similar methods to the ones above, we can prove a number of results on sorting via chip-firing. All of the results come down to the same principles outlined above. There is a diamond of moves at the end of the process that does satisfy local confluence. The chip-firing process leading up to the diamond will “mess up” the order of the chips by a small enough amount that the diamond can always “fix” the resulting errors.

3.1 n odd, new initial configuration

If the number of chips n is odd, the initial configuration with all chips at the origin need not result in a sorted final configuration. However, a change in the initial configuration can lead to such a final state. The following result was conjectured by Galashin et al. as part of a larger conjecture about central firing on root systems [6]:

Theorem 3.1 (From Conjecture 7.1 of [6])**.** *In the labeled chip-firing problem beginning with chips $-m$ through 0 at the origin, and chips* 1 *through m at site 1, the final configuration is sorted.*

The proof is virtually identical to that of Theorem 2.1. The only significant difference is that the diamond is m by $m+1$ instead of m by m.

3.2 Multiple edges

We now modify the graph itself, rather than the initial chip configuration. We first consider a version of the problem in which each edge in the 1 dimensional grid is replaced with r edges. Each firing move sends r chips to the left and r chips to the right. In the labeled setting, each firing move consists of choosing $2r$ chips at the same site and then sending the r chips with the smallest labels to the left, and the r chips with the largest labels to the right.

Starting with a multiple of $2r$ chips, the final configuration will have r chips at each nonempty site, so the notion of sorting doesn’t directly apply. Instead, we have that in the final configuration, for all chips a and b with $a < b$, the final position of a is at or to the left of the final position of b. The result, as conjectured in [9], is as follows:

Theorem 3.2 (Conjecture 24 from [9])**.** *In labeled chip-firing with r copies of each edge and $2rm$ chips, the final configuration is weakly sorted with r chips at each site from $-m$ to -1, and from 1 to m.*

As there is a one-to-one correspondence between unlabeled states in this problem and in the original problem with $2m$ chips, the firing move poset is identical to the one in Theorem 2.1, and the same location bounds apply to groups of r chips rather than individual chips. The rest of the proof remains the same.

3.3 One Self-Loop at Every Site

We now turn to a case that is further from the original problem. Consider the original 1 dimensional grid graph with a self-loop at every vertex. Thus, every firing move requires 3 chips in order to fire, with the smallest of the three moving to the left, and the largest of the three moving to the right. We get the following result, again from [9]:

Theorem 3.3 (Conjecture 21 from [9])**.** *In labeled chip-firing with self loops, if the initial number of chips at the origin is congruent to 3 mod 4, then the final configuration has all chips in weakly sorted order.*

The proof of this result is more involved. It follows from the same outline, but the diamond that appears is smaller than in the original problem, and therefore much stronger bounds on chip positions are needed in order to obtain the result.

Just as the original problem can be generalized by making r copies of each edge, this problem can also be extended by making r copies of each edge and self-loop, resulting in a similar proof of sorting. For the Hasse diagram for the firing move poset for this case, see Figure 5.

3.4 Exponentially many edges

We now vary the number of edges between each pair of adjacent nodes. We choose a t, and then for every k from 0 to t, we place 2^{t-k} edges between nodes k and $k+1$, as well as between nodes $-k$ and $-k-1$. We also place one edge each between nodes $t+1$ and $t+2$, and between $-t-1$ and $-t-2$. If a site has a leftward edges and b rightward edges, we can fire at that site by choosing $a+b$ chips, and then sending the smallest a to the left and the largest b to the right. This gives us the following result:

Theorem 3.4. *In the exponential edge problem beginning with 2^{t+2} chips at the origin, the final configuration has all chips in weakly sorted order.*

As the final configuration may have many different numbers of chips at different nodes, the proof changes accordingly, but the grid structure of the diamond extends to the entire firing move poset, rather than just the final moves. For the Hasse diagram for the firing move poset for this case, see Figure 5.

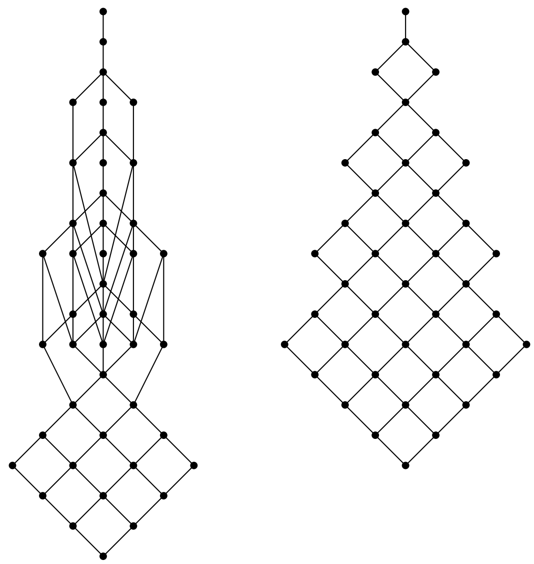

Figure 5: Left: the full firing order poset for $n = 15$ in the self-loop problem. Right: the firing order poset for the exponential edge case, with $t = 3$ and $n = 32$.

4 Applications to n odd

The main result of this paper (and of [9]) was to prove that labeled chip-firing on a 1-dimensional grid results in a sorted chip configuration where the number of chips n is even. When n is odd, aside from the trivial $n = 1$, this is no longer the case.

This is largely due to the last move in the process. While the last move in the even case took place with 2 chips present, the last move in the odd case has 3 chips present, and only one of the 3 possible firing moves result in a sorted configuration. However, it appears that this tends to be the only thing that prevents the process from sorting. We define three procedures for choosing a sequence of firing moves at random (see [10, Chapter 5]):

1. For each firing move, choose any pair of chips that can fire together uniformly at random.

2. For each firing move, choose a site with at least 2 chips uniformly at that random. Then choose 2 chips at that site uniformly at random to fire.

3. Choose a possible sequence of firing moves uniformly at random from all possible sequences.

There are two primary conjectures for the odd case of labeled chip-firing. While we have not yet resolved these conjectures, considering the form of P sheds light on the fundamental difference between the even and odd cases (see Figure 1).

Conjecture 4.1. *Under any of the above probability measures, the probability of* $2m+1$ *chips sorting converges to* $\frac{1}{3}$ *as* $n \to \infty$*.*

The Hasse diagrams for $n = 10$ and $n = 11$ are shown below. The number of firing moves at each site is the same in each case, but the odd $n = 11$ lacks the diamond structure, helping to explain why the chips can fail to sort.

Conjecture 4.2. *In an instance of labeled chip-firing with* $2m+1$ *chips, then the order of the chips in the final configuration must have at most* m *inversions.*

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