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Combinatorics of the double-dimer model

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Abstract. We prove that the partition function for tripartite double-dimer configurations of a planar bipartite graph satisfies a recurrence related to the Desnanot–Jacobi identity from linear algebra. A similar identity for the dimer partition function was established nearly 20 years ago by Kuo and has applications to random tiling theory and the theory of cluster algebras. This work was motivated in part by the potential for applications in these areas. Additionally, we discuss an application to Donaldson–Thomas and Pandharipande–Thomas theory which will be the subject of a forthcoming paper. The proof of our recurrence requires generalizing work of Kenyon and Wilson; specifically, lifting their assumption that the nodes of the graph are black and odd or white and even.

Keywords: double-dimer model, dimer model, recurrence

1 Introduction

Let $G = (V_1, V_2, E)$ be a finite edge-weighted bipartite planar graph embedded in the plane with $|V_1| = |V_2|$. Let $\mathbf{N}$ denote a set of special vertices called *nodes* on the outer face of G numbered consecutively in counterclockwise order. A *double-dimer configuration* on $(G, \mathbf{N})$ is a multiset of the edges of G with the property that each internal vertex is the endpoint of exactly two edges, and each vertex in $\mathbf{N}$ is the endpoint of exactly one edge. In other words, it is a configuration of paths connecting the nodes in pairs, doubled edges, and disjoint cycles of length greater than two (called *loops*). Define a probability measure Pr where the probability of a configuration is proportional to the product of its edge weights times 2^{ℓ}, where ℓ is the number of loops in the configuration. Kenyon and Wilson initiated the study of the double-dimer model in [6], by showing how to compute the probability that a random double-dimer configuration has a particular node pairing.

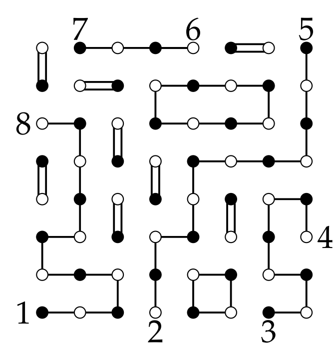

Figure 1: A double-dimer configuration on a grid graph with 8 nodes.

Before going into the details of Kenyon and Wilson's work, we will describe Kuo's recurrence for *dimer configurations*, which is the motivation for our work, and state one of

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our main results. A dimer configuration (or perfect matching) of G is a collection of the edges that covers all of the vertices exactly once. The weight of a dimer configuration is the product of its edge weights. Let $Z^D(G)$ denote the sum of the weights of all possible dimer configurations on G. In [7], Kuo proved that $Z^D(G)$ satisfies an elegant recurrence.

Theorem 1.1 ([7, Theorem 5.1])**.** *Let $G = (V_1, V_2, E)$ be a planar bipartite graph with a given planar embedding in which $|V_1| = |V_2|$. Let vertices $a, b, c,$ and d appear in a cyclic order on a face of G. If $a, c \in V_1$ and $b, d \in V_2$, then*

$$Z^D(G)Z^D(G - \{a, b, c, d\}) = Z^D(G - \{a, b\})Z^D(G - \{c, d\}) + Z^D(G - \{a, d\})Z^D(G - \{b, c\}).$$

His proof uses a technique called *graphical condensation*, named for its resemblance to *Dodgson condensation*, a method for computing the determinants of square matrices.

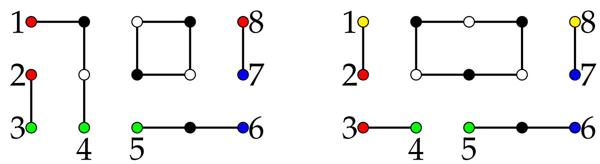


Figure 2: The pairing of the nodes on the left is a tripartite pairing because the nodes can be colored contiguously using three colors so that no pair contains nodes of the same color. The pairing on the right is not a tripartite pairing because four colors are required.

We prove that when σ is a *tripartite pairing*, a similar recurrence to Theorem 1.1 holds for $Z_\sigma^{DD}(G, \mathbf{N})$, the weighted sum of all double-dimer configurations on $(G, \mathbf{N})$ with pairing σ.

A planar pairing σ is a tripartite pairing if the nodes can be divided into three circularly contiguous sets $R, G,$ and B so that no node is paired with a node in the same set (see Figure 2). We often color the nodes in the sets red, green, and blue, in which case σ is the unique planar pairing in which like colors are not paired.

