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Ordered set partitions, Tanisaki ideals, and rank varieties

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Abstract. We introduce a family of ideals $I_{n,\lambda,s}$ in $\mathbb{Q}[x_1,\ldots,x_n]$ for λ a partition of $k \leq n$ and an integer $s \geq \ell(\lambda)$. This family contains both the Tanisaki ideals I_λ and the ideals $I_{n,k}$ of Haglund–Rhoades–Shimozono as special cases. We study the corresponding quotient rings $R_{n,\lambda,s}$ as symmetric group modules. When $n = k$ and s is arbitrary, we recover the Garsia–Procesi modules, and when $\lambda = (1^k)$ and $s = k$, we recover the generalized coinvariant algebras of Haglund–Rhoades–Shimozono.

We give a monomial basis for $R_{n,\lambda,s}$ in terms of (n,λ,s)-staircases, unifying the monomial bases studied by Garsia–Procesi and Haglund–Rhoades–Shimozono. We realize the S_n-module structure of $R_{n,\lambda,s}$ in terms of an action on (n,λ,s)-ordered set partitions. We find a formula for the Hilbert series of $R_{n,\lambda,s}$ in terms of inversion and diagonal inversion statistics on (n,λ,s)-ordered set partitions. Furthermore, we give an expansion of the graded Frobenius characteristic of our rings in terms of Gessel's fundamental basis and in terms of dual Hall–Littlewood symmetric functions.

We connect our work with Eisenbud–Saltman rank varieties using results of Weyman. As an application of our results on $R_{n,\lambda,s}$, we give a monomial basis, Hilbert series formula, and graded Frobenius characteristic formula for the coordinate ring of the scheme-theoretic intersection of a rank variety with diagonal matrices.

Keywords: Ordered set partitions, symmetric functions, Hall–Littlewood functions, Springer fibers, rank varieties

1 Introduction

The goal of this paper is to unify the representation theory and combinatorics of the *generalized coinvariant algebras* $R_{n,k}$ introduced by Haglund, Rhoades, and Shimozono [8], and the cohomology rings R_λ of the *Springer fibers* introduced by Springer [11]. On the one hand, the generalized coinvariant algebras are graded modules of the symmetric group whose combinatorics are controlled by ordered set partitions. On the other hand, the cohomology rings of the Springer fibers are graded modules of the symmetric group whose combinatorics are controlled by tabloids.

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We introduce a family of rings $R_{n,\lambda,s}$ which are graded modules of the symmetric group whose combinatorics are controlled by (n,λ,s)-*ordered set partitions*. We recover the rings $R_{n,k}$ and R_λ as special cases of our rings. We then give formulas for the dimension, Hilbert series, and graded Frobenius characteristic of $R_{n,\lambda,s}$ which generalize known formulas for $R_{n,k}$ and R_λ. In particular, we see that the graded Frobenius characteristic of $R_{n,\lambda,s}$ has a positive expansion in terms of modified Hall–Littlewood functions. One of our main techniques to prove these results is to realize $R_{n,\lambda,s}$ as the associated graded ideal of the defining ideal of a finite set of points. This also allows us to identify a monomial basis of $R_{n,\lambda,s}$ generalizing the Artin monomial basis of the coinvariant algebra. Furthermore, we show that the rings $R_{n,\lambda,s}$ have connections to the geometry of rank varieties defined by Eisenbud and Saltman [4] by using results due to Weyman [13]. These rank varieties are not to be confused with the rank varieties of Billey and Coskun [1].

2 Background on the rings $R_{n,k}$ and R_λ

Let us recall the generalized coinvariant algebras $R_{n,k}$. Fix positive integers $k \leq n$, and let $\mathbf{x}_n = \{x_1, \ldots, x_n\}$ be a set of n commuting variables. Let $\mathbb{Q}[\mathbf{x}_n]$ be the polynomial ring on the variables $\mathbf{x}_n$ with rational coefficients. We consider $\mathbb{Q}[\mathbf{x}_n]$ as a S_n-module, where S_n acts by permuting the variables. For $1 \leq d \leq n$, let $e_d(\mathbf{x}_n)$ be the *elementary symmetric polynomial of degree* d in the variables $\mathbf{x}_n$, defined by $e_d(\mathbf{x}_n) = \sum_{1\leq i_1<\cdots<i_d\leq n} x_{i_1}x_{i_2}\cdots x_{i_d}$. The ideal $I_{n,k}$ is defined to be

$$I_{n,k} = \langle x_1^k, x_2^k, \ldots, x_n^k, e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \ldots, e_{n-k+1}(\mathbf{x}_n)\rangle \subseteq \mathbb{Q}[\mathbf{x}_n]. \tag{2.1}$$