Theorem 1.2 ([3, Theorem 1.0.2])**.** *Let $G = (V_1, V_2, E)$ be a finite edge-weighted planar bipartite graph with a set of nodes $\mathbf{N}$. Divide the nodes into three circularly contiguous sets R, G, and B such that $|R|$, $|G|$ and $|B|$ satisfy the triangle inequality and let σ be the corresponding tripartite pairing*[1]*. Let x, y, w, v be nodes appearing in a cyclic order such that the set $\{x, y, w, v\}$ contains at least one node of each RGB color. If $x, w \in V_1$ and $y, v \in V_2$ then*

$$\begin{aligned} & Z_\sigma^{DD}(G, \mathbf{N})Z_{\sigma_5}^{DD}(G, \mathbf{N} - \{x, y, w, v\}) \\ = \; & Z_{\sigma_1}^{DD}(G, \mathbf{N} - \{x, y\})Z_{\sigma_2}^{DD}(G, \mathbf{N} - \{w, v\}) + Z_{\sigma_3}^{DD}(G, \mathbf{N} - \{x, v\})Z_{\sigma_4}^{DD}(G, \mathbf{N} - \{w, y\}) \end{aligned}$$

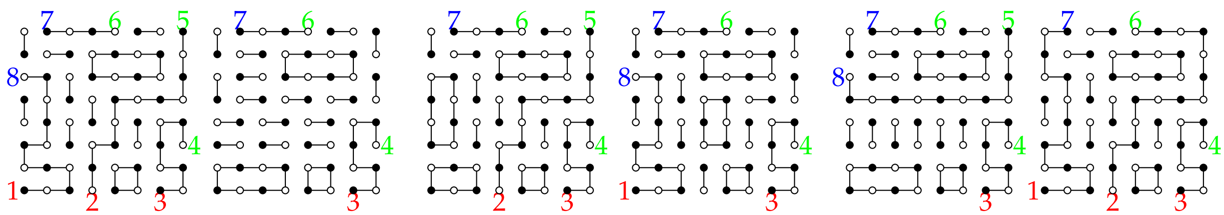
where σ_i is the unique planar pairing on the corresponding node set in which like colors are not paired together.

We illustrate Theorem 1.2 with an example.

[1] If $|R|, |G|,$ and $|B|$ do not satisfy the triangle inequality, there is no corresponding tripartite pairing σ.

Example 1.3. If G is a graph with eight nodes colored red, green, and blue as below, then $\sigma = ((1,8),(3,4),(5,2),(7,6))$. If $x = 8, y = 1, w = 2, v = 5$, then by Theorem 1.2,

$$Z^{DD}_{\sigma}(\mathbf{N})Z^{DD}_{\sigma_5}(\mathbf{N}-1,2,5,8) \;=\; Z^{DD}_{\sigma_1}(\mathbf{N}-1,8)Z^{DD}_{\sigma_2}(\mathbf{N}-2,5) + Z^{DD}_{\sigma_3}(\mathbf{N}-1,2)Z^{DD}_{\sigma_4}(\mathbf{N}-5,8)$$



We were motivated to find an analogue of Theorem 1.1 by its potential applications, which we discuss in the next section.

1.1 Applications

Kuo's work has a variety of applications. For example, Kuo uses graphical condensation to give a new proof that the number of tilings of an order-n Aztec diamond is $2^{n(n+1)/2}$ [7, Theorem 3.2] and a new proof for MacMahon's generating function for plane partitions that are subsets of a box [7, Theorem 6.1]. His results also have applications to random tiling theory (see [7, Section 4.1]) and the theory of cluster algebras.

Cluster algebras are a class of commutative rings introduced by Fomin and Zelevinsky [2] to study total positivity and dual canonical bases in Lie Theory. The theory of cluster algebras has since been connected to many areas of math, including quiver representations, Teichmüller theory, Poisson geometry, and integrable systems [17]. In [8, 9], Tri Lai and Gregg Musiker study toric cluster variables for the quiver associated to the cone over the del Pezzo surface dP_3, giving algebraic formulas for these cluster variables as Laurent polynomials. Using identities similar to Kuo's Theorem 1.1, they give combinatorial interpretations of most of these formulas [8].

We expect Theorem 1.2 to have similar applications. In addition, by using both Theorem 1.1 and Theorem 1.2 we can give a direct proof of a problem in Donaldson–Thomas and Pandharipande–Thomas theory.

1.1.1 Application to Donaldson–Thomas and Pandharipande–Thomas theory.

Donaldson–Thomas (DT) theory, Pandharipande–Thomas (PT) theory, and Gromov–Witten (GW) theory are branches of enumerative geometry closely related to mirror symmetry and string theory. The DT and GW theories give frameworks for counting curves[2] on a threefold X. A conjecture in [11, 12] gives a correspondence between the DT and GW frameworks, which has been proven in special cases, such as when X is toric [13].

[2]The frameworks differ in what is meant by a curve on X.

PT theory gives a third framework for counting curves when X is a nonsingular projective threefold that is Calabi-Yau. The correspondence between the DT and PT frameworks was first conjectured in [14] and was proven in [1], which is closely related to the work in [16]. Specifically, let X be a toric Calabi-Yau 3-fold. Define $Z_{DT}(q) = \sum_n I_n q^n$, where I_n counts length n subschemes of X, and $Z_{PT}(q) = \sum_n P_n q^n$, where P_n counts stable pairs on X (see [14]). Bridgeland proved that these generating functions coincide up to a factor of $M(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}$, which is the total q-weight of all plane partitions [10].

Theorem 1.4 ([1, Theorem 1.1])**.** $Z_{DT}(q) = Z_{PT}(q)M(-q)$.

The application of Theorem 1.2 that we describe relates to Theorem 1.4 at the level of the topological vertex. Define $V_{\lambda,\mu,\nu} = q^{c(\lambda,\mu,\nu)} \sum_\pi q^{|\pi|}$, where the sum is taken over all plane partitions π *asymptotic to* (λ, μ, ν). Maulik, Nekrasov, Okounkov, and Pandharipande [11, 12] proved that $Z_{DT}(q) = V_{\lambda,\mu,\nu}$ and thus $V_{\lambda,\mu,\nu}$ is called the DT topological vertex. Let $W_{\lambda,\mu,\nu} = q^{c(\lambda,\mu,\nu)} \sum_i d_i q^i$ where d_i is a certain weighted enumeration of *labeled box configurations* of length i [15]. In [15, Theorem/Conjecture 2] Pandharipande and Thomas conjecture that $W_{\lambda,\mu,\nu}$ is the stable pairs vertex, i.e. that $Z_{PT}(q) = W_{\lambda,\mu,\nu}$.

Conjecture 1.5 ([15, Calabi-Yau case of Conjecture 4])**.** $V_{\lambda,\mu,\nu} = W_{\lambda,\mu,\nu}M(-q)$.

Pandharipande and Thomas remark that a straightforward (but long) approach to this conjecture using DT theory exists [15]. In a forthcoming paper with Gautam Webb and Ben Young [4], we give a direct proof by showing that $V_{\lambda,\mu,\nu}$ is a single dimer model and $W_{\lambda,\mu,\nu}$ is a double-dimer model, and then using Theorems 1.1 and 1.2 to show that both sides of the above equation satisfy the same recurrence.

2 Proof of Theorem 1.2

Presently, we discuss the main ideas behind the proof of Theorem 1.2. We start by giving an overview of the results from [6, 5] that are needed for our work.

2.1 Background

Kenyon and Wilson gave explicit formulas for the probability that a random double-dimer configuration has a particular node pairing σ. When σ is a tripartite pairing, this probability is proportional to the determinant of a matrix.

To be more precise, we need to introduce some notation and definitions. Since G is bipartite, we can color its vertices black and white so that each edge connects a black vertex to a white vertex. Let G^{BW} be the subgraph of G formed by deleting the nodes except for the ones that are black and odd or white and even. Define G^{WB} analogously, but with the roles of black and white reversed. Let $G^{BW}_{i,j}$ be the graph G^{BW} with nodes i and j included if and only if they were not included in G^{BW}. For convenience, they assume the nodes alternate in color, so all nodes are black and odd or white and even. (If a graph G does not have this property, we can add edges of weight 1 to each node that has the wrong color to obtain a graph whose double-dimer configurations are in a one-to-one weight-preserving correspondence with double-dimer configurations of G.)

For each planar pairing σ, Kenyon and Wilson showed the normalized probability

$$\widehat{\Pr}(\sigma) := \Pr(\sigma)\frac{Z^D(G^{WB})}{Z^D(G^{BW})} = \frac{Z^{DD}_\sigma(G,\mathbf{N})}{(Z^D(G^{BW}))^2}$$

that a random double-dimer configuration has pairing σ is an integer-coefficient homogeneous polynomial in the quantities $X_{i,j} := \dfrac{Z^D(G^{BW}_{i,j})}{Z^D(G^{BW})}$ [6, Theorem 1.3].