Since $I_{n,k}$ is homogeneous and stable under the action of S_n, the quotient ring $R_{n,k}$ has the structure of a graded S_n-module. Haglund, Rhoades and Shimozono defined the *generalized coinvariant algebra* $R_{n,k}$ to be the quotient ring $R_{n,k} = \mathbb{Q}[\mathbf{x}_n]/I_{n,k}$. When $k = n$, then it can be shown that (see [8, Section 1])

$$I_{n,n} = \langle e_1(\mathbf{x}_n), \ldots, e_n(\mathbf{x}_n)\rangle = \langle \mathbb{Q}[\mathbf{x}_n]_+^{S_n}\rangle, \tag{2.2}$$

which is the ideal generated by the positive degree invariants of $\mathbb{Q}[\mathbf{x}_n]$. Hence, $R_{n,n}$ is the well-known *coinvariant algebra*.

We also recall some terminology from [8]. Let $\mathcal{OP}_{n,k}$ be the collection of ordered set partitions of $[n]$ into k nonempty blocks. The group S_n acts on $\mathcal{OP}_{n,k}$ by permuting the letters $1, 2, \ldots, n$. Define the usual q-analogues of numbers, factorials, and multinomial coefficients,

$$[n]_q = 1 + q + \cdots + q^{n-1}, \qquad [n]!_q = [n]_q[n-1]_q\cdots[1]_q, \tag{2.3}$$

$$\begin{bmatrix} n \\ a_1, \ldots, a_r \end{bmatrix}_q = \frac{[n]!_q}{[a_1]!_q\cdots[a_r]!_q}, \qquad \begin{bmatrix} n \\ a \end{bmatrix}_q = \frac{[n]!_q}{[a]!_q[n-a]!_q}. \tag{2.4}$$

Let $\mathbf{x} = (x_1, x_2, \dots)$ be an infinite set of variables, and let $\mathbb{Z}[[x]]$ be the formal power series ring over the integers in the variables $\mathbf{x}$. Given $f \in \mathbb{Z}[[x]][q]$, let $f = a_0 + a_1 q + \cdots + a_n q^n$ be its expansion as a polynomial in q with coefficients in $\mathbb{Z}[[x]]$. Define $\mathrm{rev}_q(f) = a_n + a_{n-1}q + \cdots + a_0 q^n$.

Given two sequences of nonnegative integers $(a_1, \dots, a_r)$ and $(b_1, \dots, b_s)$, a *shuffle* of these two sequences is an interleaving $(c_1, \dots, c_{r+s})$ of the two sequences such that the a_i appear in order from left to right and the b_i appear in order from left to right. An (n,k)-*staircase* is a shuffle of the sequence $(0, 1, \dots, k-1)$ and the sequence $((k-1)^{n-k})$ consisting of $k-1$ repeated $n-k$ many times.

Haglund, Rhoades, and Shimozono proved that $R_{n,k}$ has the following properties which generalize the well-known properties of the coinvariant algebra [8].

- The dimension of $R_{n,k}$ is given by $\dim_{\mathbb{Q}}(R_{n,k}) = |\mathcal{OP}_{n,k}| = k! \cdot \mathrm{Stir}(n,k)$. The Hilbert polynomial is

$$\mathrm{Hilb}_q(R_{n,k}) = \mathrm{rev}_q([k]!_q \cdot \mathrm{Stir}_q(n,k)) = \sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\mathrm{coinv}(\sigma)}, \tag{2.5}$$

  where $\mathrm{Stir}_q(n,k)$ is a well-known q-analogue of the Stirling number of the second kind and $\mathrm{coinv}(\sigma)$ is the coinversion statistic on ordered set partitions.

- The set of monomials

$$\mathcal{A}_{n,k} = \{x_1^{a_1} \cdots x_n^{a_n} \ : \ (a_1, \dots, a_n) \text{ is component-wise } \leq \text{ some } (n,k)\text{-staircase}\} \tag{2.6}$$

  represents a basis of $R_{n,k}$, generalizing the Artin basis of the coinvariant algebra. As a consequence, we have $|\mathcal{A}_{n,k}| = |\mathcal{OP}_{n,k}|$.

- As S_n-modules,

$$R_{n,k} \cong_{S_n} \mathbb{Q}\mathcal{OP}_{n,k}, \tag{2.7}$$

  where $\mathbb{Q}\mathcal{OP}_{n,k}$ is the vector space over $\mathbb{Q}$ whose basis is indexed by $\mathcal{OP}_{n,k}$ and whose S_n-module structure is induced from the natural action of S_n on $\mathcal{OP}_{n,k}$.

- The graded S_n-module structure of $R_{n,k}$ can be expressed in terms of the dual Hall–Littlewood symmetric functions $Q'_\mu(\mathbf{x};q)$ as follows,

$$\mathrm{Frob}_q(R_{n,k}) = \mathrm{rev}_q \left[\sum_{\mu} q^{\sum_{i \geq 1}(i-1)(\mu_i - 1)} \begin{bmatrix} k \\ m_1(\mu), \dots, m_n(\mu) \end{bmatrix}_q Q'_\mu(\mathbf{x};q) \right], \tag{2.8}$$

  where the sum is over partitions μ of n into k parts. See [9] for the definition of a dual Hall–Littlewood symmetric function.