For example, the normalized probability $\widehat{\Pr}$ that a random double-dimer configuration on eight nodes has the pairing $((1,8),(3,4),(5,2),(7,6))$ (see Figure 1) is

$$\begin{aligned}\widehat{\Pr}\left(\begin{array}{c|c|c|c}1&3&5&7\\8&4&2&6\end{array}\right) &= X_{1,8}X_{3,4}X_{5,2}X_{7,6} - X_{1,4}X_{3,8}X_{5,2}X_{7,6} + X_{1,6}X_{3,4}X_{5,8}X_{7,2}\\ &\quad -X_{1,8}X_{3,6}X_{5,2}X_{7,4} - X_{1,4}X_{3,6}X_{5,8}X_{7,2} + X_{1,6}X_{3,8}X_{5,2}X_{7,4}.\end{aligned}$$

Kenyon and Wilson gave an explicit method for computing these polynomials. They defined a matrix $\mathcal{P}^{(DD)}$ with entries $\mathcal{P}^{(DD)}_{\sigma,\tau}$; the rows are indexed by planar pairings and the columns are indexed by odd-even pairings. They showed how to calculate the columns of the matrix combinatorially and proved that for any planar pairing σ,

$$\widehat{\Pr}(\sigma) = \sum_{\text{odd-even pairings } \tau} \mathcal{P}^{(DD)}_{\sigma,\tau} X'_\tau. \tag{2.1}$$

where $X'_\tau = (-1)^{\#\text{ crosses of }\tau} \prod_{i \text{ odd}} X_{i,\tau(i)}$ [6, Theorem 1.4].

In the case where σ is a tripartite pairing, $\widehat{\Pr}(\sigma)$ is a determinant of a matrix.

Theorem 2.1 ([5, Theorem 6.1])**.** *Suppose that the nodes are contiguously colored red, green, and blue (a color may occur zero times), and that σ is the (unique) planar pairing in which like colors are not paired together. Let $\sigma(i)$ denote the item that σ pairs with item i. We have*

$$\widehat{Pr}(\sigma) = \det[1_{i,j \text{ colored differently}}\, X_{i,j}]^{i=1,3,\ldots,2n-1}_{j=\sigma(1),\sigma(3),\ldots,\sigma(2n-1)}.$$

Initially, it seems that Theorem 1.2 will follow immediately from combining Theorem 2.1 with the Desnanot–Jacobi identity.

Theorem 2.2 (Desnanot–Jacobi identity)**.** *Let $M = (m_{i,j})_{i,j=1}^{n}$ be a square matrix, and let M_i^j be the matrix resulting from M by deleting the ith row and the jth column for $1 \leq i, j \leq n$. Then*

$$\det(M)\det(M_{i,j}^{i,j}) = \det(M_i^i)\det(M_j^j) - \det(M_i^j)\det(M_j^i).$$

However, we run into some technical obstacles, which we illustrate with an example.

2.2 Example

Suppose we wish to prove the equation from Example 1.3:

$$\begin{aligned} Z_{\sigma}^{DD}(\mathbf{N})Z_{\sigma_5}^{DD}(\mathbf{N} - \{1,2,5,8\}) &= Z_{\sigma_1}^{DD}(\mathbf{N} - \{1,8\})Z_{\sigma_2}^{DD}(\mathbf{N} - \{2,5\}) \\ &\quad + Z_{\sigma_3}^{DD}(\mathbf{N} - \{1,2\})Z_{\sigma_4}^{DD}(\mathbf{N} - \{5,8\}) \end{aligned}$$

where recall that $\sigma = ((1,8),(3,4),(5,2),(7,6))$. Then the matrix M from Theorem 2.1 is

$$M = \begin{pmatrix} X_{1,8} & X_{1,4} & 0 & X_{1,6} \\ X_{3,8} & X_{3,4} & 0 & X_{3,6} \\ X_{5,8} & 0 & X_{5,2} & 0 \\ 0 & X_{7,4} & X_{7,2} & X_{7,6} \end{pmatrix}.$$

Since the first row and column of M correspond to nodes 1 and 8, respectively, and the third row and column correspond to nodes 5 and 2, we apply the Desnanot–Jacobi identity with $i = 1$ and $j = 3$:

$$\det(M)\det(M_{1,3}^{1,3}) = \det(M_1^1)\det(M_3^3) - \det(M_1^3)\det(M_3^1).$$

By Theorem 2.1, $\det(M) = \dfrac{Z_{\sigma}^{DD}(G,\mathbf{N})}{(Z^D(G^{BW}))^2}$.