- The S_n-module $R_{n,k}$ is related to the *Delta Conjecture* of Haglund, Remmel, and Wilson [7]. Precisely, they prove that

$$\mathrm{Frob}_q(R_{n,k}) = (\mathrm{rev}_q \circ \omega) C_{n,k}(\mathbf{x}; q), \tag{2.9}$$

where $C_{n,k}(\mathbf{x}; q)$ is the expression in the Delta Conjecture at $t = 0$, and ω is the involution on symmetric functions sending a Schur function s_λ to $s_{\lambda'}$.

For each partition $\lambda \vdash n$, let R_λ be the cohomology ring of the *Springer fiber* $\mathcal{F}_\lambda$ indexed by λ with rational coefficients. See [2] for more background on Springer fibers. One remarkable property of Springer fibers is that the cohomology ring R_λ has a symmetric group action, due to Springer [11], which does not come from an action on the variety $\mathcal{F}_\lambda$ itself. We refer to the graded S_n-module R_λ as the *Garsia–Procesi module* based on their seminal work in [5] on the S_n-module structure of R_λ.

By work of De Concini and Procesi [3], the ring R_λ has an explicit description in terms of generators and relations. The particular presentation for R_λ we give next is due to Tanisaki [12], who simplified the presentation as well as many of the proofs in [3]. Let the conjugate of λ be $\lambda' = (\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n \geq 0)$. Here, we pad the conjugate partition by 0s to make it length n. Let $p^n_m(\lambda) = \lambda'_n + \lambda'_{n-1} + \cdots + \lambda'_{n-m+1}$ for $1 \leq m \leq n$. Given a subset of variables $S \subseteq \mathbf{x}_n$ and a positive integer d, define $e_d(S)$ to be the sum over all squarefree monomials of degree d in the set of variables S. The *Tanisaki ideal* I_λ is defined by

$$I_\lambda = \langle e_d(S) \,:\, S \subseteq \mathbf{x}_n,\, |S| \geq d > |S| - p^n_{|S|}(\lambda) \rangle, \tag{2.10}$$

and the ring R_λ is defined by

$$R_\lambda = \mathbb{Q}[\mathbf{x}_n]/I_\lambda. \tag{2.11}$$

When $\lambda = (1^n)$, a single column, then the Springer fiber corresponding to (1^n) is the *complete flag variety* whose cohomology ring is the coinvariant algebra. Indeed, we have $R_{(1^n)} = \mathbb{Q}[\mathbf{x}_n]/\langle e_1(\mathbf{x}_n), \ldots, e_n(\mathbf{x}_n)\rangle$, which is the coinvariant algebra.

The ring R_λ has the following properties [5].

- The dimension of R_λ is the multinomial coefficient

$$\dim_{\mathbb{Q}}(R_\lambda) = \binom{n}{\lambda_1, \ldots, \lambda_k}, \tag{2.12}$$

where $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$. The Hilbert series of R_λ is given by the generating function for the *cocharge* statistic on a certain set of words, see [5, Remark 1.2]. Given $\lambda \vdash n$, we draw the Young diagram of λ in the French convention with λ_i cells in the ith row, where we number the rows from bottom to top. We have the following alternative characterization of the Hilbert series which follows from [6, Equation 36] upon setting $t = 0$,

$$\mathrm{Hilb}_q(R_\lambda) = \sum_{\sigma} q^{\mathrm{inv}(\sigma)}, \tag{2.13}$$

where the sum is over fillings of the Young diagram of λ with the number $1, \ldots, n$ which increase down each column, and inv is the number of *inversions* of σ [6].

- There is a monomial basis $\mathcal{A}_\lambda$ of R_λ which specializes to the Artin basis of the coinvariant algebra when $\lambda = (1^n)$. In [5], this basis is denoted by $\mathscr{B}(\lambda)$.

- As S_n-modules, we have

$$R_\lambda \cong_{S_n} \mathbb{Q}(S_n/S_{\lambda_1} \times \cdots \times S_{\lambda_k}), \tag{2.14}$$

  where $S_{\lambda_1} \times \cdots \times S_{\lambda_k}$ is the Young subgroup of S_n permuting $1, \ldots, \lambda_1$ among themselves, $\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2$ among themselves, and so on. Equivalently, R_λ is isomorphic to the S_n-module given by the action of S_n on tabloids of shape λ.

- The graded S_n-module structure of R_λ is given by the reversal of the dual Hall–Littlewood symmetric function, which is sometimes referred to as the modified Hall–Littlewood symmetric function $\widetilde{H}_\lambda(\mathbf{x}; q)$,

$$\mathrm{Frob}_q(R_\lambda) = \mathrm{rev}_q(Q'_\lambda(\mathbf{x}; q)) = \widetilde{H}_\lambda(\mathbf{x}; q). \tag{2.15}$$

  See [8] for background on the graded Frobenius characteristic and dual Hall–Littlewood symmetric functions.

- If $\lambda, \mu \vdash n$ such that $\lambda \geq_{\mathrm{dom}} \mu$, we have the monotonicity property

$$[s_\nu]\mathrm{Frob}_q(R_\lambda) \leq [s_\nu]\mathrm{Frob}_q(R_\mu), \tag{2.16}$$

  for all $\nu \vdash n$, where $[s_\nu]f$ stands for the coefficient of s_ν in the Schur function expansion of f, and the inequality is a coefficient-wise comparison of two polynomials in q.

3 The rings $R_{n,\lambda,s}$

Fix positive integers $k \leq n$, a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ of k, and an integers $s \geq \ell(\lambda)$, where $\ell(\lambda) = \ell$ is the length of the partition. Let the conjugate of λ be $\lambda' = (\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n \geq 0)$, where we pad the conjugate partition by 0s to make it length n, and define $p^n_m(\lambda) = \lambda'_n + \lambda'_{n-1} + \cdots + \lambda'_{n-m+1}$ for $1 \leq m \leq n$. We introduce the ring $R_{n,\lambda,s}$, defined as follows.

Definition 3.1. Define the ideal $I_{n,\lambda,s}$ and quotient ring $R_{n,\lambda,s}$ by

$$I_{n,\lambda,s} = \langle x_i^s \,:\, 1 \le i \le n \rangle + \langle e_d(S) \,:\, S \subseteq \mathbf{x}_n,\, d > |S| - p^n_{|S|}(\lambda) \rangle, \tag{3.1}$$
$$R_{n,\lambda,s} = \mathbb{Q}[\mathbf{x}_n]/I_{n,\lambda}. \tag{3.2}$$

Since the ideal $I_{n,\lambda,s}$ is generated by homogeneous polynomials, it is a homogeneous ideal. Furthermore, since the generating set is closed under the action of S_n, the ideal $I_{n,\lambda,s}$ is symmetric. Therefore, the quotient ring $R_{n,\lambda,s}$ inherits the structure of a graded S_n-module. For example, let $n = 6$, $\lambda = (3,2)$ and $s = 3$, so that $k = 5$. Then the ideal $I_{6,(3,2),3}$ is generated by the set of homogenous polynomials

$$\{x_1^3, \ldots, x_6^3\} \cup \{e_2(\mathbf{x}_6), e_3(\mathbf{x}_6), e_4(\mathbf{x}_6), e_5(\mathbf{x}_6), e_6(\mathbf{x}_6)\} \cup \{e_3(S) \,|\, S \subseteq \mathbf{x}_6, |S| = 5\} \tag{3.3}$$

$$\cup \{e_4(S) \,|\, S \subseteq \mathbf{x}_6, |S| = 5\} \cup \{e_5(S) \,|\, S \subseteq \mathbf{x}_6, |S| = 5\} \cup \{e_4(S) \,|\, S \subseteq \mathbf{x}_6, |S| = 4\}, \tag{3.4}$$

which is closed under the action of S_6.

The generalized coinvariant algebras $R_{n,k}$ and the rings R_λ are special cases of the rings $R_{n,\lambda,s}$. We have

$$R_{n,k} = R_{n,(1^k),k} \qquad \text{for } k \le n, \tag{3.5}$$
$$R_\lambda = R_{n,\lambda,\ell(\lambda)} \qquad \text{for } \lambda \vdash n, \tag{3.6}$$

where (3.5) follows from Definition 3.1, whereas (3.6) is less obvious. As a bonus, we also have $R_{n,k,s} = R_{n,(1^s),k}$, where $R_{n,k,s}$ is the ring defined in [8, Section 6].

3.1 Dimension and Hilbert series of $R_{n,\lambda,s}$

We say that an (n,λ,s)-*ordered set partition* is a weak ordered set partition $(B_1|B_2|\cdots|B_s)$ of $[n]$ into s parts such that $|B_i| \ge \lambda_i$ for all $i \le \ell(\lambda)$. Here, we allow B_i to be empty for $i > \ell(\lambda)$. Let $\mathcal{OP}_{n,\lambda,s}$ be the set of (n,λ,s)-ordered set partitions. The group S_n acts on $\mathcal{OP}_{n,\lambda,s}$ by permuting the letters $1, 2, \ldots, n$. For example, when $n = 4$, $\lambda = (2,1)$, and $s = 2$, we have

$$\mathcal{OP}_{4,(2,1),2} = \{(123|4), (124|3), (134|2), (234|1), (12|34), \tag{3.7}$$
$$(13|24), (14|23), (23|14), (24|13), (34|12)\}. \tag{3.8}$$

Our first main result is a combinatorial description of the dimension of the ring $R_{n,\lambda,s}$.