We also need to prove, for example, that

$$\det(M_3^3) = \frac{Z_{\sigma_2}^{DD}(G,\mathbf{N} - \{2,5\})}{(Z^D(G^{BW}))^2} \qquad (2.2)$$

where $M_3^3 = \begin{pmatrix} X_{1,8} & X_{1,4} & X_{1,6} \\ X_{3,8} & X_{3,4} & X_{3,6} \\ 0 & X_{7,4} & X_{7,6} \end{pmatrix}$. An example of a double-dimer configuration counted by $Z_{\sigma_2}^{DD}(G,\mathbf{N} - \{2,5\})$ is shown in Figure 3 to the right.

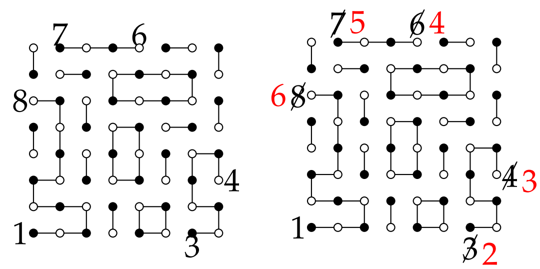


Figure 3: Left: A double-dimer configuration on a grid graph with node set $\mathbf{N} - \{2,5\}$. Right: The same double-dimer configuration after relabeling the nodes.

We cannot apply Theorem 2.1 to prove (2.2) because the nodes are not numbered consecutively. We might hope to resolve this by relabeling the nodes, as shown in Figure 3.But since Kenyon and Wilson assume that all nodes are black and odd or white and even, in order to satisfy the assumptions of Kenyon and Wilson's theorem, we need to add edges of weight 1 to nodes 2 and 3. Call the resulting graph $\widetilde{G}$ and let $\widetilde{X}_{i,j} = \frac{Z^D(\widetilde{G}^{BW}_{i,j})}{Z^D(\widetilde{G}^{BW})}$. The matrix from Theorem 2.1 is

$$\widetilde{M} = \begin{pmatrix} \widetilde{X}_{1,6} & 0 & \widetilde{X}_{1,4} \\ \widetilde{X}_{3,6} & \widetilde{X}_{3,2} & 0 \\ 0 & \widetilde{X}_{5,2} & \widetilde{X}_{5,4} \end{pmatrix}$$

To prove (2.2) it suffices to show

$$(Z^D(\widetilde{G}^{BW}))^2 \det(\widetilde{M}) = (Z^D(G^{BW}))^2 \det(M_3^3), \tag{2.3}$$

since $\det(\widetilde{M}) = \frac{Z^{DD}_{\sigma_2}(\widetilde{G}, \mathbf{N} - \{2,5\})}{(Z^D(\widetilde{G}^{BW}))^2}$ by Theorem 2.1.

Verifying (2.3) is a straightforward computation, but as we consider graphs with more nodes, the computations quickly become more involved. To be able to interpret the minors of Kenyon and Wilson's matrix outside of small examples, we need to lift their assumption that the nodes of the graph be black and odd or white and even.

Notice that under the assumption that the nodes of the graph are black and odd or white and even, $X_{i,j} = \frac{Z^D(G^{BW}_{i,j})}{Z^D(G^{BW})} = \frac{Z^D(G_{i,j})}{Z^D(G)}$. This suggests that the correct generalization of Kenyon and Wilson's matrix will have entries $\frac{Z^D(G_{i,j})}{Z^D(G)}$.

2.3 Generalization of Kenyon and Wilson

The previous remark motivates our approach, which is to define $Y_{i,j} := \frac{Z^D(G_{i,j})}{Z^D(G)}$ and $\widetilde{\Pr}(\sigma) = \frac{Z^{DD}_{\sigma}(G, \mathbf{N})}{(Z^D(G))^2}$. When G is a graph with nodes that are either black and odd or white and even, $Z^D(G) = Z^D(G^{BW})$, so $Y_{i,j} = X_{i,j}$ and $\widetilde{\Pr}(\sigma) = \widehat{\Pr}(\sigma)$.