Theorem 3.2. *The dimension of $R_{n,\lambda,s}$ is given by* $\dim_{\mathbb{Q}}(R_{n,\lambda,s}) = |\mathcal{OP}_{n,\lambda,s}|$

Furthermore, we have a monomial basis for the ring $R_{n,\lambda,s}$. For $1 \le j \le \lambda_1$, let $\beta^j(\lambda) = (0, 1, \ldots, \lambda'_j - 1)$. An (n,λ,s)-*staircase* is a shuffle of the compositions $\beta^1(\lambda)$, $\beta^2(\lambda)$,..., $\beta^{\lambda_1}(\lambda)$, and $((s-1)^{n-k})$.

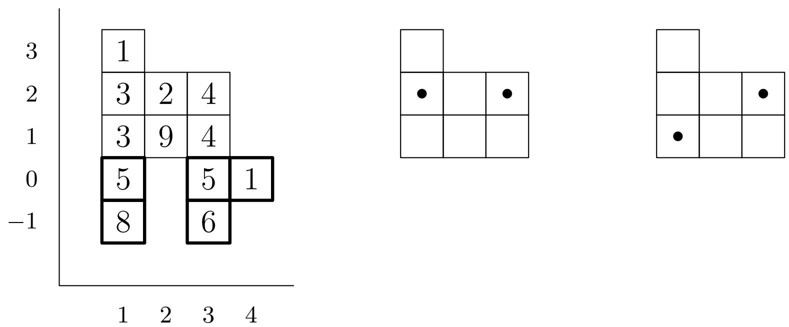

Figure 1: On the left, an extended column-increasing filling in $\mathrm{ECI}_{12,(3,2,2),4}$, where basement cells are in bold. In the middle and right, two examples of attacking pairs of $\mathrm{dg}(\lambda')$ for $\lambda = (3,2,2)$.

Theorem 3.3. *We have that*

$$\mathcal{A}_{n,\lambda,s} = \{x_1^{a_1} \cdots x_n^{a_n} : (a_1, \ldots, a_n) \text{ is component-wise } \leq \text{ some } (n,\lambda,s)\text{-staircase}\} \tag{3.9}$$

represents a basis of $R_{n,\lambda,s}$.

Let $\mathrm{Par}(n,s)$ be the set of partitions of n into at most s many parts. Given $\lambda \in \mathrm{Par}(n,s)$, recall that we draw the Young diagram $\mathrm{dg}(\lambda)$ in the French convention, where we number the rows from bottom to top. We also number the columns from left to right. We index the cells of $\mathrm{dg}(\lambda)$ in Cartesian coordinates, so that (i,j) is the cell in the ith column and jth row.

Define an *extended column-increasing filling of λ' with s columns* to consist of

- A diagram $\mathrm{D}(\varphi) = \mathrm{dg}(\lambda') \cup \mathrm{B}(\varphi)$, where $\mathrm{B}(\varphi)$ is a possibly empty collection of *basement cells* in columns $1 \leq i \leq s$ and rows $j \leq 0$, such that in each column i the basement cells are top justified so that the top basement cell is at coordinates $(i,0)$,
- A labeling of the cells of $\mathrm{D}(\varphi)$ with positive integers which weakly increases down each column.

Given a cell $(i,j) \in \mathrm{D}(\varphi)$, we denote by $\varphi_{i,j}$ the label of φ in the cell (i,j). Let $\mathrm{ECI}_{n,\lambda,s}$ be the set of extended column-increasing fillings φ of λ' with s columns and n cells. Let $\mathrm{SECI}_{n,\lambda,s}$ be the subset of $\mathrm{ECI}_{n,\lambda,s}$ consisting of *standard* extended column-increasing fillings which use the letters in $[n]$ without repetition. See the left side of Figure 1 for an example of an extended column-increasing filling in $\mathrm{ECI}_{12,(3,2,2),4}$. Given two cells (i,j) and (i',j') of $\mathrm{dg}(\lambda')$, we say that $((i,j),(i',j'))$ is an *attacking pair* if either $j = j'$ and $i < i'$, or $j = j' + 1$ and $i > i'$. See the middle and right side of Figure 1 for two examples of attacking pairs, where we indicate the cells in the attacking pair with dots.