Many of Kenyon and Wilson's results from [6, 5] have analogues in the variables $Y_{i,j}$. Following Kenyon and Wilson's approach, for any black-white pairing ρ, we define

$$Y'_\rho = (-1)^{\text{\# crosses of } \rho} \prod_{i \text{ black}} Y_{i,\rho(i)}.$$

Note that we work with black-white pairings rather than odd-even pairings since we are not requiring that the nodes are black and odd or white and even. In [6, 5], black-white pairings and odd-even pairings coincide, so $X_{i,j} = 0$ when i and j have the same parity, which occurs exactly when they have the same color. In our general setting, $Y_{i,j}$ may be nonzero when i and j have the same parity, but if i and j are the same color then there are no dimer configurations of $G_{i,j}$, so $Y_{i,j} = 0$.

Our analogue of Kenyon and Wilson's matrix $\mathcal{P}^{(DD)}$ (see (2.1)) is $\mathcal{Q}^{(DD)}$. To define $\mathcal{Q}^{(DD)}$, we use Kenyon and Wilson's work as a road map, proving analogues of Lemmas 3.1 − 3.5 and Theorem 3.6 from [6]. The rows of $\mathcal{Q}^{(DD)}$ are indexed by planar pairings and columns are indexed by black-white pairings. To prove that $\mathcal{Q}^{(DD)}$ is integer-valued, we show that the columns of this matrix can be computed combinatorially. Given a black-white pairing ρ, the corresponding column of $\mathcal{Q}^{(DD)}$ can be computed (up to a sign) by repeatedly applying the following transformation rule.

Rule 2.3 ([6])**.** If a pairing ρ is nonplanar, then there will exist items $a < b < c < d$ such that a and c are paired, and b and d are paired. Let the remaining pairs be denoted by "rest". Then the transformation rule is

$$ac|bd|\text{rest} \to -ab|cd|\text{rest} - ad|bc|\text{rest}.$$

To describe how we correct the signs of the column entries, we need two definitions.

Definition 2.4. If σ is an odd-even pairing, then $\text{sign}_{OE}(\sigma)$ is the parity of the permutation $\begin{pmatrix} \frac{\sigma(1)}{2} & \frac{\sigma(2)}{2} & \cdots & \frac{\sigma(2n-1)}{2} \end{pmatrix}$, written in one-line notation.

Definition 2.5. If ρ is a black-white pairing, then we can write $\rho = ((b_1, w_1), \ldots, (b_n, w_n))$, where $b_1 < b_2 < \cdots < b_n$. Let $r : \{w_1, \ldots, w_n\} \to \{1, \ldots, n\}$ be the map defined by $r(k) = \#\{i : w_i \le w_k\}$. Then $\text{sign}_{BW}(\rho)$ is the parity of the permutation $\begin{pmatrix} r(w_1) & r(w_2) & \cdots & r(w_n) \end{pmatrix}$, written in one-line notation.

Rule 2.6 ([3, Rule 2.5.2])**.** For a black-white pairing ρ, repeatedly apply Rule 2.3 until we have written ρ as a linear combination of planar pairings. Then multiply each planar pairing σ in this sum by $\text{sign}_{OE}(\sigma)\text{sign}_{BW}(\rho)$.

We prove the following theorem, which generalizes [6, Theorem 1.4].

Theorem 2.7 ([3, Theorem 1.3.1])**.** *Any black-white pairing ρ can be transformed into a formal linear combination of planar pairings by repeated application of Rule 2.6, and the resulting linear combination does not depend on the choices made when applying Rule 2.6, so that we may write*

$$\rho \to \sum_{\textit{planar pairings } \sigma} \mathcal{Q}^{(DD)}_{\sigma,\rho}\,\sigma.$$

For any planar pairing σ, these same coefficients $\mathcal{Q}^{(DD)}_{\sigma,\rho}$ satisfy the equation

$$\widetilde{Pr}(\sigma) := \frac{Z^{DD}_{\sigma}(G,\mathbf{N})}{(Z^{D}(G))^2} = \sum_{\text{black-white pairings } \rho} \mathcal{Q}^{(DD)}_{\sigma,\rho} Y'_{\rho}.$$

Establishing that we can compute the columns of $\mathcal{Q}^{(DD)}$ using Rule 2.6 is key in proving Theorem 2.13, which is our generalization of Kenyon and Wilson's determinant formula (Theorem 2.1). Once we have Theorem 2.13 we will be able to apply the proof method described in Section 2.1 to prove Theorem 1.2.