Given $\varphi \in \mathrm{ECI}_{n,\lambda,s}$, an *inversion* of φ is one of the following,

1. An attacking pair $((i,j),(i',j'))$ of $\mathrm{dg}(\lambda')$ such that $\varphi_{i,j} > \varphi_{i',j'}$,
2. A pair $((i,1),(i',j'))$ such that $(i,1) \in \mathrm{dg}(\lambda')$, and $(i',j') \in \mathrm{B}(\varphi)$ such that $i > i'$ and $\varphi_{i,1} > \varphi_{i',j'}$,
3. A pair $(i,(i',j'))$, where $(i',j') \in \mathrm{B}(\varphi)$ and i is an integer such that $1 \le i < i'$.

Let $\mathrm{inv}(\varphi)$ be the number of inversions of φ. For φ in Figure 1, we have the following inversions,

Type 1: $((1,2),(2,2))$, $((3,2),(1,1))$, $((2,1),(3,1))$
Type 2: $((2,1),(1,0))$, $((2,1),(1,-1))$
Type 3: $(1,(3,0))$, $(2,(3,0))$, $(1,(3,-1))$, $(2,(3,-1))$, $(1,(4,0))$, $(2,(4,0))$, $(3,(4,0))$.

In total, we have $\mathrm{inv}(\varphi) = 12$.

The extended column-increasing fillings defined here are a variation of the fillings introduced by Rhoades–Yu–Zhao in [10] during the preparation of this article. To translate from our conventions and theirs, simply flip our labelings across the horizontal axis and convert each basement label into a floating number. They prove that the Hilbert series of $R_{n,\lambda,s}$ is the generating function of a *coinversion* statistic on ordered set partitions. The inversion statistic above can be seen to be a slight variation of their coinversion statistic after identifying a standard extended column-increasing filling φ with the ordered set partition where B_i is the set of labels in the ith column of φ. We prove that the generating function of the inversion statistic on extended column-increasing fillings also gives a formula for the Hilbert series. Since the results in [10] rely on our theorems, we are careful to give independent proofs.

Theorem 3.4. *We have*

$$\mathrm{Hilb}_q(R_{n,\lambda,s}) = \sum_{\varphi \in \mathrm{SECI}_{n,\lambda,s}} q^{\mathrm{inv}(\varphi)}. \tag{3.10}$$

3.2 S_n-module structure

In this subsection, we identify $R_{n,\lambda,s}$ as a symmetric group module. Our main strategy, used by Garsia–Procesi [5] and formalized by Haglund–Rhoades–Shimozono [8, Section 4.1], is to show that $I_{n,\lambda,s}$ is the associated graded ideal of the defining ideal of a finite set of points in $\mathbb{Q}^n$.

Fix s distinct rational numbers $\alpha_1, \ldots, \alpha_s \in \mathbb{Q}$. Let $X_{n,\lambda,s}$ be the set of points $p = (p_1, \ldots, p_n) \in \mathbb{Q}^n$ such that for each $1 \le i \le n$, we have $p_i = \alpha_j$ for some j, and for each $1 \le i \le s$, we have that α_i appears as a coordinate in p at least λ_i many times. Let $\mathbf{I}(X_{n,\lambda,s})$ be the defining ideal of $X_{n,\lambda,s}$ as a variety in $\mathbb{Q}^n$, and let $\mathrm{gr}\,\mathbf{I}(X_{n,\lambda,s})$ be the associated graded ideal of $\mathbf{I}(X_{n,\lambda,s})$. Since $X_{n,\lambda,s}$ is closed under the S_n-action permuting coordinates, then $\mathbf{I}(X_{n,\lambda,s})$ and $\operatorname{gr}\mathbf{I}(X_{n,\lambda,s})$ are closed under the action permuting the variables. Hence, the ring $\mathbb{Q}[\mathbf{x}_n]/\operatorname{gr}\mathbf{I}(X_{n,\lambda,s})$ is an S_n-module.

Theorem 3.5. *We have the following chain of equalities and S_n-module isomorphisms*

$$R_{n,\lambda,s} = \frac{\mathbb{Q}[\mathbf{x}_n]}{\operatorname{gr}\mathbf{I}(X_{n,\lambda,s})} \cong_{S_n} \mathbb{Q}X_{n,\lambda,s} \cong_{S_n} \mathbb{Q}\mathcal{OP}_{n,\lambda,s}. \tag{3.11}$$

Proof Sketch. The middle isomorphism of S_n-modules in (3.11) follows by general facts about defining ideals and associated graded ideals, see [8, Section 4.1]. In order to show the first equality in (3.11), we first show directly that each generator of $I_{n,\lambda,s}$ is in $\operatorname{gr}\mathbf{I}(X_{n,\lambda,s})$, so we have the containment $I_{n,\lambda,s} \subseteq \operatorname{gr}\mathbf{I}(X_{n,\lambda,s})$. We then prove that $|\mathcal{A}_{n,\lambda,s}| = |X_{n,\lambda,s}|$, hence by Theorem 3.3, we see that $R_{n,\lambda,s}$ and $\frac{\mathbb{Q}[\mathbf{x}_n]}{\operatorname{gr}\mathbf{I}(X_{n,\lambda,s})}$ have the same dimension. Hence, we must have the equality $I_{n,\lambda,s} = \operatorname{gr}\mathbf{I}(X_{n,\lambda,s})$, and the equality of rings follows. Finally, the last isomorphism of S_n-modules in (3.11) follows by constructing an explicit S_n-equivariant bijection between $X_{n,\lambda,s}$ and $\mathcal{OP}_{n,\lambda,s}$. □

Given $\mu \in \operatorname{Par}(n,s)$, we say λ *is contained in* μ if $\lambda_i \leq \mu_i$ for all $i \leq \ell(\lambda)$. Given $\mu \in \operatorname{Par}(n,s)$ such that $\lambda \subseteq \mu$, let $n(\mu,\lambda) = \sum_i \binom{\mu_i' - \lambda_i'}{2}$.

Using the skewing operators $e_j^{\perp}$ utilized in [5, 8], we are able to identify the graded Frobenius characteristic of $R_{n,\lambda,s}$ in terms of dual Hall–Littlewood functions. It can be checked that our formula for $\operatorname{Frob}_q(R_{n,\lambda,s})$ specializes to (2.8) when $\lambda = (1^k)$ and (2.15) when $n = k$. We also give an expansion of the graded Frobenius characteristic in terms of the inversion statistic.

Theorem 3.6. *We have*

$$\operatorname{Frob}_q(R_{n,\lambda,s}) = \operatorname{rev}_q \left[\sum_{\substack{\mu \in \operatorname{Par}(n,s), \\ \lambda \subseteq \mu}} q^{n(\mu,\lambda)} \prod_{i \geq 0} \begin{bmatrix} \mu_i' - \lambda_{i+1}' \\ \mu_i' - \mu_{i+1}' \end{bmatrix}_q Q_\mu'(\mathbf{x};q) \right], \tag{3.12}$$

where we define $\mu_0' = s$.

Theorem 3.7. *We have*

$$\operatorname{Frob}_q(R_{n,\lambda,s}) = \sum_{\varphi \in \mathrm{ECI}_{n,\lambda,s}} q^{\operatorname{inv}(\varphi)} \mathbf{x}^{\varphi}, \tag{3.13}$$

where $\mathbf{x}^{\varphi} = \prod_{i \geq 1} x_i^{\#i\text{'s in } \varphi}$.

Furthermore, we prove that the coefficients in the Schur expansion of $\operatorname{Frob}_q(R_{n,\lambda,s})$ enjoy two types of monotonicity. Here, the relation $\geq_{\mathrm{dom}}$ is the usual dominance relation on partitions. That is, if $\lambda, \mu \in \operatorname{Par}(k,s)$ then $\lambda \geq_{\mathrm{dom}} \mu$ if and only if $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all $i \leq s$.

Theorem 3.8. *Let $h \leq k \leq n$ be positive integers, let $\lambda \in \mathrm{Par}(k,s)$, and let $\mu \in \mathrm{Par}(h,s)$ such that either $h = k$ and $\lambda \geq_{dom} \mu$ or $h < k$ and $\lambda \supseteq \mu$. We have the monotonicity property*

$$[s_\nu]\mathrm{Frob}_q(R_{n,\lambda,s}) \leq [s_\nu]\mathrm{Frob}_q(R_{n,\mu,s}), \tag{3.14}$$

for all $\nu \vdash n$.

Haglund, Rhoades, and Shimozono [8] use Gröbner bases to prove their results. In particular, they find Gröbner bases of the ideals $I_{n,k}$ in terms of *Demazure characters*. To the author's knowledge, such explicit Gröbner bases for the ideals I_λ are not known. Therefore, different techniques are required to prove our results. Indeed, we prove the above results without the use of Gröbner bases using techniques similar to those of Garsia and Procesi. It is an open problem to find explicit Gröbner bases for the ideals $I_{n,\lambda,s}$.

4 Rank varieties

Let $\mathfrak{gl}_n$ be the set of $n \times n$ matrices with entries in $\mathbb{Q}$. For $\lambda \vdash n$, let $O_\lambda \subseteq \mathfrak{gl}_n$ be the conjugacy class of nilpotent $n \times n$ matrices over $\mathbb{Q}$ whose Jordan blocks have sizes recorded by λ. Let $\overline{O}_\lambda$ be the closure of O_λ in $\mathfrak{gl}_n$ in the Zariski topology. Let $\mathfrak{t}$ be the set of diagonal matrices. De Concini and Procesi [3] proved that R_λ is isomorphic to the coordinate ring of the scheme-theoretic intersection $\overline{O}_{\lambda'} \cap \mathfrak{t}$.