2.3.1 Generalization of Theorem 2.1

To prove Theorem 2.1, Kenyon and Wilson use two results from their study of *groves.* If G is a finite edge-weighted planar graph with a set of nodes, a *grove* is a spanning acyclic subgraph of G such that each component tree contains at least one node. The weight of a grove is the product of the weights of the edges it contains.

The connected components of a grove partition the nodes into a planar partition. If σ is a planar partition of $1, 2, \ldots, n$, let $\Pr(\sigma)$ be the probability that a random grove of G partitions the nodes according to σ. Kenyon–Wilson showed that $\dddot{\Pr}(\sigma) := \dfrac{\Pr(\sigma)}{\Pr(1|2|\cdots|n)}$ is an integer-coefficient homogeneous polynomial in the variables $L_{i,j}$[3] [6, Theorem 1.2]. The grove polynomials $\dddot{\Pr}(\sigma)$ and the double-dimer polynomials $\widehat{\Pr}(\sigma)$ are related: when σ is a pairing, the grove polynomials specialize to the double-dimer polynomials.

Theorem 2.8 ([6, Theorem 4.2])**.** *If a planar partition σ only contains pairs and we make the following substitutions to the grove partition polynomial $\dddot{\Pr}(\sigma)$:*

$$L_{i,j} \to \begin{cases} 0, & \text{if } i \text{ and } j \text{ have the same parity,} \\ (-1)^{(|i-j|-1)/2} X_{i,j}, & \text{otherwise,} \end{cases}$$

then the result is $\mathrm{sign}_{OE}(\sigma)$ times the double-dimer pairing polynomial $\widehat{\Pr}(\sigma)$, when we interpret σ as a pairing.

In the case where σ is a partition that is a tripartite pairing, the grove polynomial $\dddot{\Pr}(\sigma)$ can be expressed as a Pfaffian.

Theorem 2.9 ([5, Theorem 3.1])**.** *Let σ be the tripartite pairing partition defined by circularly contiguous sets of nodes R, G, and B, where $|R|$, $|G|$, and $|B|$ satisfy the triangle inequality. Then*

$$\dddot{\Pr}(\sigma) = Pf \begin{pmatrix} 0 & L_{R,G} & L_{R,B} \\ -L_{G,R} & 0 & L_{G,B} \\ -L_{B,R} & -L_{B,G} & 0 \end{pmatrix}$$

[3] When G is viewed as a resistor network with conductances equal to the edge weights, $L_{i,j}$ is the current flowing into node j if node i were set to one volt with all other nodes set to zero volts [6, Appendix A].

where $L_{R,G}$ is the submatrix of L whose rows are the red nodes and columns are the green nodes.

Kenyon and Wilson's Theorem 2.1 follows quickly from Theorems 2.8 and 2.9. We prove Theorem 2.13 similarly. We can use Theorem 2.9 as stated, but we need an analogue of Theorem 2.8 to obtain our polynomials $\widetilde{\Pr}(\sigma)$ from the grove polynomials.

In Theorem 2.8, $(-1)^{(|i-j|-1)/2}$ is always an integer because $X_{i,j} = 0$ if i and j have the same parity. Without the assumption that the nodes are black and odd or white and even, $Y_{i,j}$ may be nonzero for i and j with the same parity. Therefore we need a different way to define the sign of a pair of nodes (i, j). To motivate this definition, notice that if two nodes of opposite color b and w have the same parity, it cannot be the case that the nodes between b and w alternate black and white. Therefore we must keep track of the number of consecutive nodes of the same color between b and w. Consecutive nodes of the same color appear in pairs. For example, if we have a graph with eight nodes so that nodes $1, 3, 4$, and 6 are black and nodes $2, 5, 7, 8$ are white, there are two pairs of consecutive nodes of the same color: $(3, 4)$ and $(7, 8)$. Since we frequently use the term pair when describing pairings of the nodes, we will refer to pairs of consecutive nodes as *couples of consecutive nodes* instead.

Definition 2.10 ([3, Definition 2.1.6])**.** If (b, w) is a pair of nodes, let $a_{b,w}$ be the number of couples of consecutive nodes of the same color in the interval $[\min\{b, w\}, \min\{b, w\} + 1, \ldots, \max\{b, w\}]$. Define

$$\operatorname{sign}(b, w) = (-1)^{(|b-w|+a_{b,w}-1)/2}$$

We observe that when the nodes of G alternate black and white, $a_{b,w} = 0$ for all pairs (b, w), so this agrees with Kenyon and Wilson's definition of the sign of a pair of nodes.

Have established Definition 2.10, we can state our analogue of Theorem 2.8:

Theorem 2.11 ([3, Theorem 3.1.3])**.** *If a planar partition σ only contains pairs and we make the following substitutions to the grove partition polynomial $\ddot{Pr}(\sigma)$:*

$$L_{i,j} \to \begin{cases} 0, & \textit{if } i \textit{ and } j \textit{ are the same color,} \\ sign(i,j)Y_{i,j}, & \textit{otherwise,} \end{cases}$$

then the result is $sign_c(\mathbf{N}) sign_{OE}(\sigma)\widetilde{Pr}(\sigma)$.

In Theorem 2.11, $\operatorname{sign}_c(\mathbf{N})$ depends on the order in which the couples of consecutive nodes of the same color appear (see [3, Definition 2.1.8]). We prove Theorem 2.11 by comparing Rule 2.6 to Kenyon and Wilson's transformation rule for groves (see [6, Rule 1]) and applying the following lemma.

Lemma 2.12 ([3, Lemma 2.1.13])**.** *If ρ is a black-white pairing,*

$$sign_c(\mathbf{N}) sign_{BW}(\rho) \prod_{(b,w)\in\rho} sign(b, w) = (-1)^{\#\text{ crosses of } \rho}.$$

Recall that Kenyon and Wilson's Theorem 2.1 states that when σ is a tripartite pairing,

$$\widehat{\Pr}(\sigma) = \det[1_{i,j \text{ colored differently}} X_{i,j}]^{i=1,3,\ldots,2n-1}_{j=\sigma(1),\sigma(3)\ldots,\sigma(2n-1)}.$$

By reordering the columns, we see that

$$\widehat{\Pr}(\sigma) = \text{sign}_{OE}(\sigma) \det[1_{i,j \text{ colored differently}} X_{i,j}]^{i=1,3,\ldots,2n-1}_{j=2,4,\ldots,2n}.$$

Remarkably, our generalization of this theorem has no additional global sign.

Theorem 2.13 ([3, Theorem 1.3.2]). *Let G be a finite edge-weighted planar bipartite graph with a set of nodes. Suppose that the nodes are contiguously colored red, green, and blue (a color may occur zero times), and that σ is the (unique) planar pairing in which like colors are not paired together. We have*

$$\widetilde{\Pr}(\sigma) = sign_{OE}(\sigma) \det[1_{i,j \textit{ colored differently}} Y_{i,j}]^{i=b_1,b_2,\ldots,b_n}_{j=w_1,w_2,\ldots,w_n}.$$

where $b_1 < b_2 < \cdots < b_n$ are the black nodes and $w_1 < w_2 < \cdots < w_n$ are the white nodes.

By combining Theorem 2.13 with the Desnanot–Jacobi identity, we prove our main result:

Theorem 2.14 ([3, Theorem 1.3.3]). *Let $G = (V_1, V_2, E)$ be a finite edge-weighted planar bipartite graph with a set of nodes $\mathbf{N}$. Divide the nodes into three circularly contiguous sets R, G, and B such that $|R|, |G|$, and $|B|$ satisfy the triangle inequality and let σ be the corresponding tripartite pairing. If $x, w \in V_1$ and $y, v \in V_2$ then*

$$\begin{aligned} & sign_{OE}(\sigma) sign_{OE}(\sigma'_5) Z^{DD}_{\sigma}(G, \mathbf{N}) Z^{DD}_{\sigma_5}(G, \mathbf{N} - \{x, y, w, v\}) \\ = \; & sign_{OE}(\sigma'_1) sign_{OE}(\sigma'_2) Z^{DD}_{\sigma_1}(G, \mathbf{N} - \{x, y\}) Z^{DD}_{\sigma_2}(G, \mathbf{N} - \{w, v\}) \\ & - sign_{OE}(\sigma'_3) sign_{OE}(\sigma'_4) Z^{DD}_{\sigma_3}(G, \mathbf{N} - \{x, v\}) Z^{DD}_{\sigma_4}(G, \mathbf{N} - \{w, y\}) \end{aligned}$$

where σ_i is the unique planar pairing on the corresponding node set in which like colors are not paired together and σ'_i is the pairing after the corresponding node set has been relabeled so that the nodes are numbered consecutively.

Theorem 1.2 follows as a corollary; the additional assumptions in Theorem 1.2 lead to a nice simplification of the signs in Theorem 2.14.

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