We connect the rings $R_{n,\lambda,s}$ to a generalization of these scheme-theoretic intersections as follows. Define $I_{n,\lambda}$ to be the ideal

$$I_{n,\lambda} = \langle e_d(S) \,:\, S \subseteq \mathbf{x}_n,\, d > |S| - p^n_{|S|}(\lambda) \rangle. \tag{4.1}$$

Define the quotient ring $R_{n,\lambda} = \mathbb{Q}[\mathbf{x}_n]/I_{n,\lambda}$. Observe that $R_{n,\lambda}$ has positive Krull dimension when $k < n$, and hence is infinite-dimensional as a $\mathbb{Q}$-vector space.

Let $k \leq n$, and let $\lambda \vdash k$. The *Eisenbud–Saltman rank variety* is the subvariety of $n \times n$ matrices,

$$\overline{O}_{n,\lambda} = \{X \,:\, \mathrm{rk}(X^d) \leq (n-k) + p^n_{n-d}(\lambda),\, d = 1,2,\ldots,n\}. \tag{4.2}$$

The variety $\overline{O}_{n,\lambda}$ is the same as X_r defined in [4], where r is the rank function defined by $r(d) = (n-k) + p^n_{n-d}(\lambda)$. When $n = k$, we have $\overline{O}_{n,\lambda} = \overline{O}_\lambda$. When $n > k$, then $\overline{O}_{n,\lambda}$ contains matrices which are not nilpotent. We have the following corollary of work by Weyman [13], who gave an explicit generating set for the defining ideal $I(\overline{O}_{n,\lambda})$.

Corollary 4.1. *We have an isomorphism of graded rings*

$$R_{n,\lambda} \cong \mathbb{Q}[\overline{O}_{n,\lambda'} \cap \mathfrak{t}],$$

where the right-hand side is the coordinate ring of the scheme-theoretic intersection of $\overline{O}_{n,\lambda'}$ with the diagonal matrices $\mathfrak{t}$.

Recall that $\beta^j(\lambda) = (0, 1, \ldots, \lambda'_j - 1)$ for $1 \leq j \leq \lambda_1$. Define an (n, λ)-*staircase* to be a shuffle of $\beta^1(\lambda), \beta^2(\lambda), \ldots, \beta^{\lambda_1}(\lambda)$, and (∞^{n-k}). Combining Corollary 4.1 and Theorem 3.3, we have the following characterization of a monomial basis for this coordinate ring.

Theorem 4.2. *The set*

$$\mathcal{A}_{n,\lambda} = \{x_1^{a_1} \cdots x_n^{a_n} : (a_1, \ldots, a_n) \text{ is component-wise } \leq \text{ some } (n, \lambda)\text{-staircase}\} \tag{4.3}$$

represents a basis of $R_{n,\lambda} \cong \mathbb{Q}[\overline{O}_{n,\lambda'} \cap \mathfrak{t}]$.

We also give a formula for the Hilbert series and graded Frobenius characteristic of $R_{n,\lambda}$. Let

$$\mathrm{ECI}_{n,\lambda} = \bigcup_{s \geq \ell(\lambda)} \mathrm{ECI}_{n,\lambda,s}, \tag{4.4}$$

$$\mathrm{SECI}_{n,\lambda} = \bigcup_{s \geq \ell(\lambda)} \mathrm{SECI}_{n,\lambda,s}, \tag{4.5}$$

where we identify $\varphi \in \mathrm{ECI}_{n,\lambda,s}$ with the extended column-increasing filling in $\mathrm{ECI}_{n,\lambda,s+1}$ obtained by appending an empty $(s+1)$th column to φ. We similarly identify each element of $\mathrm{SECI}_{n,\lambda,s}$ with its counterpart in $\mathrm{SECI}_{n,\lambda,s+1}$. Observe that for each $\varphi \in \mathrm{ECI}_{n,\lambda,s}$, the statistic $\mathrm{inv}(\varphi)$ does not depend on the parameter s. Hence, we may consider inv to be a statistic on elements of $\mathrm{ECI}_{n,\lambda}$.

Theorem 4.3. *For any $k \leq n$ and partition $\lambda \vdash k$,*

$$\mathrm{Frob}_q(R_{n,\lambda}) = \mathrm{Frob}_q(\mathbb{Q}[\overline{O}_{n,\lambda'} \cap \mathfrak{t}]) = \sum_{\varphi \in \mathrm{ECI}_{n,\lambda}} q^{\mathrm{inv}(\varphi)} \mathbf{x}^{\varphi}. \tag{4.6}$$

Corollary 4.4. *We have*

$$\mathrm{Hilb}_q(R_{n,\lambda}) = \mathrm{Hilb}_q(\mathbb{Q}[\overline{O}_{n,\lambda'} \cap \mathfrak{t}]) = \sum_{\varphi \in \mathrm{SECI}_{n,\lambda}} q^{\mathrm{inv}(\varphi)}. \tag{4.7}$$

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